Selected Topics in Lagrangian Geometry from Geometric Analytic Viewpoints

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Contents

Preface

The materials presented in this book is based on a graduate course at the University of British Columbia, Vancouver, which was delivered by the author in the fall semester of 2021.

The essential theme of the book is about basic properties of the stationary points of the volume functional of submanifolds in various geometric settings. After a motivative discussion on fundamental rigidity and curvature estimates results for stable hypersurfaces, we mainly focus on a distinguished class of higher co-dimensional volume critical points, namely, within the space of Lagrangian submanifolds in a symplectic manifold.

We provide a precise introduction on the background, necessary to the developments in the book, on symplectic geometry, and provide detailed proofs of some important facts, such as the Darboux's theorem, Weinstein's Lagrangian neighbourhood theorem, the Darboux coordinates with estimates of Lee-Joyce-Schoen, which will be needed in the later part of the discussion. Our main concentration is on the so-called Hamiltonian stationary Lagrangian submanifolds, in $\mathbb{C}^n$ and beyond.

Regularity of HSL submanifolds lies in the central place of the book. Two approaches are presented in full detail. The first one observes the delicate bootstrapping by viewing the HSL equation as factorized fourth order elliptic equation into a Laplace operator acting on the second order elliptic operator Θ, which admits a geometric interpretation when the ambient is $\mathbb{C}^n$ as the Lagrangian phase function and most importantly it can be written a sum of the arctangent for further linkage to the Evans-Krylov theory, now in the inhomogeneous setting as demonstrated by Cabré-Caffarelli. Secondly, we deliver a fourth order elliptic theory, which applies to a reasonably large class of elliptic equations in the double divergence form. Our regularity theory on 4th-order equations, in this less geometric approach, enables us to conclude statements in a general symplectic manifold. Getting rid off the special arctan expression of Θ in $\mathbb{C}^n$ separates the two situations (based on ambient space is $\mathbb{C}^n$ or not) is a conceptual advance for our understanding.

We will also address issues on removable singularity (in graphical setting or general setting) and on compactness of the space of compact HSL with bounded extrinsic total curvature and volume.

Riemannian Geometry of Submanifolds

1.1. First and second variations of volume

Let Σ be a manifold of dimension k and M be a manifold of dimension $n > k$. Suppose $f : \Sigma \to M$ is an immersion. We will write $\Sigma \subset M$ by identifying Σ with $f(\Sigma)$ is no confusion arises. Let h be the induced metric on Σ from (M, g). The second fundamental form of Σ of Σ in M at p is a map

$$A : T_p\Sigma \times T_p\Sigma \to (T_p\Sigma)^\perp$$

defined by

$$A(X, Y) = (\nabla_X Y)^\perp, \qquad X, Y \in T_p\Sigma.$$

Note that A is symmetric. The trace of A is the mean curvature $H = Tr_h A = h^{ij} A(\partial_i, \partial_j)$ where $\partial_1, \ldots, \partial_k$ is a basis for $T_p\Sigma$ and (h^{ij}) is the inverse of (h_{ij}) with $h_{ij} = h(\partial_i, \partial_j)$.

Let $F_t : M \to M$ be a family of diffeomorphisms with F_0 being the identity map for $t \in (-\epsilon, \epsilon)$. The variation vector field generated by F_t is denoted by $X = \frac{dF_t}{dt}|_{t=0}$. Write $\Sigma_t = F_t(f(\Sigma))$. Consider a family of maps

$$\varphi : \Sigma \times (-\epsilon, \epsilon) \to M$$

given by

$$\varphi(x, t) = F_t(f(x))$$

In the following, if no confusion arises, we will identity Σ with $f(\Sigma)$ from now on. Let g be a Riemannian metric on M. Take a coordinate system $(x^1, \ldots, x^k)$ around some $p \in \Sigma$. Denote the volume of the Σ_t in the induced metric $h = \varphi^* g$ on Σ by $|\Sigma_t|$. Then

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} |\Sigma_t| &= \int_\Sigma \frac{d}{dt}\Big|_{t=0} \sqrt{\det h}_{t=0}\ dx \\ &= \int_\Sigma h^{ij} (\nabla_{\partial_{x^i}} \varphi', \partial_{x^j})_{t=0} \sqrt{\det h}_{t=0}\, dx \\ &= \int_\Sigma div_\Sigma X \sqrt{\det h}_{t=0}\, dx. \end{aligned}$$

Therefore, we have the 1st variation formula:

$$\delta|\Sigma|(X) = \int_\Sigma div_\Sigma X\, d\mu_h. \tag{1.1}$$

Note that $div_\Sigma X = \sum_{i=1}^k \langle \nabla_{e_i} X, e_i \rangle$ with $e_1, \ldots, e_k$ o.n. basis for Σ.

Decompose $X = X^T + X^\perp$ along Σ. We have

$$div_\Sigma X = div_\Sigma X^T + div_\Sigma X^\perp$$

where, by setting $N_1, \ldots, N_{n-k}$ o.n. basis for $T\Sigma^\perp$, $X^\perp = \sum_{j=1}^{n-k}\langle X, N_j\rangle N_j$

$$div_\Sigma X^\perp = \sum_{i=1}^{k}\langle \nabla_{e_i} X^\perp, e_i\rangle = \sum_{j=1}^{n-k}\sum_{i=1}^{k}\langle X, N_j\rangle\langle \nabla_{e_i} N_j, e_i\rangle = -\langle X^\perp, H\rangle$$

by Weingarten's equations. By the divergence theorem, we get the **first variation formula of volume**:

$$\delta|\Sigma|(X) = \int_\Sigma \langle X, H\rangle d\mu + \int_{\partial\Sigma}\langle X, \eta\rangle d\nu \tag{1.2}$$

where η is the outward unit normal of $\partial\Sigma$ in Σ. We say Σ is a **minimal submanifold** in (M, g) if $\delta|\Sigma|(X) = 0$ for any C^1-regular X.

We now compute the 2nd variation of volume for a *normal* variation X at $t = 0$. First,

$$\frac{d}{dt}\sqrt{\det h} = \frac{1}{2}h^{ij}h'_{ij}\sqrt{\det h}.$$

Then

$$\frac{d^2}{dt^2}\sqrt{\det h} = \frac{1}{4}(h^{ij}h'_{ij})^2\sqrt{\det h} + \frac{1}{2}({h^{ij}}'h'_{ij})\sqrt{\det h} + \frac{1}{2}h^{ij}h''_{ij}\sqrt{\det h}.$$

At $t = 0$, we compute

$$\begin{aligned} h'_{ij} &= \langle \nabla_{\partial_t\varphi}\partial_i\varphi, \partial_j\varphi\rangle + \langle \nabla_{\partial_t\varphi}\partial_j\varphi, \partial_i\varphi\rangle \\ &= \langle \nabla_{\partial_i\varphi}\partial_t\varphi, \partial_j\varphi\rangle + \langle \nabla_{\partial_j\varphi}\partial_t\varphi, \partial_i\varphi\rangle \\ &= \langle \nabla_{\varphi_i}X, \partial_j\varphi\rangle + \langle \nabla_{\varphi_j}X, \partial_i\varphi\rangle \\ &= -\langle X, \nabla_{\varphi_j}\partial_i\varphi\rangle - \langle X, \nabla_{\varphi_i}\partial_j\varphi\rangle \\ &= -\langle A_{ij}, X\rangle - \langle A_{ji}, X\rangle \\ &= -2\langle A_{ij}, X\rangle \end{aligned}$$

Therefore, at $t = 0$,

$$h^{ij}h'_{ij} = -2\langle H, X\rangle \quad (= 0 \text{ if } \Sigma \text{ is minimal})$$

Next,

$$\begin{aligned} {h^{ij}}'h'_{ij} = -2{h^{ij}}'\langle A_{ij}, X\rangle &= 2h^{ik}h^{lj}h'_{kl}\langle A_{ij}, X\rangle \\ &= -4h^{ik}h^{jl}\langle A_{kl}, X\rangle\langle A_{ij}, X\rangle \\ &= -4|\langle A, X\rangle|^2. \end{aligned}$$

Thirdly,

$$\begin{aligned} h''_{ij} &= \langle \nabla_{\partial_t\varphi}\nabla_{\partial_i\varphi}\partial_t\varphi, \partial_j\varphi\rangle + \langle \nabla_{\partial_t\varphi}\partial_i\varphi, \nabla_{\partial_t\varphi}\partial_j\varphi\rangle + \langle \nabla_{\partial_t\varphi}\nabla_{\partial_j\varphi}\partial_t\varphi, \partial_i\varphi\rangle + \langle \nabla_{\partial_t\varphi}\partial_j\varphi, \nabla_{\partial_t\varphi}\partial_i\varphi\rangle \\ &= \langle R(\partial_t\varphi, \partial_i\varphi)\partial_t\varphi, \partial_j\varphi\rangle + \langle \nabla_{\partial_i\varphi}\varphi'', \partial_j\varphi\rangle + \langle \nabla_{\partial_t\varphi}\partial_i\varphi, \nabla_{\partial_t\varphi}\partial_j\varphi\rangle + i, j \text{ swapped terms} \\ &= -R(X, \partial_i\varphi, X, \partial_j\varphi) + \langle \nabla_{\partial_i\varphi}\varphi'', \partial_j\varphi\rangle + \langle \nabla_{\partial_i\varphi}X, \nabla_{\partial_j\varphi}X\rangle + i, j \text{ swapped terms.} \end{aligned}$$

It follows, at $t = 0$,

$$\begin{aligned} h^{ij}h''_{ij} &= -2R(X, \partial_i\varphi, X, \partial_j\varphi) + 2div_\Sigma\varphi'' + 2|\nabla^\perp X|^2 + 2h^{ij}(\nabla^T_{\partial_i}X, \nabla^T_{\partial_j}X) \\ &= -2R(X, \partial_i\varphi, X, \partial_j\varphi) + 2div_\Sigma\varphi'' + 2|\nabla^\perp X|^2 + 2|\langle A, X\rangle|^2. \end{aligned}$$

Hence,

$$\frac{d^2}{dt^2}\Big|_{t=0}\sqrt{\det h} = div_\Sigma\varphi'' + |\nabla^\perp X|^2 - |\langle A, X\rangle|^2 - h^{ij}R(X, \partial_i\varphi, X, \partial_j\varphi).$$

Integrating, we get the **second variation formula of volume**:

$$\delta^2|\Sigma|(X,X) = \int_\Sigma |\nabla^\perp X|^2 - |\langle A, X\rangle|^2 - \sum_{i=1}^k R(X,e_i,X,e_i) + \langle X, H\rangle^2 \tag{1.3}$$

where $e_1,\dots,e_k$ is am o.n. basis for the tangent space of Σ and X is normal to Σ.

A minimal submanifold is *stable* is its second variation is nonnegative for all X. When Σ is complete noncompact, we use compactly supported X. Σ is *volume minimizing* if any compact region $\Omega \subset \Sigma$ has least volume among k-dimensional submanfolds in M with the same boundary $\partial\Omega$. When $\partial\Sigma$ is nonempty, we can consider the so-called Morse index. Suppose the normal v.f. X vanishes on $\partial\Sigma$. Integration by parts allows us to write

$$\delta^2|\Sigma|(X,X) = -\int_\Sigma \langle LX, X\rangle d\mu_\Sigma$$

where

$$LX = \Delta^\perp X + trR^M(\cdot, X, \cdot, X) + \sum_{i,j}\langle A(e_i,e_j), X\rangle A(e_i,e_j)$$

and

$$\Delta^\perp = \sum_{i=1}^k (\nabla_{e_i}\nabla_{e_i}X)^\perp - \sum_{i=1}^k (\nabla_{(\nabla_{e_i}e_i)^T}X)^\perp$$

is the normal Laplacian along Σ. We call L *stability operator* or *Jacobi operator* (defined on the normal bundle of Σ). A *Jacobi field* is a normal vector field X with $LX = 0$. We can verify that L is self-adjoint. It has discrete real eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \to \infty$ on compact $\Omega \subset \Sigma$ for nontrivial normal v.f. X vanishing on $\partial\Sigma$ such that

$$LX + \lambda X = 0.$$

The number of negative eigenvalues of L (counting multiplicity) acting on the space of smooth sections of the normal bundle which vanish on $\partial\Sigma$ is called the *Morse index* of the minimal submanifold Σ.

1.2. Minimal Submanifolds

1.2.1. Examples and basic facts.

(1) Graphs in $\mathbb{R}^{n+1}$. Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Its graph $(x, f) = \{(x, f(x) : x \in \Omega\}$ is a hypersurface in $\mathbb{R}^{n+1}$ whose mean curvature is

$$H = div\left(\frac{Df}{\sqrt{1+|Df|^2}}\right)$$

in turn the divergence form of the minimal surface equation (MSE) is

$$\frac{\partial}{\partial x^i}\frac{u_i}{\sqrt{1+|Df|^2}} = 0.$$

Work out $n = 2$, the classical MSE. Catenoid is a complete embedded minimal surface in $\mathbb{R}^3$ with two ends. Search for other minimal surfaces in $\mathbb{R}^3$ on-line!

(2) $\mathbb{S}^n \cap P_k$ is minimal in $\mathbb{S}^n$ with the standard metric, where P_k is a k-plane through 0.

(3) M, N are minimal submanifolds in $(M \times N, g_1 \times g_2)$. In fact, they are *totally geodesic*, i.e. $A \equiv 0$.
(4) Every compact surface can be minimally immersed into $\mathbb{S}^n$. (Calabi)
(5) Any minimal submanifold in $\mathbb{S}^n$ is unstable! (Lawson-Simons)
(6) Simons cone. The 7-dimensional cone

$$\{(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x| = |y|\}$$

has zero mean curvature outside 0.
(7) If $M \subset \mathbb{S}^n$ is an immersed minimal submanifold, then the cone over M

$$C(M) = \{tx : t > 0, x \in M\}$$

is minimal in $\mathbb{R}^{n+1}$.
(8) Every complex submanifold, for example the 0-locus of a holomorphic polynomial, in $\mathbb{C}P^n$ with the Fubini-Study metric is minimal and in fact is volume minimizing in its homology class. More generally, every complex submanifold ino a Kähler manifold minimizes volume in its homology class, therefore is a minimal submanifold. (Wirtinger's inequality)
(9) There is no *compact* immersed minimal submanifolds in $\mathbb{R}^n$ (any co-dimension!). This is due to the Liouville property: for a minimal immersion $F : \Sigma \to \mathbb{R}^n$

$$\Delta F = H = 0.$$

(10) The Plateau Problem: Show the existence of a minimal surface with a given boundary, a problem raised by Lagrange in 1760. However, it is named after Plateau who experimented with soap films. The affirmative answer to this question was credited to Douglas (awarded the Fields medal for this work) and Rado. It does not say the solution is unique.
(11) Minimal surfaces. Let $f : \Sigma^2 \to (M, g)$ be an immersion. The, $f(\Sigma)$ is a minimal submanifold in (M, g) if and only if f is *conformal*, i.e. $f^*g = e^{2u}h$ where h is a metric on Σ, and f is a *harmonic map* from (Σ, h) to (M, g) defined by

$$\tau(f)^\alpha := h^{ij}\partial^2_{ij} f^\alpha - h^{ij}(\Gamma^\Sigma)^k_{ij}\partial_k f^\alpha + h^{ij}(\Gamma^M)^\alpha_{\beta\gamma}\partial_i f^\beta \partial_j f^\gamma = 0,$$

where $i, j = 1, \dots, \dim\Sigma$ and $\alpha = 1, \dots, \dim M$. $\tau(f)$ is called the tension field of f.

1.2.2. Simons identity and Simons inequality. This refers to the important formula, first derived by J. Simons (1967), that computes the Laplacian of the second fundamental form A of a minimal submanifold in a Riemannian manifold. We will follow S. S. Chern ["Minimal submanifolds of a Riemannian manifold", lecture notes, Univ. Kansas, 1969], as recorded in Schoen-Simon-Yau (1975).

Let Σ^n be an oriented minimal submanifold in (M^{n+1}, g). Choose local o.n. frame $e_1, \dots, e_{n+1}$ on N such that $e_i|_\Sigma$ is tangential to Σ. Let $\omega_1, \dots, \omega_{n+1}$ be the dual frames, i.e. $\omega_i(e_j) = \delta_{ij}$. The structure equations of M are given by

$$d\omega_i = -\sum_{j=1}^{n+1} \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0. \tag{1.4}$$

$$d\omega_{ij} = -\sum_{k=1}^{n+1} \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}. \tag{1.5}$$

defining the connection matrix ω_{ij} and the curvature matrix

$$\Omega_{ij} = \frac{1}{2}\sum_{k,l=1}^{n+1} K_{ijkl}\omega_k \wedge \omega_l, \qquad K_{ijkl} + K_{ijlk} = 0,$$

respectively. Restricting to Σ,

$$\omega_{n+1} = 0.$$

So

$$0 = d\omega_{n+1} = -\sum_{i=1}^{n} \omega_{(n+1)i} \wedge \omega_i$$

and by Cartan's lemma we can write

$$\omega_{(n+1)i} = \sum_{j=1}^{n} A_{ij}\omega_j, \qquad A_{ij} = A_{ji}. \tag{1.6}$$

Then

$$\begin{aligned} d\omega_i &= -\sum_{j=1}^{n} \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0 \\ d\omega_{ij} &= -\sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{k,l=1}^{n} R_{ijkl}\omega_k \wedge \omega_l \end{aligned}$$

where

$$R_{ijkl} = K_{ijkl} + A_{ik}A_{jl} - A_{il}A_{jk}.$$

This is the Gauss equations, relating curvature R of Σ to K and the 2nd fundamental form $A = \sum_{i,j=1}^{n} A_{ij}\omega_i\omega_j$. The mean curvature $nH = \sum_{i=1}^{n} A_{ii}$. By exterior differenting (1.6),

$$d\omega_{(n+1)i} = \sum_{j=1}^{n} dA_{ij} \wedge \omega_j + \sum_{j=1}^{n} A_{ij} d\omega_j = \sum_{j=1}^{n} dA_{ij} \wedge \omega_j - \sum_{j,k=1}^{n} A_{ij}\omega_{jk} \wedge \omega_k$$

and recalling $\omega_{(n+1)(n+1)} = 0$, so

$$\begin{aligned} d\omega_{(n+1)i} &= \sum_{j=1}^{n+1} \omega_{(n+1)j} \wedge \omega_{ji} + \frac{1}{2}\sum_{j,k=1}^{n} K_{(n+1)ijk}\omega_j \wedge \omega_k \\ &= \sum_{j,k=1}^{n} A_{jk}\omega_k \wedge \omega_{ji} + \frac{1}{2}\sum_{j,k=1}^{n} K_{(n+1)ijk}\omega_j \wedge \omega_k \end{aligned}$$

$$\sum_{k=1}^{n} A_{ijk}\omega_k := d\omega_{(n+1)i}(e_j) = dA_{ij} - \sum_{k=1}^{n} A_{ik}\omega_{kj} - \sum_{k=1}^{n} A_{kj}\omega_{ki}. \tag{1.7}$$

Then

$$A_{ijk} - A_{ikj} = K_{(n+1)ikj} = -K_{(n+1)ijk}. \tag{1.8}$$

Exterior differentiatie (1.7) and define

$$\sum_{l=1}^{n} A_{ijkl}\omega_i = dA_{ijk} - \sum_{l=1}^{n} \left(A_{ljk}\omega_{li} + A_{ijl}\omega_{lk} + A_{ilk}\omega_{lj}\right).$$

Then

$$\sum_{k,l=1}^{n}\left(A_{ijkl}-\frac{1}{2}\sum_{m=1}^{n}A_{im}R_{mjkl}-\frac{1}{2}\sum_{m=1}^{n}A_{mj}R_{mikl}\right)\omega_k\wedge\omega_l=0$$

and

$$A_{ijkl}-A_{ijlk}=\sum_{m=1}^{n}\left(A_{im}R_{mjkl}+A_{mj}R_{mikl}\right). \tag{1.9}$$

Covariantly differentiating the 4-tensor K_{ijkl} then restricting to Σ

$$K_{(n+1)ijk;l}=K_{(n+1)ijkl}-K_{(n+1)i(n+1)k}A_{jl}-K_{(n+1)ij(n+1)}A_{kl}-\sum_{m=1}^{n}A_{ml}K_{mijk}$$

where

$$\sum_{l=1}^{n}K_{(n+1)ijkl}\omega_l=dK_{(n+1)ijk}-\sum_{m=1}^{n}\left(K_{(n+1)mjk}\omega_{mi}+K_{(n+1)imk}\omega_{mj}+K_{(n+1)ijm}\omega_{mk}\right).$$

Define

$$\Delta A_{ij}=\sum_{k=1}^{n}A_{ijkk}.$$

Then, by (1.8)

$$\Delta A_{ij}=\sum_{k=1}^{n}\left(A_{ikjk}-K_{(n+1)ijkk}\right)=\sum_{k=1}^{n}\left(A_{kijk}-K_{(n+1)ijkk}\right).$$

By (1.9)

$$A_{kijk}=A_{kikj}+\sum_{m=1}^{n}\left(A_{km}R_{mijk}+A_{mi}R_{mkjk}\right).$$

Combining the previous formulas

$$\begin{aligned}\Delta A_{ij} &= \sum_{k=1}^{n}\left(A_{kkij}-K_{(n+1)kik;j}-K_{(n+1)ijk;k}-A_{kk}K_{(n+1)ij(n+1)}-A_{ij}K_{(n+1)k(n+1)k}\right)\\ &\quad+\sum_{m,k=1}^{n}\left(A_{mj}K_{mkik}+A_{mi}K_{mkjk}+2A_{mk}K_{mijk}\right)\\ &\quad+\sum_{m,k=1}^{n}\left(A_{mi}A_{mj}A_{kk}+A_{km}A_{ki}A_{mj}-A_{km}A_{km}A_{ij}-A_{mi}A_{mk}A_{kj}\right).\end{aligned}\tag{1.10}$$

When $nH=\sum_{k=1}^{n}A_{kk}=0$, we have

$$\begin{aligned}\sum_{i,j=1}^{n}A_{ij}\Delta A_{ij} &= -\sum_{i,j,k=1}^{n}\left(A_{ij}K_{(n+1)kik;j}+A_{ij}K_{(n+1)ijk;k}+A_{ij}^2K_{(n+1)k(n+1)k}\right)\\ &\quad+\sum_{m,i,j,k=1}^{n}\left(2A_{mj}A_{ij}K_{mkik}+2A_{mk}A_{ij}K_{mijk}\right)-|A|^4.\end{aligned}\tag{1.11}$$

For simplicity, we will assume the curvature tensor K of M is 0. More general situation is treated in Schoen-Simon-Yau. Now, (1.11) becomes

$$\sum_{i,j=1}^{n} A_{ij}\Delta A_{ij} = -|A|^4. \tag{1.12}$$

It follows

$$\frac{1}{2}\Delta|A|^2 = \sum_{i,j=1}^{n} A_{ij}\Delta A_{ij} + \sum_{i,j=1}^{n} |\nabla A_{ij}|^2 = -|A|^4 + \sum_{i,j=1}^{n} |\nabla A_{ij}|^2.$$

In turn, we obtain the so-called *Simons identity*:

$$\Delta|A|^2 = 2\sum_{i,j=1}^{n} |\nabla A_{ij}|^2 - 2|A|^4 \tag{1.13}$$

where $|A|^2 = \sum_{i,j} A_{ij}^2$.

At any $p \in \Sigma$, we can choose our frame $\{e_1, \ldots, e_n\}$ so that the 2nd fundamental form is diagonalized at p (one can only do this for codimension one case):

$$A_{ij} = \lambda_i \delta_{ij}.$$

At such a point p, we have

$$\begin{aligned} 4|A|^2|\nabla|A||^2 &= |\nabla|A|^2|^2 \\ &= \sum_k \left((\sum_{i,j} A_{ij}^2)_k \right)^2 \\ &= 4\sum_k \left(\sum_{i,j} \lambda_i A_{iik} \right)^2 \\ &\leq 4|A|^2 \sum_{i,k} A_{iik}^2. \end{aligned}$$

Assume $|A|(p) \neq 0$. Then we see

$$\begin{aligned}
|\nabla |A||^2 &\leq \sum_{i,k} A_{ijk}^2 \\
&= \sum_{i\neq k} A_{iik}^2 + \sum_i A_{ii;i}^2 \\
&= \sum_{i\neq k} A_{iik}^2 + \sum_i \left(\sum_{i\neq j} A_{jji} \right)^2 \qquad \text{(by minimality)} \\
&\leq \sum_{i\neq k} A_{iik}^2 + (n-1) \sum_i \sum_{i\neq j} A_{jji}^2 \\
&= n \sum_{i\neq k} A_{iik}^2 \\
&= n \sum_{i\neq k} A_{iki}^2 \qquad \text{(by (1.8) and } K=0) \\
&= \frac{1}{2} \left(\sum_{i\neq k} A_{iki}^2 + \sum_{i\neq k} A_{kii}^2 \right).
\end{aligned}$$

Therefore, we can sum up the observations above:

$$\left(1+\frac{2}{n}\right)|\nabla|A||^2 \leq \sum_{i,k} A_{iik}^2 + \sum_{i\neq k} A_{iki}^2 + \sum_{i\neq k} A_{kii}^2 \leq \sum_{i,j,k} A_{ijk}^2. \tag{1.14}$$

We arrive at *Simons inequality* for a minimal submanifold in a flat Riemannian manifold M:

$$\Delta |A|^2 \geq -2|A|^4 + 2\left(1+\frac{2}{n}\right)|\nabla |A||^2. \tag{1.15}$$

1.2.3. Stability inequality and Curvature estimate. Let $\Sigma^n \subset M^{n+1}$ be a orientable stable minimal hypersurface. There exists a unit normal vector field N along Σ and every section $(TX)^\perp$ can be written as $X = fN$ for some smooth function f on Σ with compact support. Then

$$0 \leq -\int_\Sigma \langle L(fN), fN\rangle = -\int_\Sigma f\Delta f + |A|^2 f^2 + Ric(N,N) f^2.$$

Integrating by parts, we obtain the *stability inequality*

$$\int_\Sigma |A|^2 f^2 + Ric(N,N) f^2 \leq \int_\Sigma |\nabla f|^2. \tag{1.16}$$

In particular, when $M = \mathbb{R}^{n+1}$, the stability reads

$$\int_\Sigma |A|^2 f^2 \leq \int_\Sigma |\nabla f|^2. \tag{1.17}$$

We now derive the integral curvature estimates of Schoen-Simon-Yau.

Step 1. Replacing f by $|A|^{1+q} f$ with $q \geq 0$ in the stability inequality leads to

$$\int_\Sigma |A|^{4+2q} f^2 \leq (1+q)^2 \int_\Sigma |A|^{2q} |\nabla |A||^2 f^2 + \int_\Sigma |A^{2+2q}|\nabla f|^2 + (2+2q)\int_\Sigma |A|^{1+2q} f\, |\nabla |A| \cdot \nabla f. \tag{1.18}$$

On the other hand, multiplying the Simons inequality

$$|A|\Delta|A| + |A|^4 \geq \frac{2}{n}|\nabla|A||^2$$

by $f^2|A|^{2q}$ and integrating by parts

$$\frac{2}{n}\int_\Sigma |A|^{2q}|\nabla|A||^2 f^2 \leq \int_\Sigma |A|^{4+2q}f^2 - 2\int_\Sigma f|A|^{2q+1}\nabla f\nabla|A| - (1+2q)\int_\Sigma f^2|A|^{2q}|\nabla|A||^2.$$

Combining the above inequalities

$$\left(\frac{2}{n} - q^2\right)\int_\Sigma |A|^{2q}|\nabla|A||^2 f^2 \leq \int_\Sigma |A|^{2+2q}|\nabla f|^2 + 2q\int_\Sigma f|A|^{1+2q}\nabla f\nabla|A|$$

and observing

$$2q\int_\Sigma f|A|^{2q+1}\nabla f\nabla|A| \leq \epsilon q^2\int_\Sigma^2 f^2|A|^{2q}|\nabla|A||^2 + \frac{1}{\epsilon}\int_\Sigma |A|^{2+2q}|\nabla f|^2,$$

we can arranger terms to obtain

$$\left(\frac{2}{n} - (1+\epsilon)q^2\right)\int_\Sigma^2 f^2|A|^{2q}|\nabla|A||^2 \leq \left(1+\frac{1}{\epsilon}\right)\int_\Sigma |A|^{2+2q}|\nabla f|^2 \tag{1.19}$$

Step 2. The cross term in (1.18) can be absorbed by the Cauchy-Schwarz inequality for any

$$\epsilon < \frac{2/n - q^2}{q}, \qquad q^2 < 2/n$$

leading to

$$\begin{aligned}\int_\Sigma |A|^{4+2q}f^2 &\leq 2(1+q)^2\int_\Sigma f^2|A|^{2q}|\nabla|A||^2 + 2\int_\Sigma |A|^{2+2q}|\nabla f|^2\\ &\leq \left(\frac{2(1+q)^2(1+q/\epsilon)}{2/n - q^2 - \epsilon q} + 2\right)\int_\Sigma |A|^{2+2q}|\nabla f|^2.\end{aligned}$$

Let $p = 2+q$ and $f = \eta^p$. Then $2 \leq p < 2+\sqrt{2/n}$ and apply Hölder's inequality to (1.19) gives

$$\int_\Sigma |A|^{2p}\eta^{2p} \leq C(n,p)\left(\int_\Sigma |A|^{2p}\eta^{2p}\right)^{\frac{p-1}{p}}\left(\int_\Sigma |\nabla\eta|^{2p}\right)^{\frac{1}{p}}.$$

Theorem 1.2.1 (Schoen-Simon-Yau). Suppose that Σ^n is an orientable stable minimal surface in $\mathbb{R}^{n+1}$. Then for all $2 \leq p < 2+\sqrt{2/n}$ and Lipschitz continuous function η with compact support, it holds

$$\int_\Sigma |A|^{2p}\eta^{2p} \leq C(n,p)\int_\Sigma |\nabla\eta|^{2p}. \tag{1.20}$$

This result has important applications when $\mathbf{n \leq 5}$. A ratio bound in the following form is useful in many problems on minimal submanifolds

$$\sup_R \frac{|\Sigma \cap B_R|}{R^n} \leq C_0 < +\infty \tag{1.21}$$

where $|\Sigma \cap B_R|$ denotes the volume of $\Sigma \cap B_R(0)$ for the euclidean ball in $\mathbb{R}^{n+1}$ of radius R centred at the origin.

Suppose that Σ is volume minimizing. Then $\partial B_R(0) \cap \Sigma$ divides $\partial B_R(0)$ into two components and at least one of them has volume at most half of the volume of $\partial B_R(0)$. Thus

$$|\Sigma \cap B_R(0)| \leq \frac{\omega_n}{2} R^n.$$

By a calibration argument, it can be shown that minimal graphs in euclidean space are volume minimizing.

THEOREM 1.2.2 (Schoen-Simon-Yau). If Σ^n is a complete orientable stable minimal hypersurface in $\mathbb{R}^{n+1}$, $n \leq 5$ and (1.21) holds, then Σ is a hyperplane.

PROOF. Take a radial function $\eta = 1$ on $B_R(0)$, $\eta = 0$ on $B_{2R}(0)$ and $|\eta'| \leq C/R$ on $B_{2R}(0) \backslash B_R(0)$, and take

$$2p = 4 + \sqrt{7/5} < 4 + \sqrt{8/n}.$$

By (1.20),

$$\int_{\Sigma \cap B_R(0)} |A|^{4+\sqrt{7/5}} \leq C(n,p) R^{-4-\sqrt{7/5}} (2R)^n \to 0 \qquad \text{as } R \to \infty$$

since $n - 4 - \sqrt{7/5} < 0$ as $n \leq 5$. Therefore $A \equiv 0$. □

Pointwise curvature estimates are available and can be used to prove result above:

THEOREM 1.2.3 (Schoen-Simon-Yau). If Σ^n is an orientable stable minimal hypersurface in $\mathbb{R}^{n+1}$, $n \leq 5$ and (1.21) holds, then

$$\sup_{B_{\theta R} \cap \Sigma} |A| \leq \frac{C(n, \theta, C_0)}{R}$$

for each $\theta \in (0,1)$.

PROOF. Using Simons inequality, we can check that the function $u = |A|^2 + C_0^2/R^2$ satisfies an inequality of the form

$$\Delta u + C(n)(|A|^2 + R^{-2}) u \geq 0.$$

Recall a well known result from the theory of elliptic equations (Theorem 5.3.1 in "Multiple integrals of in calculus of variations", C.B. Morrey): If $\phi \geq 0$ satisfies

$$\Delta \phi + c\phi \geq 0$$

on $B_R(0)$, then for any $\theta \in (0,1)$ and $\epsilon > 0$

$$\sum_{B_{\theta R}(0)} \phi \leq c_1 \left(R^{-n} \int_{B_R(0)} \phi^2 dx \right)^{1/2}$$

where $c_1 = c_1(n, \epsilon, \theta, R^{\epsilon} \int_{B_R(0)} |c|^{(n+\epsilon)/2} dx)$. Then

$$\sup_{B_{\theta R}(0)} |A|^2 \leq \sup_{B_{\theta R}(0)} u \leq c_2 \left(R^{-n} \int_{B_R(0)} (R^{-2} C_0 + |A|^2)^2 \right)^{1/2}, \tag{1.22}$$

where c_2 depends on $n, \epsilon, R^\epsilon \int_{B_R(0)}[C(n)(|A|^2 + R^{-2})]^{(n+\epsilon)/2}$. Choose $\epsilon > 0$ so that $n + \epsilon < 4 + \sqrt{8/n}$ (can be done for $n \leq 5$). Using (1.20) for $p \in [2, 2 + \sqrt{2/n})$, we have

$$\int_{B_R(0)} |A|^{2p} \leq CR^{-2p}|\Sigma \cap B_{2R}(0)| \leq CR^{n-2p}.$$

Taking $2p = n + \epsilon$, we see

$$R^\epsilon \int_{B_R(0)} |A|^{2p} \leq CR^\epsilon R^{n-(n+\epsilon)} \leq C$$

and

$$R^\epsilon \int_{B_R(0)} R^{-(n+\epsilon)} \leq C,$$

so, c_2 is bounded. Finally, taking $p = 2$ in (1.20) gives the desired result from (1.22). □

Bernstein conjecture: The only entire solutions of MSE are linear functions.

THEOREM 1.2.4 (Bernstein $n = 2$, De Girogi $n = 3$, Almgren $n = 4$, Simons $n = 5, 6, 7$; Bombieri-De Giorgi-Giusti $n > 7$). The only entire solutions of MSE over $\mathbb{R}^n$are linear functions for $n \leq 7$. There are non-linear entire solutions for $n > 7$.

Using the above estimates Schoen-Simon-Yau gave a simplified proof of Simons' result on minimal cones in $\mathbb{R}^n, n \leq 6$, but not for the last dimension $n = 7$.

THEOREM 1.2.5. Every 6-dimensional stable minimal cone in $\mathbb{R}^7$ is a hyperplane.

THEOREM 1.2.6 (De Giorgi for all n in the non-parametric case and Fleming for $n \leq 6$). If there is no non-trivial stable minimal cones in $\mathbb{R}^n$ then the only entire solutions of MSE in are the linear functions on $\mathbb{R}^n$.

Combining the above two theorems yields the affirmative answer to the Bernstein conjecture for $n \leq 6$.

1.3. Closed stable geodesics in a hyperkähler surface

THEOREM 1.3.1 (Bourguingnon-Yau). Let M be a hyperkähler manifold of dimension 4. If there exists a nontrivial closed stable geodesic γ in M, then the Riemann curvature tensor of M vanishes along γ. In particular, there are no nontrivial closed stable geodesics in the Eguchi-Hanson space (in fact, in any hyperkähler 4-manifold whose Riemann curvature tensor is nowhere vanishing).

PROOF. Let R be the Riemannian curvature tensor of M and let $I, J, K = IJ$ be the parallel complex structures with respect to the covariant derivative ∇ on M which determine the hyperähler structure compatible with the Riemannian metric on M. Parametrize γ by its arc-length and consider the unit vector fields $I\gamma', J\gamma', K\gamma'$ along γ.

From the second variation formula of length at a closed geodesic and the assumption that γ is stable, we have the stability inequality for any variation vector field X of γ:

$$0 \leq \int_\gamma \left(|\nabla_{\gamma'} X|^2 - \langle R(\gamma', X)\gamma', X\rangle\right) := \mathrm{Ind}(X, X).$$

Substituting X with $I\gamma', J\gamma', K\gamma'$ into the stability inequality above and taking summation:

$$\begin{aligned} 0 &\leq \operatorname{Ind}(I\gamma', I\gamma') + \operatorname{Ind}(J\gamma', J\gamma') + \operatorname{Ind}(K\gamma', K\gamma') \\ &= \int_\gamma \left(|\nabla_{\gamma'}(I\gamma')|^2 + |\nabla_{\gamma'}(J\gamma')|^2 + |\nabla_{\gamma'}(K\gamma')|^2\right) \\ &\quad - (\langle R(\gamma', I\gamma')\gamma', I\gamma'\rangle + \langle R(\gamma', J\gamma')\gamma', J\gamma'\rangle + \langle R(\gamma', K\gamma')\gamma', K\gamma'\rangle) \\ &= -\int_\gamma Ric\,(\gamma', \gamma') \\ &= 0 \end{aligned}$$

where we have used

$$\begin{aligned} \nabla_{\gamma'}(I\gamma') &= I\,\nabla_{\gamma'}\gamma' = 0, \\ \nabla_{\gamma'}(J\gamma') &= J\,\nabla_{\gamma'}\gamma' = 0, \\ \nabla_{\gamma'}(K\gamma') &= K\,\nabla_{\gamma'}\gamma' = 0 \end{aligned}$$

since I, J, K are parallel w.r.t. ∇ and γ is a geodesic and that $I\gamma', J\gamma', K\gamma', \gamma'$ form an orthonormal frame along γ and M is Ricci flat. Therefore, since each of the three terms on the right hand side of the inequality above is nonnegative due to stability of γ, we have

$$\operatorname{Ind}(I\gamma', I\gamma') = \operatorname{Ind}(J\gamma', J\gamma') = \operatorname{Ind}(K\gamma', K\gamma') = 0.$$

Stability of γ then implies that the first eigenvalue λ_0 of the Jacobi operator is nonnegative:

$$\lambda_0 = \inf_X \frac{\operatorname{Ind}_\gamma(X, X)}{\int_\gamma |X|^2} \geq 0$$

for any variation field X along γ which is not identically zero. Therefore $\lambda_0 = 0$ and $I\gamma', J\gamma', K\gamma'$ are eigenfunctions for λ_0, hence they are Jacobi vector fields along γ. Since the vector fields $I\gamma', J\gamma', K\gamma'$ are parallel along γ, the first term in the Jacobi equation

$$\frac{D^2 X}{dt^2} + R(\gamma'(t), X(t))\gamma'(t) = 0$$

vanishes for $X = I\gamma', J\gamma', K\gamma'$, and this leads to

$$R(\gamma', I\gamma')\gamma' = R(\gamma', J\gamma')\gamma' = R(\gamma', K\gamma')\gamma' = 0.$$

By the Kähler identities for curvature [**21**]:

$$R(JX, JY) = J \circ R(X, Y) \quad \text{and} \quad R(X, Y) \circ J = J \circ R(X, Y)$$

we see the sectional curvature

$$\begin{aligned} \langle R(J\gamma', I\gamma')J\gamma', I\gamma'\rangle &= \langle R(J\gamma', JK\gamma')J\gamma', I\gamma'\rangle \\ &= -\langle R(\gamma', K\gamma')\gamma', I\gamma'\rangle \\ &= 0. \end{aligned}$$

Similarly,

$$\langle R(K\gamma', I\gamma')K\gamma', I\gamma'\rangle = 0.$$

We conclude that the sectional curvatures vanish on the sections containing $I\gamma'$. The same reasoning shows all sectional curvatures vanish on sections containing $J\gamma'$ and on those containing $K\gamma'$. It then follows that the Riemann curvature tensor R vanishes along γ [**?**].

Since the Eguchi-Hanson space is a hyperkähler 4-dimensional manifold whose Riemann curvature tensor is nowhere vanishing [**14**], it does not admits any non-trivial closed stable geodesics. □

1.4. Volume minimizing via calibrations

We begin with a fundamental observation of Harvey-Lawson ["Calibrated geometries", Acta Math. (1982)]. Let (M, g) be a Riemannian manifold. Suppose φ is a closed exterior p-form on M which satisfies

$$\varphi|_P \le vol_P$$

for all oriented tangent p-planes P on M. Suppose that Σ is an oriented p-dimensional submanifold of M with the property that

$$\varphi|_\Sigma = vol_\Sigma. \tag{1.23}$$

Then Σ is volume minimizing in its homology class, i.e., $vol(\Sigma) \le vol(\Sigma')$ for any $\Sigma' \subset M$ so that $\partial\Sigma' = \partial\Sigma$ and $[\Sigma - \Sigma'] = 0$ in $H_p(M;\mathbb{R})$. To see this,

$$vol(\Sigma) = \int_\Sigma \varphi = \int_{\Sigma'} \varphi \le vol(\Sigma').$$

Note that

$$\int_\Sigma \varphi - \int_{\Sigma'} \varphi = \int_{\Sigma-\Sigma'} \varphi = \int_{\partial K} \varphi = \int_K d\varphi = 0$$

since $d\varphi = 0$, where $\partial K = \Sigma - \Sigma'$ from $[\Sigma - \Sigma'] = 0$. A closed p-form φ satisfying (1.23) is called a *calibration* and (M, g, φ) is a *calibrated manifold*.

A k-dimensional submanifold Σ in M is a *φ-submanifold* associated to an exterior k-form φ (not necessarily closed) if $\varphi|_\Sigma = vol_\Sigma$. If $d\varphi = 0$ and $\varphi|_P \le vol_P$ for every tangent k-plane on M, then a φ-submanifold is homologically volume minimizing.

1.4.1. φ-submanifolds and differential forms of comass one.

We begin with an example from complex geometry. Let (M, J, g, ω) be a Hermitian complex manifold of complex dimension n. Set

$$\varphi = \frac{1}{k!}\omega^k.$$

The φ-submanifodsl Σ, i.e. $\varphi|_\Sigma = vol_\Sigma$, are complex submanifolds of dimension k by Wirtinger's theorem:

$$vol(\Sigma) = \frac{1}{k!}\int_\Sigma \omega^k.$$

When M is Kähler, $d\varphi = 0$ and Σ is volume minimizing in its homology class (Federer).

Now, let M be an n-dimensional Riemannian manifold, and let $\varphi \in \Lambda^k$ be a k-form on M, $k < n$. At each $x \in M$, define the *comass of φ_x* to be

$$\|\varphi\|^*_x = \sup\{\langle \varphi_x, \xi_x\rangle : \ \xi_x \text{ is a unit simple } k\text{-vector in } T_xM\}. \tag{1.24}$$

For any $A \subset M$, define *comass of φ on A* to be

$$\|\varphi\|^*_A = \sup_{x\in A} \|\varphi\|^*_x.$$

LEMMA 1.4.1. Suppose that φ is of comass one on M. Let Σ be a k-dimensional compact oriented submanifold (possibly with boundary) in M. Then

$$\int_\Sigma \varphi \leq vol(\Sigma)$$

with equaility if and only if Σ is a φ-submanifold.

PROOF. Let $e_1, \ldots, e_k$ be o.n. tangent vectors in $T_x\Sigma$. Then $d\mu_\Sigma(e_1, \ldots, e_k) = 1$. $e_1 \wedge \cdots \wedge e_k$ is a unit simple k-vector in T_xM. At $x \in \Sigma$

$$\begin{aligned}
\varphi|_\Sigma &= \varphi(e_1, \ldots, e_k) e_1^* \wedge \cdots \wedge e_k^* \\
&\leq \|\varphi\|_x^* e_1^* \wedge \cdots \wedge e_k^* \\
&\leq \|\varphi\|_M^* e_1^* \wedge \cdots \wedge e_k^* \\
&= e_1^* \wedge \cdots \wedge e_k^*.
\end{aligned}$$

When "=" holds, $\varphi(e_1, \ldots, e_k) = 1$ (almost, hence) everywhere on Σ, i.e. $\varphi|_\Sigma = vol_\Sigma$; so Σ is a φ-submanifold in M. □

1.4.2. Complex manifolds. The calibrated manifolds root in complex geometry. A *complex manifold* of complex dimension n is a real $2n$-dimensional manifold whose transition functions are holomorphic. This means that there is local coordinates $z^j : U_j \to \mathbb{C}^n$ such that

(1) $M = \cup_j U_j$.
(2) The maps $z^j \circ (z^k)^{-1}$ are holomorphic for all j, k with $U_j \cap U_k \neq \emptyset$.

Two such complex coordinate systems $\{z^j\}, \{w^k\}$ are equivalent if the maps $z^j(p) \to w^k(p)$ are biholomorphic, i.e. holomorphic and the inverse is holomorphic, when and where defined. A *complex structure* on a manifold is an equivalence class of complex coordinate systems on it.

An *almost complex structure* on a manifold M is a smooth field of automorphisms J of TM so that the linear map $T_x : T_xM \to T_xM$ satisfies

$$J_x^2 = -I_x, \qquad x \in M.$$

An almost complex structure is *integrable* if its Nijenhuis tensor N defined by

$$4N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY], \qquad X, Y \in T_xM, x \in M$$

vanishes. A manifold with an almost complex is called an *almost complex manifold.* Its dimensional is necessarily even.

A famous theorem of Newlander-Nirenberg asserts that an integrable complex structure is induced by a unique complex structure.

A *Hermitian* metric on complex manifold M is a Riemannian metric g such that

$$g(JX, JY) = g(X, Y), \qquad X, Y \in T_xM, x \in M.$$

From a Riemannian metric on (M, J), we can always obtain a Hermitian metric by setting

$$h(X, Y) = g(X, Y) + g(JX, JY).$$

Locally, a Hermitian metric can be written as

$$ds^2 = g_{\alpha\bar{\beta}} dz^\alpha dz\bar{z}^\beta$$

for some positive definite Hermitian matrix $(g_{\alpha\bar{\beta}})$.

The *Kähler form* (or the fundamental 2-form) ω is defined as

$$\omega(X, Y) = g(X, JY).$$

Note

$$\omega(JX, JY) = g(JX, JJY) = g(X, JY) = \omega(X, Y).$$

When $\nabla J = 0$ where ∇ is the Levi-Civita connection of g, the Hermitian metric is called a *Kähler metric* and (M, g, J, ω) is a *Kähler manifold*. This is equivalent to that ω is closed.

2

Lagrangian Submanifolds

2.1. Basic symplectic geometry

2.1.1. Symplectic manifolds. A *symplectic manifold* M is a smooth $2n$-dimensional manifold equipped with a closed 2-form ω which is non-degenerate in the sense $\omega^n = \omega \wedge \cdots \wedge \omega$ is nowhere zero, and ω is called a *symplectic 2-form* or a *symplectic structure* on M. As ω^n nowhere vanishes, any symplectic manifold is orientable.

Examples of symplectic manifolds include

(1) $\mathbb{R}^{2n}$ with $\omega_0 = \sum_{i=1}^n dx^i \wedge dy^i$ where $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ is the standard coordinates.
(2) Any Kähler manifold (M, ω). In particular, $\mathbb{S}^2$ viewed as $\mathbb{C}P^1$.
(3) The cotangent bundle $\pi : T^*M \to M$ of a smooth manifold M admits a natural symplectic structure. For $x \in T^*M, V \in T_xT^*M$, define a 1-form (Liouville form) β on T^*M by
$$\beta(V)_x = x(\pi_* V).$$
Let $(x^1, \ldots, x^n)$ be local coordinates on M, set $q^i = x^i \circ \pi$ and the fibre coordinates $p^1, \ldots, p^n$ for local coordinates on T^*M. Then $\beta = \sum_{i=1}^n p^i dq^i$, and $\omega = d\beta$ is a symplectic 2-form on T^*M.

Let (M^{2n}, ω) be a symplectic manifold. Then ω^n is a closed $2n$-form on M. Suppose that M is compact and without boundary. Thus ω represents a cohomology class in $H^2(M, \mathbb{R})$ and $\omega^n \in H^{2n}(M, \mathbb{R})$. The non-degeneracy of ω implies
$$\int_M \omega^n \neq 0.$$
It follows that ω and ω^n cannot be exact exterior differential forms. Since $H^{2n}(\mathbb{S}^{2n}, \mathbb{R}) = 0$ for $n > 1$, there do not exist any symplectic structures on $\mathbb{S}^4, \mathbb{S}^6, \ldots$.

A *symplectomorphism* between two symplectic manifolds $(M_1, \omega_1), (M_2, \omega_2)$ is a diffeomorphism $f : M_1 \to M_2$ such that $f^*\omega_2 = \omega_1$.

Let (M, ω) be a symplectic manifold and let (U, ϕ) be a local coordinate chart with $\phi : U \subseteq M \to \phi(U) \subseteq \mathbb{R}^{2n}$. Then $(\phi(U), \phi^{-1*}\omega)$ is a symplectic manifold. Note that $(\phi(U), \omega_0)$ is also symplectic. A local chart is called a *Darboux chart* if $\phi^*\omega_0 = \omega$.

Let V be a finite dimensional vector space over $\mathbb{R}$. Let $\omega : V \times V \to \mathbb{R}$ be a bilinear map that is skew-symmetric, i.e. $\omega(X, Y) = -\omega(Y, X)$, and non-degenerate, i.e. $\omega(X, Y) = 0$ for all $Y \in V$ only if $X = 0$. The non-degeneracy of ω implies V must have even dimension, say $2n$, since 0 is an eigenvalue of a real skew-symmetric matrix of odd size. The pair (V, ω) is called a *symplectic vector space*. The following is a linear version of the Darboux theorem.

THEOREM 2.1.1. Let (V,ω) be a symplectic vector space of dimension $2n$. Then V has a basis $u_1,\dots,u_n,v_1,\dots,v_n$ with

$$\omega(u_i,u_j)=0=\omega(v_i,v_j) \quad \text{and} \quad \omega(u_i,v_j)=\delta_{ij}. \tag{2.1}$$

There is an isomorphism $\phi:\mathbb{R}^{2n}\to V$ satisfying $\phi^*\omega=\omega_0$.

PROOF. We use induction on n. For $n=1$, there exist $u_1,v_1\in V$ such that $\omega(u_1,v_1)=1$ by the non-degeneracy of ω. Thus $\omega=u_1^*\wedge v_1^*$. Assume the theorem holds for $n=k$. For $n=k+1$, take u_1,v_1 with $\omega(u_1,v_1)=1$ and set $W=\{w\in V:\omega(w,u_1)=0=\omega(w,v_1)\}$. Then $(W,\omega|_W)$ is a symplectic vector space of dimension $2k$. The induction hypothesis then asserts that there exist a basis $u_2,\dots,u_k,v_2,\dots,v_k$ for W so that $\omega|_W(u_i,u_j)=0=\omega(v_i,v_j),\omega(u_i,v_j)=\delta_{ij}$ for $2\le i,j\le k+1$. Therefore $u_1,\dots,u_{k+1},v_1,\dots,v_{k+1}$ has the same property.

We can take

$$\phi\Big(\sum_{i=1}^n(x_ie_i+y_ie_{i+n})\Big)=\sum_{i=1}^n(x_iu_i+y_iv_i).$$

for $\phi:\mathbb{R}^{2n}\to V$. □

We now use the so-called Moser's trick to prove

THEOREM 2.1.2 (Darboux). Every point in a symplectic manifold (M,ω) has a Darboux chart.

PROOF. For any $p\in M$, let $(u_1,\dots,u_n,v_1,\dots,v_n)$ be local coordinates such that (2.1) holds at p and denote the chart by (U,φ) where $\varphi:U\subseteq M\to\mathbb{R}^{2n}$. We assume U is simply connected. Take $\omega_0=\sum_{i=1}^n du_i\wedge dv_i$ on $\mathbb{R}^{2n}$. Then $\varphi^*\omega-\omega_0$ is closed and vanishes at p. By Poincaré lemma, there is a 1-form η on $\varphi(U)$ such that

$$d\eta=\varphi^*\omega-\omega_0.$$

Define

$$\omega_t=\omega_0+td\eta.$$

As $d\eta=0$ at p, ω_t is non-degenerate in a neighbourhood of p (still denote by U). For each t let X_t be the uniquely determined vector field on M such that

$$\iota_{X_t}\omega_t=-\eta.$$

Then we consider the diffeomorphisms ψ_t on M generated by X_t via

$$\frac{d}{dt}\psi_t=X_t(\psi_t), \quad \psi_0=id.$$

Then by using Cartan's magic formula,

$$\begin{aligned}\frac{d}{dt}\psi_t^*\omega_t&=\psi_t^*(L_{X_t}\omega_t+\frac{d}{dt}\omega_t)\\&=\psi_t^*(\iota_{X_t}d\omega_t+d\iota_{X_t}\omega_t+\frac{d}{dt}\omega_t)\\&=\psi_t^*(-d\eta+\frac{d}{dt}\omega_t)\\&=0.\end{aligned}$$

Since $\psi_0=id$, we have

$$\psi_1^*\omega_1=\omega_0.$$

Therefore

$$\omega_0 = \psi_1^*(\omega_0 + d\eta) = \psi_1^*(\varphi^*\omega).$$

$\square$

In fact, we can modify the argument to show

LEMMA 2.1.1. Let M be a $2n$-dimensional smooth manifold and α_1, α_2 are closed 2-forms on M such that $\alpha_1 = \alpha_2$ on a compact submanifold Σ and they are non-degenerate on $T_qM, q \in \Sigma$. Then there exists neighbourhoods N_1, N_2 of Σ and a diffeomorphism $F : N_1 \to N_2$ such that $F^*\alpha_2 = \alpha_1$ and F equals the identity map on Σ.

An almost complex structure J on a symplectic manifold (M, ω) is *compatible with* ω if

$$\omega(JX, JY) = \omega(X, Y), \quad \omega(X, JX) > 0$$

for all $X, Y \in T_xM, x \in M$. Let $\mathcal{J}_\omega(M)$ be the set of almost complex structures on (M, ω) that are compatible with ω.

THEOREM 2.1.3. $\mathcal{J}_\omega(M)$ is non-empty and contractible.

THEOREM 2.1.4. Let (V^{2n}, ω) be a symplectic vector space and $g : V \times V \to \mathbb{R}$ be positive definite symmetric bilinear form (i.e. an inner product). Then there exists a symplectic basis $u_1, ..., u_n, v_1, ..., v_n$ which also satisfies $g(u_i, u_j) = \delta_{ij} g(v_i, v_j)$ and $g(u_i, v_j) = 0$.

PROOF. Define $f : V \to \mathbb{R}^{2n}$ by $f(X) = \vec{x}$ such that

$$g(X, Y) = \langle f(X), f(Y) \rangle = \langle \vec{x}, \vec{y} \rangle.$$

Define a $2n \times 2n$ matrix A by

$$\omega(X, Y) = \langle \vec{x}, A\vec{y} \rangle, \quad \forall X, Y \in V.$$

Since ω is non-degenerate and $\omega(X, Y) = -\omega(Y, X)$, the real matrix A is non-degenerate and skew symmetric. Then $\sqrt{-1}A \in GL(2n, \mathbb{C})$ is a positive definite Hermitian matrix, hence it is unitarily diagonalizable with pure imaginary eigenvalues $\pm\sqrt{-1}\lambda_j, \lambda_j > 0, j = 1, \ldots, n$, *i.e.*, there are eigenvectors $\vec{z}_j = \vec{u}_j + \sqrt{-1}\vec{v}_j \in \mathbb{C}^{2n}$ where $\vec{u}_j, \vec{v}_j \in \mathbb{R}^{2n}$ satisfying

$$A\vec{z}_j = \sqrt{-1}\lambda_j \vec{z}_j, \quad \langle \vec{z}_i, \bar{\vec{z}}_j \rangle = \delta_{ij}.$$

Taking complex conjugate, we have

$$A\bar{\vec{z}}_j = -\sqrt{-1}\lambda_j \bar{\vec{z}}_j, \quad \langle \vec{z}_i, \vec{z}_j \rangle = 0$$

because $\vec{z}_1, \ldots, \vec{z}_n, \bar{\vec{z}}_1, \ldots, \bar{\vec{z}}_n$ form a unitary basis of $\mathbb{C}^{2n}$. In real form:

$$\begin{aligned} A\vec{u}_j &= -\lambda_j \vec{v}_j \\ A\vec{v}_j &= \lambda_j \vec{u}_j \end{aligned}$$

and

$$|\vec{u}_j|^2 = |\vec{v}_j|^2 = \frac{1}{2}, \quad \langle \vec{u}_j, \vec{v}_j \rangle = 0, \quad \forall j$$
$$\langle \vec{u}_j, \vec{v}_k \rangle = \langle \vec{u}_j, \vec{u}_k \rangle = \langle \vec{v}_j, \vec{v}_k \rangle = 0, \quad j \neq k.$$

Let Λ be the $n \times n$ diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$ and set

$$Q = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}, \quad J_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

It follows
$$A = QJ_0, \quad QJ_0 = J_0Q, \quad A^2 = -Q^2.$$
Then $\{U_j' = f^{-1}(\lambda_j^{-1/2} J_0\vec{u}_j), V_j' = f^{-1}(\lambda_j^{-1/2} J_0\vec{v}_j) : 1 \le j \le n\}$ is a symplectic basis:
$$\omega(U_j', V_j') = 1 = -\omega(V_j', U_j'), \quad \forall j$$
$$\omega(U_j', V_k') = \omega(U_j', U_k') = \omega(V_j', V_k') = 0, \quad j \neq k.$$
Now we talk about g:
$$g(U_i', U_j') = \lambda_i^{-1/2}\lambda_j^{-1/2}\delta_{ij} = g(V_i', V_j')$$
$$g(U_i', V_j') = 0.$$
$$\langle f(U_i'), f(U_j')\rangle = g(U_i', U_j') = \lambda_i^{-1/2}\lambda_j^{-1/2} g(U_i, U_j) = \lambda_i^{-1/2}\lambda_j^{-1/2}\langle \vec{u}_i, \vec{u}_j\rangle = \lambda_i^{-1}\delta_{ij}.$$
Thus, the symplectic basis if also orthogonal w.r.t. g with the above property. □

Remark 1. Geometrically, this means: Any ellipsoid
$$E = \{w \in \mathbb{R}^{2n} : \sum_{i,j}^{2n} a_{ij} w_i w_j \le 1\}$$
can be symplectically (by a linear symplectomorphism) mapped to an ellipsoid
$$\{z \in \mathbb{C}^n : \sum |z_j/r_j|^2 \le 1\}$$
where r_j are uniquely determined by E. Non-squeezing of the unit ball through a small hole! Demonstrating symplectic rigidity vs diffeomorphism.

Now, given g and ω on V, define $B : V \to V$ by
$$g(BX, Y) = \omega(X, Y) = \langle \vec{x}, A\vec{y}\rangle = \langle -A\vec{x}, \vec{y}\rangle.$$
Then
$$g(BX, Y) = \langle f(BX), f(Y)\rangle = \langle fBf^{-1}\vec{x}, \vec{y}\rangle$$
In the g-orthogonal symplectic basis $\{U_j', V_j' : 1 \le j \le n\}$ from the previous result, for $f : V(= \mathbb{R}^{2n}) \to \mathbb{R}^{2n}$ with $f(X) = \vec{x}$, we see
$$g(U_i', U_j') = \langle f(U_i'), f(U_j')\rangle = \langle Q^{-1/2}\vec{u}_i, Q^{-1/2}\vec{u}_j\rangle = \lambda_i^{-1}\delta_{ij}$$
so $g = f^*g_0 = (Q^{-1/2}J_0)^*g_0$ and $f = -Q^{-1/2}J_0 \in Sp(V, \omega_0)$ (linear symplectomorphisms of V, $f^*\omega_0 = \omega_0$). It follows that $(\omega, J_0, (Q^{1/2}J_0)^*g)$ is compatible. Then
$$J_g = J_{f^*(f^{-1*}g)} = f^{-1}J_0 f = Q^{1/2}J_0Q^{-1/2} = Q^{-1}A.$$
Here we have used the fact: If g is replaced by ϕ^*g for a $\phi \in Sp(V, \omega)$ then $J_{\phi^*g} = \phi^{-1}J_g\phi$ as B is replaced by $\phi^{-1}B\phi$. We conclude (ω, g, J_g) compatible.

A vector field X on a symplectic manifold (M, ω) is called *symplectic* (or *Lagrangian*) if $\iota_X\omega$ is closed, and X is *Hamiltonian* if $\iota_X\omega$ is exact:
$$\iota_X\omega = dH$$
for some smooth function $H : M \to \mathbb{R}$. The vector field X generates a family of diffeomorphisms ϕ_H of M from solving
$$\frac{d}{dt}\phi_H = X(\phi_H), \quad \phi_H(\cdot, 0) = id.$$

This is called the *Hamiltonian flow associated to* H. The derivative of H in the X direction vanishes:

$$dH(X) = \iota_X\omega(X) = \omega(X,X) = 0$$

so the Hamiltonian vector field X is tangent to the level sets of its generating function H.

2.1.2. Lagrangian submanifolds. We now discuss Lagrangian submanifolds. Let (M,ω) be a symplectic manifold and J an almost complex structure compatible with ω and g a Riemannian metric on M induced by ω, J in the sense

$$g(X,JY) = \omega(X,Y).$$

A submanifold L of dimension $n = \frac{1}{2}\dim M$ is called *Lagrangian* if $\omega|_L = 0$. This means: for any $X, Y \in T_xL, x \in L$

$$g(X,JY) = \omega(X,Y) = 0.$$

In other words, J maps T_xL to its g-orthogonal complement $(T_xL)^\perp$.

We now discuss two classical examples of Lagrangian submanifolds.

LEMMA 2.1.2. The zero section of T^*L is a Lagrangian submanifold in T^*L with $\omega_{can} = d\beta$.

PROOF. At a point $x \in T^*L, V \in T_xT^*L$, the canonical 1-form is $\beta(V) = x(\pi_*V)$ and $\omega_{can} = d\beta$ is a symplectic form on T^*L. We wrote (cf. Example) $\beta = \sum_{i=1}^n p_i dq_i$, which equals 0 along the zero section, i.e. $p_i = 0$. Define $L \to T^*L$ by the inclusion ℓ. So $\ell^*\omega_{can} = 0$. $\square$

LEMMA 2.1.3. Let α be a 1-form on L. The graph of α is a Lagrangian submanifold of T^*L if and only if α is closed.

PROOF. A 1-form α on L can be viewed as a mapping $L \to T^*L$ as a graph over L. Then for $V \in T_xT^*L, x \in T^*M$ we have

$$\begin{aligned}\alpha^*\beta(V) &= \beta(\alpha_*V)\\ &= x(\pi_*\alpha_*V)\\ &= \alpha(V)\end{aligned}$$

because $\pi \circ \alpha = id$ as η is a section of T^*M and $x = \alpha$ along the image of the mapping η.

Observe

$$\begin{aligned}\alpha^*\omega_{can} &= \alpha^* d\beta\\ &= d(\alpha^*\beta)\\ &= d\alpha\end{aligned}$$

where we used $\alpha^*\beta = \sum p_i(\alpha)dq_i = \alpha$. So the section α is Lagrangian in $(T^*M, \omega_{can}$) if and only if $d\alpha = 0$. $\square$

The next result shows that symplectic vector fields generate symplectic diffeomorphisms and the flow moves Lagrangian submanifolds to Lagrangian submanifolds.

PROPOSITION 2.1.1. Let X be a symplectic vector field and ϕ_t be the diffeomorphisms generated by X with $\phi_0 = id$. Then $\phi_t^*\omega = \omega$. If L is a Lagrangian submanifold in M, so is $L_t = \phi_t(L)$.

PROOF. To see this,

$$\begin{aligned}\frac{d}{dt}\phi_t^*\omega &= \phi_t^* \lim_{t\to 0}\frac{\phi_t^*\omega - \omega}{t}\\ &= \phi_t^* L_X\omega\\ &= \phi_t^*(\iota_X d\omega + d\iota_X\omega)\\ &= \phi_t^* d\iota_X\omega\\ &= 0\end{aligned}$$

because X is symplectic. It follows $\phi_t^*\omega = \phi_0^*\omega = \omega$. Next, we verify $\omega|_{L_t} = \phi_t^*\omega|_{L_t} = \omega|_L = 0$. So L_t is Lagrangian in (M,ω). □

The second example we include is a classical fact (cf. Harvey-Lawson):

LEMMA 2.1.4. Suppose Ω is an open dopmain in $\mathbb{R}^n$ and $f:\Omega\to\mathbb{R}^n$ is a C^1 mapping. Then the graph $\Sigma = (x,f)$ in $\mathbb{C}^n = \mathbb{R}^n + \sqrt{-1}\mathbb{R}^n$ is Lagrangian w.r.t. ω_0 if and only if the Jacobian matrix of f is symmetric. When Ω is simply connected, Σ is Lagrangian if and only if $f = Du$ for some $u \in C^2(\Omega)$.

PROOF. Write $(x,y), J_0, \omega_0$ for the standard coordinates, the complex structure and symplectic form of $\mathbb{R}^n \oplus \mathbb{R}^n$, respectively. A basis of the tangent space of Σ is given by $X_i := \partial_i(x, f(x)) = (0,...,1,...,0,\partial_i f)$, and Σ is Lagrangian w.r.t. ω_0 if and only if $\langle X_i, J_0 X_j\rangle = 0$ for $i,j = 1,...,n$, which is if and only if $\partial_i f^j = \partial_j f^i$, i.e. the Jacobi matrix of f is symmetric. If Ω is simply connected, the symmetry of the Jacobian of f is equivalent to there is a potential function $u \in C^2(\Omega)$ such that $f = Du$, by the Poincaré lemma. □

The following Lagrangian neighbourhood theorem is due to A. Weinstein.

THEOREM 2.1.5. Let (M,ω) be a symplectic manifold and L is a Lagrangian submanifold. Then there exists a neighbourhood $N(L_0)$ of the zero section $L_0 = L$ of T^*L and a neighbourhood $N(L)$ of L in M and a diffeomorphism $\phi : N(L_0) \to N(L)$ such that

$$\phi^*\omega = -d\beta, \qquad \phi|_L L = id$$

where β is the canonical 1-form on T^*L.

PROOF. Let J be a ω-compatible almost complex structure on M and g a Riemannian metric compatible with ω, J. Recall the natural coordinates (q,p) for T^*L where $q\in L$. Let $\Phi_q : T_q^*L \to T_qL$ be the isomorphism defined by

$$g(X, \Phi_q(p)) = p(X).$$

Consider the map

$$\phi(q,p) = \exp_q(J\Phi_q(p)).$$

Along the zero-section in T^*L, $T_{(q,0)}T^*L$ is isomorphic to $T_qL \oplus T_q^*L$, so $V \in T_{(q,0)}T^*L$ can be identified with $(v,\eta) \in T_qL \oplus T_q^*L$. Then

$$d\phi_{(q,0)}(v,\eta) = v + J\Phi_q(\eta).$$

Since L is Lagrangian,

$$\begin{aligned}\phi^*\omega_{(q,0)}((v_1,\eta_1),(v_2,\eta_2)) &= \omega_q(v_1 + J\Phi_q(\eta_1), v_2 + J\Phi_q(\eta_2))\\ &= \omega_q(v_1, J\Phi_q(\eta_2)) - \omega_q(v_2, J\Phi_q(\eta_1))\\ &= g(v_1, \Phi_q(\eta_2)) - g(v_2, \Phi_q(\eta_1))\\ &= \eta_1(v_2) - \eta_2(v_1)\\ &= -d\beta_{(q,0)}((v_1,\eta_1),(v_2,\eta_2)).\end{aligned}$$

This shows that along the zero-section

$$\phi^*\omega = -d\beta.$$

We complete the proof by using Lemma 2.1.1. □

2.2. Mean curvature form of a Lagrangian submanifold

The mean curvature vector H of a Lagrangian submanifold L gives rise to a 1-form α_H when taking interior product with the symplectic 2-form. When the ambient space M is Kähler, the exterior differentiation of α_H is Ricci 2-form of the Kähler metric restricted to L; when M is Kähler-Einstein α_H is closed hence represents a class in $H^1(L,\mathbb{R})$; when M is a Calabi-Yau space, α_H is determined by the gradient of the so-called Lagrangian phase Θ of L acted upon by the complex structure J.

2.2.1. Kähler ambient space. On a Kähler manifold, the Riemannian curvature R and the Ricci tensor S possess the following properties ([Kobayashi-Nomizu]):

(1) $R(X,Y)\circ J = J\circ R(X,Y)$ and $R(JX,JY) = R(X,Y)$;
(2) $S(JX,JY) = S(X,Y)$ and $S(X,Y) = \frac{1}{2}\operatorname{trace}_g(J\circ R(X,JY))$.

The Kähler curvature R is a real 2-form of type (1,1) with values in $\Lambda^{1,1}M$. It may be viewed as a symmetric endomorphism of $\Lambda^{1,1}M$. The Kähler curvature operator is the same as the Riemannian curvature operator but viewed as an endomorphism of $\Lambda^{1,1}M$, instead of $\Lambda^2 M$. The Ricci 2-form is $R(\omega)$.

Recall

$$S(X,Y) = \operatorname{trace}_g(V \to R(X,V)Y).$$

The Ricci 2-form Ric of a Kähler manifold is the skew-symmetric 2-form

$$Ric(X,Y) = S(X,JY)$$

and

$$\begin{aligned}S(X,Y) &= \sum_i g(R(e_i,X)e_i,Y) + \sum_i g(R(Je_i,X)Je_i,Y)\\ &= \sum_i g(R(e_i,X)Je_i,JY) + \sum_i g(R(Je_i,X)e_i,Y)\\ &= \sum_i g(R(e_i,Je_i)X,JY).\end{aligned}$$

Therefore

$$Ric(e_B,e_C) = -\sum_i g(R(e_i,Je_i)e_B,e_C) = -R_{i(i+n)BC}.$$

THEOREM 2.2.1. Let M be a Kähler manifold with Kähler form ω, complex structure J and Kähler metric g. Let $\ell : L \to M$ be a Lagrangian immersion and denote the mean curvature form by $\iota_H\omega$ where H is the mean curvature vector of L in M. Then

$$d\iota_H\omega = \ell^* Ric$$

where Ric is the Ricci 2-form of g. In particular, when M is Kähler-Einstein, i.e. $Ric = c\omega$ for some real constant c, $\iota_H\omega$ is closed.

PROOF. Along the Lagrangian submanifold L, we can take positively oriented local orthonormal frame $e_1, ..., e_n, Je_1, ..., Je_n$ for M where e_j are tangential to L and Je_i are normal to L w.r.t. g. Let θ_j, θ_{j+n} be the duals of e_j, Je_j respectively. We have $\theta_{j+n} = -\theta_j \circ J$. The Kähler form can be written as

$$\omega = \sum_j^n \theta_j \wedge \theta_{j+n}.$$

The connection 1-forms for the Kähler metric g possess the following symmetry ([Kobayashi-Nomizu, p.153]):

$$\omega_{ij} = \omega_{(i+n)(j+n)},\ \omega_{i(j+n)} = \omega_{j(i+n)},\ \omega_{ij} = \omega_{(i+n)(j+n)}. \tag{2.2}$$

On L,

$$\theta_{j+n} = 0.$$

(strictly speaking, the pullback from ℓ)

The structure equations (the fundamental theorem of local Riemannian geometry)

$$0 = d\theta_{j+n} = -\sum_k^n \omega_{(j+n)k} \wedge \theta_k - \sum_k^n \omega_{(j+n)(k+n)} \wedge \theta_{k+n} = -\sum_k^n \omega_{(j+n)k} \wedge \theta_k$$

$$\omega_{(j+n)k} + \omega_{k(j+n)} = 0.$$

By Cartan's lemma

$$\omega_{(j+n)k} = \sum_i^n h_{ik}^{j+n}\theta_i, \quad h_{ik}^{j+n} = h_{ki}^{j+n}.$$

Moreover, from (2.2), the second fundamental form of the Lagrangian submanifold enjoys total symmetry in its indices, in the sense

$$h_{ik}^{j+n} = h_{ij}^{k+n} = h_{jk}^{i+n}. \tag{2.3}$$

This can also be seen by using $\nabla J = 0$ from the calculation

$$h_{ij}^{k+n} = g(\nabla_{e_i}e_j, Je_k) = -g(e_j, \nabla_{e_i}Je_k) = -g(e_j, J\nabla_{e_i}e_k) = g(Je_j, \nabla_i e_k) = h_{ik}^{j+n}.$$

The mean curvature vector field is (without dividing n for simplicity of writing)

$$H = \sum_{i,j}^n h_{ii}^{j+n} Je_j := \sum_j^n H^{j+n} Je_j.$$

$$\begin{aligned} d\omega_{(j+n)k} &= \sum_i (h_{ik}^{j+n} d\theta_i + dh_{ik}^{j+n} \wedge \theta_i) \\ &= \sum_{i,A} h_{ik}^{j+n} \theta_A \wedge \omega_{Ai} + \sum_i dh_{ik}^{j+n} \wedge \theta_i \\ &= \sum_i dh_{ik}^{j+n} \wedge \theta_i \end{aligned}$$

where we assumed $\omega_{ij} = 0$ at a fixed point $p \in L$.

Thus the mean curvature 1-form of a Lagrangian L is

$$\iota_H \omega = \sum_{i,j}^{n} h_{ii}^{j+n} \theta_j. \tag{2.4}$$

Exterior differentiation, assuming $\nabla_{e_j} e_i = 0$ at the point under consideration, i.e. $\omega_{ij} = 0$,

$$\begin{aligned} d\iota_H \omega &= \sum_{i,j} dh_{ii}^{j+n} \wedge \theta_j \\ &= \sum_{i,j} dh_{ij}^{i+n} \wedge \theta_j \\ &= \sum_i d\omega_{(i+n)i} \\ &= \sum_i \left(-\frac{1}{2} \sum_{B,C}^{2n} R_{(i+n)iBC} \theta_B \wedge \theta_C + \sum_A^{2n} \omega_{(i+n)A} \wedge \omega_{Ai} \right) \\ &= \sum_i \sum_{k,l}^{n} \frac{1}{2} R_{i(i+n)kl} \theta_k \wedge \theta_l \\ &= \sum_{k,l}^{n} Ric(e_k, e_l) \theta_k \wedge \theta_l \\ &= \ell^* Ric. \end{aligned}$$

When M is Kähler-Einstein, $Ric = c\omega$ restricts to zero along the Lagrangian submanifold L, hence $\iota_H \omega$ is a closed 1-form on L, in particular, it represents an elment in $H^1(L, \mathbb{R})$. $\square$

2.2.2. Calabi-Yau ambient space. A *Calabi-Yau* manifold is a Kähler manifold of complex dimension n with a nowhere vanishing holomorphic n-form Ω. It is stronger than vanishing of the first Chern class $c_1(M) \in H^{1,1}(M, \mathbb{R})$. It holds, up to a scaling constant,

$$\Omega \wedge \overline{\Omega} = (-1)^{n(n-1)/2} \frac{2^n}{\sqrt{-1}^n n!} \omega^n. \qquad (\frac{\omega^n}{n!}??) \tag{2.5}$$

THEOREM 2.2.2 (Yau). *Any compact Kähler manifold with zero first Chern class admits a Kähler metric g with zero Ricci form.*

Let L be a Lagrangian submanifold in a Calabi-Yau manifold (M, ω, g, Ω). Then

$$\Omega|_L = e^{\sqrt{-1}\Theta} d\mu_L \tag{2.6}$$

where $d\mu_L$ is the volume form of L in the induced metric from g and $\Theta : L \to \mathbb{S}^1$. To see this, locally we can write $\Omega = f dz^1 \wedge \cdots dz^n$ for some local holomorphic function f. Write the holomorphic coordinates $z^j = x^j + \sqrt{-1} y^j, j = 1, ..., n$ where $x^1, ..., x^n$ are coordinates on L. Then

$$\Omega|_L = f dx^1 \wedge \cdots \wedge dx^n.$$

$$d\mu_L = \sqrt{\det g} dx^1 \wedge \cdots \wedge dx^n.$$

$$\omega = g_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

where g is a positive definite Hermitian matrix

$$g_{i\bar{j}} = g_{j\bar{i}}.$$

In real coordinates,

$$\begin{aligned} g_{i\bar{j}} &= g(\partial_{x^i} + \sqrt{-1}\partial_{y^i}, \partial_{x^j} - \sqrt{-1}\partial_{y^j}) \\ &= g(\partial_{x^i} - \sqrt{-1}J\partial_{x^i}, \partial_{x^j} + \sqrt{-1}J\partial_{x^j}) \\ &= g(\partial_{x^i}, \partial_{x^j}) + g(J\partial_{x^i}, J\partial_{x^j}) + \sqrt{-1}\left(g(\partial_{x^i}, J\partial_{x^j}) - g(J\partial_{x^i}, \partial_{x^j})\right) \\ &= 2g(\partial_{x^i}, \partial_{x^j}) + 2\sqrt{-1}g(\partial_{x^i}, J\partial_{x^j}) \end{aligned}$$

$$\Omega \wedge \overline{\Omega} = |f|^2 dz^1 \wedge d\bar{z}^1 \cdots dz^n \wedge d\bar{z}^n$$

$$\omega^n = \det(g_{i\bar{j}}) dz^1 \wedge d\bar{z}^1 \cdots dz^n \wedge d\bar{z}^n$$

From (2.5)

$$|f|^2 = 2^{-n} \det(g_{i\bar{j}}).$$

It follows that along L,

$$|f|^2 = \det(g_{ij}).$$

Therefore,

$$\begin{aligned} \Omega|_L &= f dx^1 \wedge \cdots \wedge dx^n \\ &= e^{\sqrt{-1}\Theta} \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n \\ &= e^{\sqrt{-1}\Theta} d\mu_L. \end{aligned}$$

The following dates back at least to [Harvey-Lawson], it can also be found in [Oh], [Schoen-Wolfson?], [Thomas-Yau] and perhaps others.

LEMMA 2.2.1. Let L be a Lagrangian submanifold in a Calabi-Yau manifold (M, Ω, ω). Then

$$H = J\nabla\Theta.$$

PROOF. Let $e_1(p), ..., e_n(p)$ be local orthonormal basis of T_pL. Parallel transport them along L to get a local orthonormal frame $e_1, ..., e_n$. Then $e_1, ..., e_n, Je_1, ..., Je_n$ form a local orthonormal frame for T_pM. Since Ω is parallel,

$$\begin{aligned} 0 &= \nabla_{e_k}\Omega \\ &= \nabla_{e_k}\left(f\ (\theta_1 + \sqrt{-1}J\theta_1) \wedge \cdots \wedge (\theta_n + \sqrt{-1}J\theta_n)\right) \\ &= \nabla_{e_k} f f^{-1}\Omega + f \sum (\theta_1 + \sqrt{-1}J\theta_1) \wedge ... \wedge (\nabla_{e_k}\theta_j + \sqrt{-1}J\nabla_{e_k}\theta_j) \wedge ... \wedge (\theta_n + \sqrt{-1}J\theta_n). \end{aligned}$$

Now restricting along L, we have $f = e^{\sqrt{-1}\Theta}\sqrt{\det(g_{ij})}$ and $\nabla^L_{e_k} e_j = 0$

$$\begin{aligned}
0 &= \nabla_{e_k} f f^{-1}\Omega|_L + f\sum \theta_1 \wedge ... \wedge \left(-\sqrt{-1}h^{l+n}_{kj}\right)\theta_l \wedge ... \wedge \theta_n \\
&= \nabla_{e_k} f f^{-1}\Omega|_L - \sqrt{-1}\sum h^{l+n}_{kl}\Omega|_L \\
&= \nabla_{e_k} f f^{-1}\Omega|_L - \sqrt{-1}\sum h^{k+n}_{ll}\Omega|_L \\
&= \left(\nabla_{e_k} f f^{-1} - \sqrt{-1}H^k\right)\Omega|_L \\
&= \left(\sqrt{-1}\nabla_{e_k}\Theta - \sqrt{-1}H^k\right)\Omega|_L.
\end{aligned}$$

Therefore

$$g(\nabla\Theta, e_k) = \nabla_{e_k}\Theta = H^k = g(H, Je_k) = -g(JH, e_k)$$

in turn

$$H = J\nabla\Theta.$$

This proves the desired statement. □

2.2.3. Phase function Θ for a gradient graph in $\mathbb{R}^{2n}$. Consider the graphic Lagrangian submanifold $L = (x, Du) \subset \mathbb{R}^n \oplus \mathbb{R}^n$ for $u \in C^2(K)$ where K is an open domain in $\mathbb{R}^n$. The induced metric on L from the euclidean metric on $\mathbb{R}^{2n}$ is

$$g = I + D^2uD^2u$$

and the induced volume form is

$$d\mu_L = \sqrt{\det(I + D^2uD^2u)}\; dx^1 \wedge \cdots \wedge dx^n = \sqrt{(1+\lambda_1^2)\cdots(1+\lambda_n^2)}\; dx^1 \wedge \cdots \wedge dx^n$$

where λ_i's are the eigenvalues of the Hessian D^2u when diagonalized at a point $x_0 \in \mathbb{R}^n$. The holomorphic n-form

$$\Omega = dz^1 \wedge \cdots \wedge dz^n \tag{2.7}$$

restricts to L is

$$\begin{aligned}
\Omega|_L &= (dx^1 + \sqrt{-1}u_{1k}dx^k) \wedge \cdots \wedge (dx^n + \sqrt{-1}u_{nk}dx^k) \\
&= (1+\sqrt{-1}\lambda_1)\cdots(1+\sqrt{-1}\lambda_n)\; dx^1 \wedge \cdots \wedge dx^n.
\end{aligned} \tag{2.8}$$

Therefore, we have

$$e^{\sqrt{-1}\Theta}\sqrt{(1+\lambda_1^2)\cdots(1+\lambda_n^2)} = (1+\sqrt{-1}\lambda_1)\cdots(1+\sqrt{-1}\lambda_n)$$

leading to

$$\begin{aligned}
e^{\sqrt{-1}\Theta} &= \frac{1+\sqrt{-1}\lambda_1}{\sqrt{1+\lambda_1^2}}\cdots\frac{1+\sqrt{-1}\lambda_n}{\sqrt{1+\lambda_n^2}} \\
&= e^{\sqrt{-1}(\arctan\lambda_1 + \cdots \arctan\lambda_n)}.
\end{aligned}$$

Therefore

$$\Theta = \arctan\lambda_1 + \cdots \arctan\lambda_n \tag{2.9}$$

In particular, the graph L is minimal if $u \in C^3$ and

$$\Theta = F(D^2u) = \arctan\lambda_1 + \cdots \arctan\lambda_n = C. \tag{2.10}$$

In particular, when $n = 3$, for $C = n\pi$, (2.10) becomes

$$\det D^2u = \Delta u.$$

The ellipticity of (2.10) can be seen from

$$\frac{\partial F(D^2u)}{\partial \lambda_j} = \frac{1}{1+\lambda_j^2} > 0. \tag{2.11}$$

The operator F is concave if u is convex as

$$\frac{\partial^2 F(D^2u)}{\partial \lambda_i \partial \lambda_j} = -\frac{2\lambda_j \delta_{ij}}{(1+\lambda_j^2)^2}. \tag{2.12}$$

2.2.4. Darboux coordinates with estimates. We summarize a result of [Lee-Joyce-Schoen]. Let (M, ω) be a symplectic $2n$-dimensional manifold, J an almost complex structure compatible with ω and g a Riemannian metric compatible with ω, so that $g(X,Y) = \omega(X, JY)$. For $p \in M$, the Darboux theorem for symplectic vector space (T_pM, ω_p) asserts existence of a linear mapping

$$v : \mathbb{R}^{2n} \to T_pM$$

such that

$$v^*\omega_p = \omega_0 \text{ and } v^*g_p = g_0. \tag{2.13}$$

Denote

$$U = \{(p,v) : p \in M, v \text{ satisfies } (2.13)\}$$

and $\pi : U \to M$ with $\pi(p,v) = p$. For any $\gamma \in U(n)$, the mapping $v \circ \gamma$ also satisfies (2.13). Therefore $\pi : U \to M$ is the $U(n)$-frame bundle of M. The action of $U(n)$ on the right is free and $\pi : U \to M$ is a principal $U(n)$-bundle. As $U(n)$ is a compact Lie group, it follows that U is compact when M is compact.

Proposition 2.2.1 (Joyce-Lee-Schoen). *Let (M, ω) be a compact symplectic manifold of dimension $2n$ with a compatible almost complex structure J and a Riemannian metric g compatible with ω, J. Let $\pi : U \to M$ be the $U(n)$-bundle as above. Then for small $\epsilon > 0$ there exist a family of embeddings $\Upsilon_{p,v} : B_\epsilon \subset \mathbb{C}^n \to M$ depending smoothly on $(p,v) \in U$, such that*

1. $\Upsilon_{p,v}(0) = p$ *and* $d\Upsilon_{p,v}|_0 = v$;
2. $\Upsilon_{p,v\circ\gamma} = \Upsilon_{p,v} \circ \gamma$;
3. $\Upsilon^*_{p,v}(\omega) = \omega_0$;
4. $\Upsilon^*_{p,v} g = g_0 + O(|z|)$.

Moreover, if (M, ω) is Kähler, then (4) can be improved with $O(|z|)$ replaced by

$$\frac{1}{2} Re \left(R_{i\bar{j}k\bar{l}} z^i z^k \bar{z}^j \bar{z}^l d\bar{z}^j d\bar{z}^l \right) + O(|z|^3).$$

Proof. Let $\epsilon' > 0$ and $B_{\epsilon'}$ be the ball of radius ϵ' about $0 \in \mathbb{C}^n$. For each $(p,v) \in U$ define $\Upsilon'_{p,v} : B_{\epsilon'} \to M$ by

$$\Upsilon'_{p,v} = \exp_p \circ v|_{B_{\epsilon'}}$$

which is diffeomorphic if ϵ' is small. It holds that $\Upsilon'_{p,v}$ is smooth in p, v and

$$\begin{aligned}
&\Upsilon'_{p,v}(0) = p \\
&d\Upsilon'_{p,v}|_0 = v \\
&\Upsilon'_{p,v} \circ \gamma = \Upsilon'_{p,v} \circ \gamma, \quad \gamma \in U(n) \\
&\Upsilon'^*_{p,v} g|_0 = g_0 \\
&\Upsilon'^*_{p,v} \omega|_0 = \omega_0.
\end{aligned}$$

By Taylor's theorem,

$$\Upsilon'^*_{p,v}\omega = \omega_0 + O(|z|) \tag{2.14}$$

$$\Upsilon'^*_{p,v}g = g_0 + O(|z|) \tag{2.15}$$

where $O(|z|)$ means that the coefficients of the 2-forms in (2.14) and the $(0,2)$ tensors in (2.15), in terms of dz and $d\bar{z}$, are functions satisfying the estimate $O(|z|)$, respectively.

We now modify $\Upsilon'_{p,v}$ to a symplectomorphism $\Upsilon_{p,v}$ on B_ϵ so that $\Upsilon^*_{p,v}\omega = \omega_0$ while keeping all of the other requirements in the proposition that $\Upsilon'_{p,v}$ already achieved. Define closed 2-forms on $B_{\epsilon'}$ by

$$\omega^t_{p,v} = (1-t)\omega_0 + t\Upsilon'^*_{p,v}\omega \tag{2.16}$$

for $t \in [0,1]$. Note $\omega^t_{p,v}$ is non-degenerate in a neighbourhood of $0 \in B_{\epsilon'}$ since $\Upsilon'^*_{p,v}\omega|_0 = \omega_0|_0$. The $U(n)$-frame bundle U is compact as M is compact, in turn, $[0,1]\times U$ is compact. Hence by choosing ϵ' small enough, we may assume $\omega^t_{p,v}$ is non-degenerate on $B_{\epsilon'}$ for all $t \in [0,1]$ and $(p,v) \in U$. By Poincaré's lemma, there exist 1-forms $\beta_{p,v}$ on $B_{\epsilon'}$ so that

$$d\beta_{p,v} = \omega_0 - \Upsilon'^*_{p,v}\omega.$$

We may further assume that each $\beta_{p,v}$ smoothly depends on p, v and

$$|\beta_{p,v}| = O(|z|^2) \tag{2.17}$$

in light of (2.14), and this estimate is uniform in (p,v) as U is compact.

Each $\gamma \in U(n) : \mathbb{C}^n \to \mathbb{C}^n$ restricts to a mapping $B_{\epsilon'} \to B_{\epsilon'}$ and defines a map $(p,v) \to (p, v\circ\gamma)$ which we will still denote γ. Let

$$\alpha_{p,v} = \frac{1}{|U(n)|}\int_{\gamma\in U(n)} (\gamma^{-1})^*\beta_{p,v\circ\gamma}\, d\mu_{U(n)} \tag{2.18}$$

where we use the bi-invariant metric on the Lie group $U(n)$. For any $\gamma_0 \in U(n)$,

$$\begin{aligned}
\alpha_{p,v\circ\gamma_0} &= \frac{1}{|U(n)|}\int_{\gamma\in U(n)} (\gamma^{-1})^*\beta_{p,v\circ\gamma_0\circ\gamma}\, d\mu_{U(n)} \\
&= \frac{1}{|U(n)|}\int_{\gamma_0\circ\gamma\in U(n)} \gamma_0^*((\gamma_0\circ\gamma)^{-1})^*\beta_{p,v\circ\gamma_0\circ\gamma}\, d\mu_{U(n)} \\
&= \gamma_0^*\left(\frac{1}{|U(n)|}\int_{\gamma_0\circ\gamma\in U(n)} (\gamma_0\circ\gamma)^{-1})^*\beta_{p,v\circ\gamma_0\circ\gamma}\, d\mu_{U(n)}\right) \\
&= \gamma_0^*\alpha_{p,v}.
\end{aligned}$$

Thus, the 1-form $\alpha_{p,v}$ is $U(n)$-equivariant:

$$(\gamma_0|_{B_{\epsilon'}})^*\alpha_{p,v} = \alpha_{p,v\circ\gamma_0}$$

Moreover $\alpha_{p,v}$ are closed non-degenerate 1-forms and satisfy (2.14).

Since $\alpha_{p,v}$ is non-degenerate, there is a unique vector $v^t_{p,v}$ on $B_{\epsilon'}$ satisfying

$$\iota_{v^t_{p,v}}\omega^t_{p,v} = \alpha_{p,v}.$$

As $\alpha_{p,v}$ satisfies (2.17), we have

$$|v^t_{p,v}| = O(|z|^2). \tag{2.19}$$

By (2.14),

$$
\begin{aligned}
d\left(\iota_{v^t_{p,v}}\omega^t_{p,v}\right) &= d\alpha_{p,v} \\
&= \frac{1}{|U|}\int_{\gamma\in U(n)} d\left((\gamma^{-1})^*\beta_{p,v\circ\gamma}\right) d\mu_{U(n)} \\
&= \frac{1}{|U|}\int_{\gamma\in U(n)} (\gamma^{-1})^* d\beta_{p,v\circ\gamma} d\mu_{U(n)} \\
&= \frac{1}{|U|}\int_{\gamma\in U(n)} (\gamma^{-1})^*(\omega_0 - \Upsilon'^*_{p,v\circ\gamma}) d\mu_{U(n)} \\
&= \omega_0 - \Upsilon'^*_{p,v}\omega \\
&= O(|z|).
\end{aligned}
\tag{2.20}
$$

Since $[0,1]\times U$ is compact, there is $0<\epsilon\le\epsilon'$, such that we can solve the following initial value problem on $[0,1]\times B_\epsilon$ to get a family of diffeomorphisms (onto the images)

$$\phi^t_{p,v} : B_\epsilon \to B_{\epsilon'}$$

$$\frac{d}{dt}\phi^t_{p,v} = v^t_{p,v}\circ\phi^t_{p,v}, \qquad \phi^0_{p,v} = id : B_\epsilon \to B_\epsilon \subset B_{\epsilon'}. \tag{2.21}$$

Then by Cartan's formula

$$
\begin{aligned}
\frac{d}{dt}\left((\phi^t_{p,v})^*\omega^t_{p,v}\right) &= (\phi^t_{p,v})^*\left(L_{v^t_{p,v}}\omega^t_{p,v} + \frac{d}{dt}\omega^t_{p,v}\right) \\
&= (\phi^t_{p,v})^*\left(\iota_{v^t_{p,v}}d\omega^t_{p,v} + d\iota_{v^t_{p,v}}\omega^t_{p,v} + \frac{d}{dt}\omega^t_{p,v}\right) \\
&= (\phi^t_{p,v})^*\left(\alpha^t_{p,v} + \frac{d}{dt}\omega^t_{p,v}\right) \\
&= 0
\end{aligned}
$$

where the last step follows from (2.16) and (2.20). Therefore, as $\phi^0_{p,v} = id$ we have

$$(\phi^1_{p,v})^*\Upsilon'^*_{p,v}\omega = \omega_0. \tag{2.22}$$

As $|v^t_{p,v}| = O(|z|^2)$ from (2.19), we see, in fact, we may assume that both $v^t_{p,v}$ and $dv^t_{p,v}$ are uniformly Lipschitz in B_ϵ, then by the Picard-Lindelöf theorem ([Hartman]) solution for (2.21), and for uniqueness of (2.23), uniquely exists in $[0,1]$

$$\frac{d}{dt}d\phi^t_{p,v} = dv^t_{p,v}\circ d\phi^t_{p,v}, \quad d\phi^0_{p,v} = id : TB_\epsilon \to TB_\epsilon. \tag{2.23}$$

In particular,

$$
\begin{aligned}
\phi^t_{p,v}(0) &= 0 \\
d\phi^t_{p,v}|_0 &= id
\end{aligned}
$$

We now verify

$$\Upsilon_{p,v} = \Upsilon'_{p,v}\circ\phi^1_{p,v}$$

fufills the requirements in the proposition: For (1) we have

$$\Upsilon_{p,v}(0) = \Upsilon'_{p,v}(\phi^1_{p,v}(0)) = \Upsilon'_{p,v}(0) = p$$

and
$$d\Upsilon^*_{p,v}|_0 = d\Upsilon'^*_{p,v} \circ d\phi^1_{p,v}(0) = d\Upsilon'^*_{p,v}(0) = v.$$
For (2), the $U(n)$-equivariance of $\alpha_{p,v}$ implies the $U(n)$-equivariance of $v^t_{p,v}$, and it then follow that $\phi^t_{p,v} = \gamma^{-1} \circ \phi^t_{p,v} \circ \gamma$ which together with the $U(n)$-invariance of $\Upsilon'_{p,v}$ yields the $U(n)$-invariance of $\Upsilon_{p,v}$. (3) follows directly from (2.22); and (4) follows from
$$\phi^1_{p,v}(0) = 0, \quad d\phi^1_{p,v}|_0 = id$$
and (2.15). □

The following result is an application of Proposition 2.2.1 on scaled metrics, and it is useful for regularity estimates when the local behaviour of g in a Darboux coordinates is concerned.

PROPOSITION 2.2.2 (Lee-Joyce-Schoen). Let $g^t_{p,v} = t^{-2}(\Upsilon_{p,v} \circ T(t))^* g$ where $T(t)$ is the dilation $z \to tz$.

$$\|g^t_{p,v} - g_0\|_{C^0} \le C_0 t \quad \text{and} \quad \|\partial^k g^t_{p,v}\|_{C^0} \le C_k t^k \tag{2.24}$$

for any positive integer k, where the norms are w.r.t. g_0 and ∂ is the Levi-Civita connection of g_0.

PROOF. For each $(p,v) \in U$, by (4) in Proposition 2.2.1 and since U is compact, there is a constant $C > 0$ independent of $(p,v) \in U$ so that $|\Upsilon^*_{p,v} g - g_0| < C|z|$ on $\overline{B}_{\epsilon/2}$, where $|\cdot|$ is in g_0. For $t \in (0, \frac{\epsilon}{2R}]$, it is clear $T(t)(B_R) \subseteq B_{\epsilon/2} \subset \overline{B}_{\epsilon/2}$.

$$\begin{aligned}
(\Upsilon_{p,v} \circ T(t))^* g\ (X,Y) &= g((\Upsilon_{p,v})_* tX, (\Upsilon_{p,v})_* tY) \\
&= t^2 g((\Upsilon_{p,v})_* X, (\Upsilon_{p,v})_* Y) \\
&= t^2 \Upsilon^*_{p,v} g\ (X,Y) \\
&= t^2 g_0(X,Y) + t^3 O(|z|)
\end{aligned}$$

where t^3 arises from a factor t from coefficient functions and a factor t^2 from the $(0,2)$ tensors, as noted in the Taylor expansion for (2.15). Therefore
$$\left|g^t_{p,v} - g_0\right| \le Ct|z| \le CRt$$
on B_R. Setting $C_0 = CR$ we get the first inequality in Proposition 2.2.2.

For the second inequality, for each fixed $k = 1, 2...$, note that $\partial^k \Upsilon^*_{p,v} g(z)$ is C^0 in the compact set $U \times \overline{B}_{\epsilon/2}$, so over there, there is a constant $C_k > 0$ such that $\left|\partial^k \Upsilon^*_{p,v} g(z)\right| \le C_k$. A scaling argument for $t \in (0, \frac{\epsilon}{2R}], z \in B_R$ leads to
$$\left|\partial^k (t^{-2}(\Upsilon_{p,v} \circ T(t))^* g(z)\right| = t^k \left|\partial^k \Upsilon^*_{p,v} g(tz)\right| \le C_k t^k$$
as $tz \in B_{\epsilon/2}$. □

3

Minimal Lagrangian Submanifolds

3.1. Special Lagrangian calibrations in $\mathbb{C}^n$

We follow [Harvey-Lawson].

Let $z = (z_1, ..., z_n)$ be the coordinate on $\mathbb{C}^n$ and $z = x + \sqrt{-1}y$ where $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{R}^n$. Take $\Omega = dz^1 \wedge ... dz^n$. Consider

$$\alpha_\tau = \text{Re } (e^{\sqrt{-1}\tau}\Omega)$$

for $0 \le \tau < 2\pi$ yields a $\mathbb{S}^1$-family of closed holomorphic n-forms of constant length, each of them defines α_τ-submanifolds in $\mathbb{C}^n$. Without loss of generality, we shall only consider $\tau = 0$. Our aim is to show α_0 gives rise to a calibration.

Let *Lag* be the set of all Lagrangian n-planes (therefore oriented positively ...) in $\mathbb{C}^n$, and $Gr(n, 2n)$ the Grassmannian of oriented n-planes in $\mathbb{R}^{2n}$. $U(n)$ acts on *Lag*, i.e. sending *Lag* to *Lag*, and the action is transitive. Let $L_1 = e_1 \wedge ... \wedge e_n$ and $L_2 = f_1 \wedge ... \wedge f_n$ be two Lagrangian planes in *Lag* where $\{e_1, ..., e_n\}$ and $\{f_1, ..., f_n\}$ are orthonormal bases. Then the linear map $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given by

$$A(e_j) = f_j, \quad A(Je_j) = Jf_j$$

is unitary. The isotropy subgroup of $U(n)$ at $L_0 = \mathbb{R}^n \oplus \{0\}$, i.e. the actions fixing the plane L_0, is $SO(n)$ acting diagonally on $\mathbb{R}^n \oplus \mathbb{R}^n$. Therefore we have the identification

$$Lag \cong U(n)/SO(n).$$

A Lagrangian plane $L \in Lag$ is *special Lagrangian* if $L = AL_0$ for some $A \in U(n)$ with $\det A = 1$. So the fibre at 1 of the fibration

$$U(n)/SO(n) \xrightarrow{\det} S^1$$

consists of all the special Lagrangian n-planes.

Proposition 3.1.1 (Larvey-Lawson). For any $P \in Gr(n, 2n)$, we have

1. $|\Omega(P)|^2 = |(Re\,\Omega)(P)|^2 + |(Im\,\Omega)(P)|^2 = |P \wedge JP|^2$.
2. $|P \wedge JP| \le |P|^2$ with equality if and only if $P \in$ Lag.
3. $Re\,\Omega$ has comass one. In fact, $Re\,\Omega(P) \le |P|$ with equality if and only if P is special Lagrangian n-plane.
4. Let $L \in$ Lag. Then L is special Lagrangian if and only if $Im\,\Omega(L) = 0$.

Proof. $\square$

$Im\,\Omega(L) = 0$ implies

PROPOSITION 3.1.2. Let $u \in C^2(K)$ where K is an open domain in $\mathbb{R}^n$ satisfying

$$\sum_{k=0}^{[(n-2)/2]} (-1)^k \sigma_{2k+1}(D^2 u) = 0. \tag{3.1}$$

Then the graph (x, Du) is volume minimizing in $\mathbb{R}^{2n}$.

3.2. Special Lagrangian submanifolds in a hyperkähler manifold

A *hyperkähler manifold* is a $4n$-dimensional Riemannian manifold (M, g) with parallel complex structures I, J, K with $IJ = -JI = K$. Often, it is also defined as a $4n$-dimensional Riemannian manifold whose holonomy group is contained in $Sp(n)$.

THEOREM 3.2.1 (Berger). A hyperkähler manifold is Ricci-flat.

Compact hyperkähler manifolds of real dimensional four are the $K3$ surfaces and the complex tori. A $K3$ surface is a complex surface with vanishing first Chern class and no global holomorphic 1-forms. It can also be defined as a simply connected compact complex surface with a nowhere vanishing holomorphic 2-form.

THEOREM 3.2.2 (Siu). Every K3 surface is Kähler.

Calabi-Yau manifolds generalized $K3$ surfaces to arbitrary complex dimension.

Using a hyperkähler rotation of complex structures, every I-holomorphic curve in a hyperkáhler surface is minimal and Lagrangian w.r.t. ω_J and ω_K. An I-holomorphic curve is a real 2-dimensional surface Σ such that $IT_p\Sigma = T_p\Sigma$ for all $p \in \Sigma$.

3.3. Minimal Lagrangian submanifolds in a Kähler-Einstein manifold

The standard embedding $\mathbb{R}P^n \subset \mathbb{C}P^n$.

Wolfson, Chen-Tian, Webster

4

Hamiltonian Stationary Lagrangian Submanifolds

First, we recall some basic facts about the Hodge $*$ operator acting on differential forms. The Hodge dual of a k-form α is a $n-k$-form $*\alpha$ such that

$$\beta \wedge *\alpha = \langle \beta, \alpha \rangle_g d\mu_g$$

where $d\mu_g$ is the volume form of (M, g), in particular,

$$d\mu_g = *1$$

and

$$* * \beta = (-1)^{k(n-k)}\beta.$$

It is worth noting that $** = -1$ for 1-forms when $n = 2$.

In local coordinates

$$*(dx^{i_1} \wedge \cdots dx^{i_k}) = \frac{\sqrt{\det g}}{(n-k)!} g^{i_1 j_1} \cdots g^{i_k j_k} \varepsilon_{j_1 \cdots j_n} dx^{j_{k+1}} \wedge \cdots \wedge dx^{j_n}.$$

The Hodge star $*$ can be used to define the operator

$$\delta : \wedge^k(M) \to \wedge^{k-1}(M)$$

by

$$\delta = (-1)^{n(k-1)+1} * d * .$$

As M is assumed to be compact without boundary, Stokes's theorem implies

$$0 = \int_M d(\beta \wedge *\alpha) = \int_M d\beta \wedge *\alpha - \beta \wedge (-1)^{k+1} d*\alpha = \int_M \langle d\beta, \alpha \rangle_g d\mu_g - \int_M \langle \beta, \delta\alpha \rangle_g d\mu_g.$$

The operator δ satisfies

$$\int_M \langle \beta, \delta\alpha \rangle_g d\mu_g = \int_M \langle d\beta, \alpha \rangle_g d\mu_g$$

and

$$\delta^2 = 0.$$

So δ is the L^2 adjoint of d. The Hodge Laplacian operator on differential forms is

$$\Delta = (\delta + d)^2 = \delta d + d\delta.$$

The Hodge theory asserts that the k-th de Rham cohomology, which is independent of the Riemannian metric g on M, is isomorphic to the space of harmonic k-forms.

Let L be a Lagrangian submanifold in a symplectic manifold (M, ω). For any Hamiltonian vector field $X = J\nabla f$ for some $f \in C^1(M)$. Recall

$$df(Y) = \iota_X \omega(Y) = \omega(X, Y) = g(X, JY) = g(\nabla f, Y).$$

The first variation formula reads

$$\begin{aligned}
\frac{d}{dt}\mathrm{vol}(L_t)\big|\big|_{t=0}(X) &= -\int_L \langle H, X\rangle_g d\mu_L \\
&= -\int_L \iota_H\omega(X) d\mu_L \\
&= -\int_L \langle \alpha_H, df\rangle_g d\mu_L \\
&= -\int_L \langle \delta\alpha_H, f\rangle_g d\mu_L
\end{aligned} \tag{4.1}$$

where δ is the Hodge adjoint operator of d w.r.t. the induced metric on L from g.

A Lagrangian submanifold L is *Hamiltonian stationary* if

$$\frac{d}{dt}\mathrm{vol}(L_t)\big|\big|_{t=0}(X) = 0$$

for any Hamiltonian vector field X on M. It was formally introduced by Oh. From (4.1), this is equivalent to α_H is co-closed:

$$\delta\alpha_H = 0.$$

In particular, we have

PROPOSITION 4.0.1. Suppose that L is a Hamiltonian stationary Lagrangian submanifold in a Kähler-Einstein manifold (M,ω). Then the mean curvature form α_H of L is a harmonic 1-form on L. If (M,ω) is a Calabi-Yau manifold, the phase function $\Theta : L \to \mathbb{S}^1$ satisfies

$$\Delta\Theta = 0 \tag{4.2}$$

where Δ is the Laplacian operator of L in the induced metric.

PROOF. When M is Calabi-Yau, we know $H = J\nabla\Theta$. From (4.1),

$$0 = \int_L \langle J\nabla\Theta, J\nabla f\rangle d\mu_L = \int_L \langle \nabla\Theta, \nabla f\rangle d\mu_L$$

for all $f \in C^1(M)$. □

4.0.1. The Euler-Lagrange equations. For a bounded domain $\Omega \subset \mathbb{R}^n$, let $u : \Omega \to \mathbb{R}$ be a smooth function. The gradient graph $\Gamma_u = \{(x, Du(x)) : x \in \Omega\}$ is Lagrangian in $\mathbb{C}^n = \mathbb{R}^n \oplus \sqrt{-1}\mathbb{R}^n$ with $z = x + \sqrt{-1}y$. Consider the volume functional on the space of C^2 functions on a bounded domain Ω in $\mathbb{R}^n$

$$F_\Omega(u) = \int_\Omega \sqrt{\det\left(I + (D^2u)^T D^2u\right)}\, dx. \tag{4.3}$$

Note that for the gradient graph of a function u, we have the induced metric

$$g_{ij} = \delta_{ij} + u_{ik}\delta^{kl}u_{lj} \tag{4.4}$$

in which case the above functional becomes

$$F_\Omega(u) = \int_{\Gamma_u} \sqrt{\det g}\, dx. \tag{4.5}$$

Note that we can define the volume $F_\Omega(u)$ whenever $u \in W^{2,n}(\Omega)$. We will seek critical points in this Sobolev space.

DEFINITION 4.0.1. A function $u \in W^{2,n}(\Omega)$ is a critical point for $F_\Omega(u)$ under compactly supported variations of the scalar function u if u satisfies the Euler-Lagrange equation

$$\int_\Omega \sqrt{\det g}\, g^{ij}\delta^{kl}u_{ik}\eta_{jl}\, dx = 0, \qquad \text{for all } \eta \in C_c^\infty(\Omega). \tag{4.6}$$

We call this equation the *variational Hamiltonian stationary equation* and u a weak solution if D^2u exists almost everywhere and (4.6) holds.

Here, summation convention is applied over repeated indices, δ^{kl} is the Kronecker delta, and g is the induced metric from the Euclidean metric on $\mathbb{R}^{2n}$, which can be written as

$$g = I + (D^2u)^T D^2u.$$

If the potential u is in $C^4(\Omega)$, the equation (4.6) is equivalent to the following *geometric Hamiltonian stationary equation*

$$\Delta_g \theta = 0 \tag{4.7}$$

where Δ_g is the Laplace-Beltrami operator on Γ_u for the induced metric g and θ is the Lagrangian phase function for the gradient graph Γ_u.

DEFINITION 4.0.2. We say a function u is a *weak solution* of (4.7) if

(1) $u \in W^{2,n}(\Omega)$;
(2) $\theta \in W^{1,2}(\Omega)$ is weakly harmonic in the sense that for all $\eta \in C_c^\infty(\Omega)$

$$\int_{\Gamma_u} \langle \nabla\theta, \nabla\eta\rangle d\mu_g = 0. \tag{4.8}$$

PROPOSITION 4.0.2. Suppose that $u \in C^3(\Omega)$. Then u is a weak solution to (4.6) on Ω if and only if u is a weak solution to (4.7) on Ω, in which case (4.6) and (4.7) are each the Euler-Lagrange equation for the functional (4.3).

PROOF. First we consider the case where u solves (4.6). Take a variation generated by $\eta \in C_c^\infty(\Omega)$, which varies the manifold along the y-direction in $\mathbb{C}^n$. Computing the volume for the path of potentials

$$\gamma[t](x) = u(x) + t\eta(x), \tag{4.9}$$

we get

$$\begin{aligned} \frac{d}{dt}F_\Omega(\gamma[t])\Big|_{t=0} &= \int_\Omega \frac{1}{2}\sqrt{g[t]}g^{ij}[t]\ \frac{d}{dt}g_{ij}[t]\Big|_{t=0} dx \\ &= \frac{1}{2}\int_\Omega \sqrt{g}g^{ij}\left(u_{ik}\delta^{kl}\eta_{lj} + \eta_{ik}\delta^{kl}u_{lj}\right) dx \\ &= \int_\Omega \sqrt{g}g^{ij}u_{ik}\delta^{kl}\eta_{lj}dx. \end{aligned}$$

Thus, the first variation of F_Ω at u is given by

$$\delta F_\Omega(\eta) = \int_\Omega \sqrt{g}g^{ij}u_{ik}\delta^{kl}\eta_{lj}dx.$$

We note that while defining $F_\Omega(u)$ requires only that $u \in W^{2,n}(\Omega)$.

On the other hand, we may compute the variation using the standard first variational formula for (4.5), when $u \in C^3$:

$$\frac{d}{dt}F_\Omega(\gamma[t])\bigg|_{t=0} = \frac{d}{dt}Vol(\Gamma_u) = -\int_\Omega \langle \vec{H}, V\rangle d\mu_g$$

where $\vec{H}$ is the mean curvature vector, and V is the variational field. Recall that the variation V is Hamiltonian if $V = JDf$ for some compactly supported function f in $\mathbb{C}^n$. For a Lagrangian submanifold, we also have [**18**, 2.19]

$$\vec{H} = -J\nabla\theta.$$

Therefore, a C^2 Lagrangian submanifold is critical for the volume functional under Hamiltonian variations if and only if its Lagrangian phase is weakly harmonic.

For the gradient graph of $u \in C^3(\Omega)$, we have a vertical variational field, i.e. the x-component is 0, that is Hamiltonian:

$$V(x) = \frac{d}{dt}\left(x, Du(x) + tD\eta(x)\right)\bigg|_{t=0} = \left(0, D\eta(x)\right). \tag{4.10}$$

We claim that u is a weak solution to (4.7) is equivalent to that the gradient graph is critical for all vertical variations. In fact,

$$\begin{aligned}\delta F_\Omega(\eta) &= \int_\Omega \langle J\nabla\theta, (0, D\eta)\rangle\, d\mu_g \\ &= \int_\Omega \langle \nabla\theta, -J(0, D\eta)\rangle\, d\mu_g \\ &= \int_\Omega \langle \nabla\theta, (D\eta, 0)\rangle\, d\mu_g.\end{aligned}$$

with all inner products thus far being computed with respect to the ambient Euclidean metric. Now

$$\nabla\theta = g^{ij}\theta_i\partial_j$$

where

$$\begin{aligned}\partial_1 &= (1, 0, \ldots, 0, u_{11}, u_{21}, \ldots, u_{n1}), \\ &\cdots \\ \partial_n &= (0, 0, \ldots, 1, u_{1n}, u_{2n}, \ldots, u_{nn}),\end{aligned}$$

so we have

$$\begin{aligned}\delta F_\Omega(\eta) &= \int_\Omega \left\langle g^{ij}\theta_i\partial_j, (D\eta, 0)\right\rangle d\mu_g \\ &= \int_\Omega g^{ij}\theta_i\eta_j\, d\mu_g \\ &= \int_\Omega \langle \nabla\theta, \nabla\eta\rangle_g\, d\mu_g.\end{aligned}$$

Thus we have

$$\delta F_\Omega(\eta) = 0 \quad \text{ for all } \eta \in C_0^\infty(\Omega)$$

if and only if

$$\int_\Omega \langle \nabla\theta, \nabla\eta\rangle\, d\mu_g = 0 \quad \text{for all } \eta \in C_c^\infty(\Omega).$$

This equation has the weak form

$$\int_\Omega \eta \Delta_g \theta \, d\mu_g = 0 \text{ for all } \eta \in C_c^\infty(\Omega)$$

that is

$$\Delta_g \theta = 0. \tag{4.11}$$

It follows that for $u \in C^3(\Omega)$, the volume (4.5) is stationary under Hamiltonian variations precisely when (4.7) is satisfied. Because (4.3) and (4.5) are the same functional, if follows that for $u \in C^3(\Omega)$, (4.6) and (4.7) are equivalent. □

Observe that, for the gradient graph $\Gamma_u = \{(x, Du(x)) : x \in \Omega\}$, the vertical variations constructed by (4.100) are in 1-1 correspondence with $C_c^\infty(\Omega)$. Note that one can also construct a variational field, $V = J\nabla\eta$ for each $\eta \in C_c^\infty(\Gamma_u)$. This is the traditional way of producing Hamiltonian variations along any Lagrangian submanifold, graphical or not. If the potential u is smooth, then $C_c^\infty(\Gamma_u) = C_c^\infty(\Omega)$ where Ω is identified with Γ_u by $F_u(x) = (x, Du(x))$, and the sets of variations are in 1-1 correspondence. One can then compute geometrically

$$\begin{aligned} \frac{d}{dt} F_\Omega(\gamma(t))\Big|_{t=0} &= \int_\Omega \left\langle -\vec{H}, V \right\rangle d\mu_g \\ &= \int_\Omega \langle J\nabla\theta, J\nabla\eta\rangle \, d\mu_g \\ &= \int_\Omega \langle \nabla\theta, \nabla\eta\rangle \, d\mu_g. \end{aligned} \tag{4.12}$$

In particular, the first variational formula is the same.

When u is not smooth, in general $C_c^\infty(\Gamma_u) \neq C_c^\infty(\Omega)$. For example if the submanifold Γ_u is smooth but the gradient graph has vertical tangents (for instance, the curve $\Gamma_u = \{(x, x^{\frac{1}{3}}) : x \in (-1,1)\}$ and $u = \frac{3}{4}x^{\frac{4}{3}}$ is the same smooth curve (y^3, y) for $y \in (-1,1)$), one would expect some nearby Lagrangian manifolds that are not graphical over x: These clearly cannot be reached through a path of vertical variations. In this case, we have strict containment

$$C_c^\infty(\Omega) \subsetneq C_c^\infty(\Gamma_u).$$

Thus a Hamiltonian stationary Lagrangian submanifold, whose volume by definition is stationary under the larger set of variations, satisfies the equation (4.6) as well. In this sense, (4.6) is formally weaker than (4.7). It is worth asking when these equations are the same. We delve into this in the next section.

We note, as it will become useful later, that if D^2u is bounded by a fixed constant almost everywhere, then from (4.4) we see that the operator

$$\Delta_g = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{ij} \partial_j \right)$$

is uniformly elliptic.

4.1. Regularity of HSL submanifolds in $\mathbb{C}^n$

First, we collect some properties of Sobolev functions for the reader's convenience. Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function. We have

(1) L^∞ is the space of Lebesques measurable functions bounded a.e. with the norm $\|f\|_{L^\infty(\Omega)} = \inf\{C \geq 0 : |f(x| \leq C \quad a.e.\ x \in \Omega\}$. For $1 \leq p \leq \infty$, $L^p(\Omega, \|\cdot\|_{L^p(\Omega)})$ is a Banach space.
(2) f is locally Lipschitz in Ω if and only if $f \in W^{1,\infty}_{loc}(\Omega)$.
(3) $f \in W^{1,p}(\Omega)$ for some $1 \leq p < \infty$ can be approximated by $f_k \in W^{1,p}(\Omega) \cap C^\infty(\Omega)$ in $W^{1,p}(\Omega)$.
(4) If $f \in W^{1,p}(\Omega)$ for some $n < p \leq \infty$, then f is differentiable a.e. in Ω and its gradient equals the weak gradient a.e.
(5) $f \in W^{1,\infty}(\Omega)$ if and only if f is Lipschitz continuous in Ω.
(6) Trace operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega, \mathcal{H}^{n-1})$ such that $Tf = f$ on $\partial\Omega$ can be defined when Ω is bounded with Lipschitz boundary, $1 \leq p < \infty$. It holds

$$\int_\Omega f \operatorname{div} X\, dx = -\int_\Omega Df \cdot X\, dx + \int_{\partial\Omega} (X \cdot \nu) Tf\, d\mathcal{H}^{n-1}, \quad \forall X \in C^1(\Omega, \mathbb{R}^n).$$

The main results in this chapter are

Theorem 4.1.1 (Chen-Warren). Let Ω be a domain in $\mathbb{R}^n$ and let $Q \subset \Omega$ be a compact subset (possible empty) with capacity zero. There is a $c(n) > 0$ such that if $u \in C^{1,1}(\Omega \backslash Q)$ is a weak solution to (4.6) on $\Omega \backslash Q$ satisfying

$$\|u\|_{C^{1,1}(\Omega\backslash Q)} \leq c(n),$$

then u is a smooth solution of both (4.7) and (4.6) on Ω.

Recall that the capacity of a set Q is defined as

$$\operatorname{Cap}(Q) = \inf_{\substack{\phi \in C_c^\infty(\mathbb{R}^n),\\ 0 \leq \phi \leq 1,\\ \phi = 1 \text{ near } Q}} \int |D\phi|^2\, dx.$$

In particular, if the Hausdorff dimension of Q is less than $n-2$ then $\operatorname{Cap}(Q)$ is zero.

Theorem 4.1.2 (Chen-Warren). Suppose that $u \in C^{1,1}(\mathbb{B}_1(0))$ and u is a weak solution of (4.7). If either

$$\theta \geq \delta + \frac{\pi}{2}(n-2) \quad \text{a.e.} \tag{4.13}$$

for some constant $\delta \in (0, \pi)$; or

$$u - \delta \frac{|x|^2}{2} \text{ is convex} \tag{4.14}$$

for some constant $\delta > 0$; or

$$\|u\|_{C^{1,1}(\mathbb{B}_1(0))} \leq 1 - \delta \tag{4.15}$$

for some constant $\delta \in (0,1)$, then for $k \geq 2$ we have

$$\|u\|_{C^{k,\alpha}(\mathbb{B}_{1/2}(0))} \leq C(k, n, \|u\|_{C^{1,1}(\mathbb{B}_1(0))}, \delta).$$

The conclusion still holds if $\mathbb{B}_1(0)$ is replaced by $\mathbb{B}_1(0) \backslash Q$, where Q is a compact subset of $\mathbb{B}_1(0)$ with capacity zero.

THEOREM 4.1.3. Any C^1 Hamiltonian stationary Lagrangian submanifold of $\mathbb{C}^n$ is real analytic. More generally, suppose $u \in W^{2,n}(\Omega)$, and u satisfies equation (4.6) on Ω. There is a constant $c_0(n)$ such that if the image of the tangent planes (where defined) of the gradient graph

$$\Gamma_u = \{(x, Du(x)) : x \in \Omega\}$$

lies in a ball of radius $c_0(n)$ in the Grassmannian $Gr(n, 2n)$, then Γ_u is a real analytic submanifold of $\mathbb{R}^{2n}$.

In particular, if D^2u is within distance $c(n)$ to a continuous function, then u must be smooth, hence real analytic [**?**, p.203]. For example, while we cannot rule out non-flat tangent cones occurring, we can rule out non-flat tangent cones that are nearly flat.

In two dimensions, regularity results have been obtained by Schoen and Wolfson [**28**, Theorem 4.7] in a general Kähler manifold setting, where singularities are known to occur. The examples of singularities are non-graphical over an open domain [**?**, Section 7]. On the other hand, the Euclidean case of [**?**, Proposition 4.6] states that u solving (4.7) is smooth whenever $u \in C^{2,\alpha}$. Theorem 4.1.3 is a generalization of this result when the ambient space is $\mathbb{C}^n$, see Corollary 4.3.1.

4.1.1. $W^{3,2}$ estimates. Now we shall consider a general fourth order equation of the form

$$\int_\Omega a^{ijkl}(D^2u)\, u_{ik}\, \eta_{jl}\, dx = 0 \tag{4.16}$$

for all $\eta \in C_c^\infty(\Omega)$, where each a^{ijkl} is a smooth function defined on the Hessian space, i.e. the space $S^{n\times n}$ of real symmetric $n \times n$ matrices, and u_{ik} are the second order weak partial derivatives. A function $u \in W^{2,\infty}(\Omega)$ is called a *variational solution* to (4.16) on Ω, if (4.16) is satisfied for all $\eta \in C_c^\infty(\Omega)$. We will be mainly concerned with $u \in C^{1,1}(\Omega)$ for our geometric applications.

We will write a matrix $B \in S^{n\times n}$ as (b_{ij}) for $1 \le i, j \le n$. When b_{ij} appears in the denominator of a partial derivative it means the variable at the (i, j) position of the $n(n+1)/2$ dimensional vector space $S^{n\times n}$, not the second derivatives.

The key link between the two Euler-Lagrange equations is supplemented by the following $W^{3,2}$ estimates:

PROPOSITION 4.1.1 (Chen-Warren). Suppose that $u \in W^{2,\infty}(\Omega)$ is a weak solution to (4.16) on Ω for $n \ge 2$, and that there is a convex neighborhood $U \subset S^{n\times n}$ such that for all $M, M^*, M' \in U$, all $W \in S^{n\times n}$ and some constant $\beta > 0$

$$\frac{\partial a^{ijkl}}{\partial u_{pq}}(M^*)M'_{ik}W_{pq}W_{jl} + a^{ijkl}(M)W_{ik}W_{jl} \ge \beta \sum_{r,s} W_{rs}^2. \tag{4.17}$$

If $D^2u(\Omega) \subset U$ wherever D^2u is defined, then $u \in W^{3,2}_{loc}(\Omega)$.

PROOF. By approximation, the equation (4.16) must hold for compactly supported test functions in $W_0^{2,\infty}(\Omega)$; in particular, it must hold for the double difference quotient

$$\eta = -\left[\zeta^4 u^{(h_m)}\right]^{(-h_m)}$$

where $\zeta \in C_c^\infty(\Omega)$ is a cutoff function that is 1 on some interior set, and the upper (h_m) refers to the difference quotient

$$f^{(h_m)}(x) := \frac{f(x + he_m) - f(x)}{h}$$

and we have chosen h small enough (depending on ζ) so that η is well defined and compactly supported. We have

$$\int_\Omega a^{ijkl}(D^2u)u_{ik}\left(-\left[\zeta^4 u^{(h_m)}\right]^{(-h_m)}\right)_{jl} dx = 0. \tag{4.18}$$

For h small, we can "integrate by parts" with respect to the difference quotient, i.e.

$$\int_\Omega \left[a^{ijkl}(D^2u)u_{ik}\right]^{(h_m)}\left(\zeta^4 u^{(h_m)}\right)_{jl} dx = 0.$$

Now the "product rule" for difference quotients gives

$$\begin{aligned}
\left[a^{ijkl}(D^2u)\ u_{ik}\right]^{(h_m)}(x) &= u_{ik}(x+he_m)a^{ijkl}(D^2u)^{(h_m)}(x) + a^{ijkl}(D^2u(x))u_{ik}^{(h_m)}(x) \\
&= u_{ik}(x+he_m)\frac{1}{h}\int_0^1 \frac{d}{dt}a^{ijkl}\left((1-t)D^2u(x) + tD^2u(x+he_m)\right)dt \\
&\quad + a^{ijkl}(D^2u(x))u_{ik}^{(h_m)}(x) \\
&= u_{ik}(x+he_m)\int_0^1 \frac{\partial a^{ijkl}}{\partial u_{pq}}\left((1-t)D^2u(x) + tD^2u(x+he_m)\right)u_{pq}^{(h_m)}(x)dt \\
&\quad + a^{ijkl}(D^2u(x))u_{ik}^{(h_m)}(x) \\
&= A^{ijkl,pq}(x)u_{ik}(x+he_m)v_{pq}(x) + a^{ijkl}(D^2u(x))v_{ik}(x)
\end{aligned}$$

where

$$v = u^{(h_m)}$$

and

$$A^{ijkl,pq}(x) = \int_0^1 \frac{\partial a^{ijkl}}{\partial u_{pq}}\left((1-t)D^2u(x) + tD^2u(x+he_m)\right)dt = \frac{\partial a^{ijkl}}{\partial u_{pq}}\left(M^*(x)\right)$$

where

$$M^*(x) := (1-t^*)D^2u(x) + t^*D^2u(x+he_m)$$

for some t^* by the mean value theorem. (Note that for a fixed h, D^2u exists at both x and $x + he_m$, for almost every x, so all of the above quantities are defined for x almost everywhere.) So equation (4.18) becomes

$$\int_\Omega \left(\frac{\partial a^{ijkl}}{\partial u_{pq}}\left(M^*(x)\right)u_{ik}(x+he_m)v_{pq}(x) + a^{ijkl}(D^2u(x))v_{ik}(x)\right)\left(\zeta^4 v(x)\right)_{jl} dx = 0.$$

Now differentiating the second factor,

$$\int_\Omega \left(\begin{array}{c}\left(\frac{\partial a^{ijkl}}{\partial u_{pq}}\left(M^*(x)\right)u_{ik}(x+he_m)v_{pq}(x) + a^{ijkl}(D^2u(x))v_{ik}(x)\right) \\ \times\left(\zeta^4 v_{jl} + 4\zeta^3\zeta_j v_l + 4\zeta^3\zeta_l v_j + 4v(\zeta^3\zeta_{jl} + 3\zeta^2\zeta_j\zeta_l)\right)(x)\end{array}\right) dx = 0. \tag{4.19}$$

By the condition (4.17) in the hypothesis we have that

$$\int_\Omega \left(\frac{\partial a^{ijkl}}{\partial u_{pq}}\left(M^*(x)\right)u_{ik}(x+he_m)v_{pq}(x) + a^{ijkl}(D^2u(x))v_{ik}(x)\right)\zeta^4 v_{jl}dx \geq \beta\int_\Omega \zeta^4\sum_{r,s} v_{rs}^2 dx.$$

For the remaining terms, note that for the second term in the expansion of (4.19) we have by Young's inequality

$$\left|\frac{\partial a^{ijkl}}{\partial u_{pq}}\left(M^*(x)\right) u_{ik}(x+he_m)v_{pq}(x)4\zeta^3(x)\zeta_j(x)v_l(x)\right| \leq$$
$$C(n)\frac{1}{\varepsilon}\left(\frac{\partial a^{ijkl}}{\partial u_{pq}}\left(M^*(x)\right)\right)^2\left(u_{ik}(x+he_m)\right)^2\zeta^2(x)\left|D\zeta(x)\right|^2\left|Dv(x)\right|^2+\varepsilon\zeta^4(x)v_{pq}^2(x).$$

A similar expression can be made for each of the terms. Noting that D^2u is bounded and v is the different quotient of u, we obtain

$$\begin{aligned}&\int_\Omega\left(\begin{array}{c}\left(\frac{\partial(a^{ijkl}}{\partial u_{pq}}\left(M^*(x)\right)u_{ik}(x+he_m)v_{pq}(x)+a^{ijkl}(D^2u(x))v_{ik}(x)\right)\\ \times\left(4\zeta^3\zeta_jv_l+4\zeta^3\zeta_lv_j+4v(\zeta^3\zeta_{jl}+3\zeta^2\zeta_j\zeta_l)\right)(x)\end{array}\right)dx\\ &\leq C\left(|Du|,|D^2u|,\left|D\zeta\right|,\left|D^2\zeta\right|^2,\left|Da^{ijkl}\right|\right)\frac{1}{\varepsilon}\int_\Omega|Dv|^2dx+\varepsilon\int_\Omega\sum_{r,s}\zeta^4v_{rs}^2dx\end{aligned}$$

where $\left|Da^{ijkl}\right|$ is a norm on the total derivative of the functions a^{ijkl} on the space of symmetric matrices.

We conclude that by choosing ε appropriately, we have

$$\begin{aligned}\frac{\beta}{2}\int_\Omega\zeta^4\sum_{r,s}v_{rs}^2dx&\leq C\left(|Du|,|D^2u|,\left|D\zeta\right|,\left|D^2\zeta\right|^2,\left|Da^{ijkl}\right|\right)\frac{1}{\varepsilon}\int_\Omega|Dv|^2dx\\ &\leq C\left\|v\right\|_{W^{1,2}(\Omega)}\\ &\leq C\left\|u\right\|_{W^{2,2}(\Omega)}.\end{aligned}$$

Thus

$$\|v\|_{W^{2,2}(\{x|\zeta(x)=1\})}\leq C.$$

Now this estimate is uniform in h and direction e_m so we conclude that the derivatives are in $W^{2,2}(\Omega)$ and thus $u\in W^{3,2}(\{x|\zeta(x)=1\})$. $\square$

Proposition 4.1.2. There is a bound $c(n)$ such that if

$$\|u\|_{C^{1,1}(\Omega)}\leq c(n)$$

for a weak solution u to the variational Hamiltonian stationary equation (4.6), then $u\in W^{3,2}_{loc}(\Omega)$.

Proof. First recall (cf. [**15**, section 5.8.2]) that D^2u is defined almost everywhere and bounded where it is defined in terms of the $C^{1,1}$ norm. Considering (4.6) in the notation of (4.16) we have

$$a^{ijkl}=\sqrt{g}g^{ij}\delta^{kl}.$$

Our goal is to show that the condition (4.17) is satisfied on the set

$$U=\left\{M\in S^{n\times n}:\|M\|_\infty\leq c(n)\right\}.$$

For simplicity, we shall write $|M|$ for $\|M\|_\infty$, especially when Hessian is involved.

Computing, we see

$$\begin{aligned}\frac{\partial a^{ijkl}}{\partial u_{mp}} &= \frac{1}{2}\sqrt{g}g^{ab}\frac{\partial}{\partial u_{mp}}g_{ab}g^{ij}\delta^{kl} - \sqrt{g}g^{ia}g^{bj}\frac{\partial}{\partial u_{mp}}g_{ab}\delta^{kl} \\ &= \left(\frac{1}{2}g^{ab}g^{ij}\delta^{kl} - g^{ia}g^{bj}\delta^{kl}\right)\sqrt{g}\frac{\partial}{\partial u_{mp}}g_{ab} \\ &= \left(\frac{1}{2}g^{ab}g^{ij}\delta^{kl} - g^{ia}g^{bj}\delta^{kl}\right)\sqrt{g}\frac{\partial}{\partial u_{mp}}\left(\delta_{ab} + u_{ac}\delta^{cd}u_{db}\right) \\ &= \left(\frac{1}{2}g^{ab}g^{ij}\delta^{kl} - g^{ia}g^{bj}\delta^{kl}\right)\sqrt{g}\left(\delta_{mp,ac}\delta^{cd}u_{db} + u_{ac}\delta^{cd}\delta_{mp,db}\right).\end{aligned} \tag{4.20}$$

In particular,

$$\left|\frac{\partial a^{ijkl}}{\partial u_{pq}}(D^2u)\right| \leq C(n)\left|D^2u\right|\left(1+\left|D^2u\right|^2\right)^{n/2}. \tag{4.21}$$

Next, note that if we let

$$G_{ij} = \sqrt{g}g^{ij},$$

we can write

$$\sqrt{g}g^{ij}\delta^{kl}W_{ik}W_{jl} = \mathrm{Trace}(G^TWI_nW^T).$$

But G can be diagonalized by an orthogonal matrix O :

$$G^T = O^TDO$$

where

$$D = \sqrt{g}\begin{pmatrix} \frac{1}{1+\lambda_1^2} & & \\ & \ddots & \\ & & \frac{1}{1+\lambda_n^2}\end{pmatrix}.$$

Then

$$\begin{aligned}\sqrt{g}g^{ij}\delta^{kl}W_{ik}W_{jl} &= \mathrm{Trace}(O^TDOWW^T) \\ &= \mathrm{Trace}(OO^TDOWW^TO^T) \\ &= \mathrm{Trace}(D\,(OW)\,(OW)^T) \\ &\geq \min_i D_{ii}\cdot\mathrm{Trace}\left((OW)\,(OW)^T\right) \\ &= \min_i D_{ii}\left\|OW\right\|_{HS}^2 \\ &= \min_i D_{ii}\left\|W\right\|_{HS}^2,\end{aligned}$$

where we are using the Hilbert-Schmidt norm on matrices. Thus

$$\sqrt{g}g^{ij}\delta^{kl}W_{ik}W_{jl} \geq \frac{1}{1+c(n)^2}\left\|W\right\|_{HS}^2. \tag{4.22}$$

Combining (4.21) and (4.22) and plugging this into (4.17) we see for M^*, M', and M in U we have

$$\begin{aligned}&\frac{\partial a^{ijkl}}{\partial u_{pq}}(M^*)M'_{ik}W_{pq}W_{jl} + a^{ijkl}(M)W_{ik}W_{jl} \\ &\geq \frac{1}{1+c(n)^2}\left\|W\right\|_{HS}^2 - C(n)\left|c(n)\right|^2\left(1+c(n)^2\right)^{n/2}\left\|W\right\|_\infty^2 \\ &\geq \beta\left\|W\right\|_{HS}^2\end{aligned}$$

for some $\beta > 0$, using the equivalence of norms, when $c(n)$ is chosen small. The conclusion follows from Lemma 4.1.1. $\square$

PROPOSITION 4.1.3. There is $c(n)$, such that if u is a weak solution of (4.6) and $\|u\|_{C^{1,1}(\Omega)} \leq c(n)$, then θ is a weak solution of the uniformly elliptic equation (4.7).

PROOF. First, let us consider the case when Q is the empty set. Because $u \in W^{3,2}_{loc}(\Omega) \cap C^{1,1}(\Omega)$ we may use a standard mollification construction, letting

$$u^{\varepsilon} = \rho_{\varepsilon} * u$$

for an appropriate function ρ_{ε} as in [**15**, Appendix C.4]. In particular (see [**15**, Appendix C, Theorem 6])

$$\lim_{\varepsilon \to 0} \|u^{\varepsilon} - u\|_{W^{3,2}_{loc}(\Omega)} = 0$$

and each u^{ε} is smooth.

Now we define functionals on $C^{\infty}_{c}(\Omega)$ by

$$\begin{aligned} F^{\varepsilon}(\eta) &= \int_{\Omega} \left[\sqrt{g} g^{ij} \delta^{kl} u_{ik}\right]^{\varepsilon} \eta_{jl} dx \\ F(\eta) &= \int_{\Omega} \sqrt{g} g^{ij} \delta^{kl} u_{ik} \eta_{jl} dx \end{aligned}$$

with the notation $\left[\sqrt{g} g^{ij} \delta^{kl} u_{ik}\right]^{\varepsilon}$ means "expression constructed from u^{ε} using (4.4)" (in particular, this does *not* mean the mollification of the expression).

First we check that for each η,

$$F(\eta) = \lim_{\varepsilon \to 0} F^{\varepsilon}(\eta).$$

We have

$$\begin{aligned} F^{\varepsilon}(\eta) - F(\eta) &= \int_{\Omega} \left(\left[\sqrt{g} g^{ij} u_{ik}\right]^{\varepsilon} - \sqrt{g} g^{ij} u_{ik}\right) \delta^{kl} \eta_{jl} dx \\ &= \int_{\Omega} \left(\left[\sqrt{g} g^{ij} u_{ik}\right]^{\varepsilon} - \left[\sqrt{g} g^{ij}\right]^{\varepsilon} u_{ik} + \left[\sqrt{g} g^{ij}\right]^{\varepsilon} u_{ik} - \sqrt{g} g^{ij} u_{ik}\right) \delta^{kl} \eta_{jl} dx \\ &= \int_{\Omega} \left(\left[\sqrt{g} g^{ij}\right]^{\varepsilon} (u^{\varepsilon}_{ik} - u_{ik}) + \left(\left[\sqrt{g} g^{ij}\right]^{\varepsilon} - \sqrt{g} g^{ij}\right) g^{ij} u_{ik}\right) \delta^{kl} \eta_{jl} dx \end{aligned}$$

Now because $u \in C^{1,1}$ and η_{jl} is bounded, we simply have to check that

$$\begin{aligned} u^{\varepsilon}_{ik} - u_{ik} &\to 0 \text{ in } L^1_{loc} \\ \left[\sqrt{g} g^{ij}\right]^{\varepsilon} - \sqrt{g} g^{ij} &\to 0 \text{ in } L^1_{loc}. \end{aligned}$$

The first assertion is clear as $u \in W^{3,2}_{loc}(\Omega)$.

Next,

$$\left| \left[\sqrt{g} g^{ij}\right]^{\varepsilon} - \sqrt{g} g^{ij} \right| \leq \sup_{i,j} \left| \frac{\partial \left(\sqrt{g} g^{ij}\right)}{\partial u_{ab}} \right| (u^{\varepsilon}_{ab} - u_{ab}).$$

Mimicking computations following (4.20) we see

$$\left| \frac{\partial \left(\sqrt{g} g^{ij}\right)}{\partial u_{ab}} \right| \leq C(n) \left|D^2 u\right| \left(1 + \left|D^2 u\right|^2\right)^{n/2} \leq C.$$

Thus

$$\left|\left[\sqrt{g}g^{ij}\right]^{\varepsilon}-\sqrt{g}g^{ij}\right|\leq C\left|D^2u^{\varepsilon}-D^2u\right| \tag{4.23}$$

and the second assertion then follows from the first.

We conclude that

$$F(\eta)=\lim_{\varepsilon\to 0}F^{\varepsilon}(\eta).$$

Next, we define functionals

$$G^{\varepsilon}(\eta)=\int_{\Omega}\left[\sqrt{g}g^{ij}\theta_i\right]^{\varepsilon}\eta_j\,dx$$
$$G(\eta)=\int_{\Omega}\sqrt{g}g^{ij}\theta_i\eta_j dx=\int_{\Omega}\sqrt{g}g^{ij}g^{ab}u_{abi}\eta_j\,dx$$

recalling that

$$\theta_i=\left(\operatorname{Im}\log\det\left(I+iD^2u\right)\right)_i=g^{ab}u_{abi}$$

and noting that since $u\in W^{3,2}_{loc}(\Omega)$, the third order derivatives exist almost everywhere.

Applying the first variational formulae for smooth submanifolds in section 2 to the smooth $\Gamma_{u^{\varepsilon}}$, we see that

$$\delta F_{\Omega}(\eta)=\int_{\Omega}\left[\sqrt{g}g^{ij}\delta^{kl}u_{ik}\right]^{\varepsilon}\eta_{jl}\,dx=\int_{\Omega}\left[\sqrt{g}g^{ij}\theta_i\right]^{\varepsilon}\eta_j\,dx$$

that is

$$G^{\varepsilon}(\eta)=F^{\varepsilon}(\eta).$$

So clearly, from our observations on $F^{\varepsilon}(\eta)$ we see that

$$\lim_{\varepsilon\to 0}G^{\varepsilon}(\eta)=0.$$

All that remains is to show that

$$\lim_{\varepsilon\to 0}G^{\varepsilon}(\eta)=G(\eta).$$

We follow the same procedure as above:

$$\begin{aligned}G^{\varepsilon}(\eta)-G(\eta)&=\int_{\Omega}\left(\left[\sqrt{g}g^{ij}\theta_i\right]^{\varepsilon}-\sqrt{g}g^{ij}\theta_i\right)\eta_j\,dx\\&=\int_{\Omega}\left(\left[\sqrt{g}g^{ij}\theta_i\right]^{\varepsilon}-\left[\sqrt{g}g^{ij}\right]^{\varepsilon}\theta_i+\left[\sqrt{g}g^{ij}\right]^{\varepsilon}\theta_i-\sqrt{g}g^{ij}\theta_i\right)\eta_j\,dx\\&=\int_{\Omega}\left(\left[\sqrt{g}g^{ij}\right]^{\varepsilon}\left([\theta]^{\varepsilon}_i-\theta_i\right)+\left(\left[\sqrt{g}g^{ij}\right]^{\varepsilon}-\sqrt{g}g^{ij}\right)\theta_i\right)\eta_j\,dx\end{aligned}$$

where $[\theta]^{\varepsilon}$ stands for the angle function in (**??**) using u^{ε}. Now we have to be slightly more careful, but proceed as before: Starting with the last term, we use (4.23)

$$\begin{aligned}\int_{\Omega}\left(\left[\sqrt{g}g^{ij}\right]^{\varepsilon}-\sqrt{g}g^{ij}\right)\theta_i\eta_j\,dx&\leq\|D\theta\|_{L^2}\|D\eta\|_{L^{\infty}}\left\|\left[\sqrt{g}g^{ij}\right]^{\varepsilon}-\sqrt{g}g^{ij}\right\|_{L^2}\\&\leq\|D\theta\|_{L^2}\|D\eta\|_{L^{\infty}}C\left\|D^2u^{\varepsilon}-D^2u\right\|_{L^2}\\&\to 0\end{aligned}$$

as

$$\|D\theta\|_{L^2(K)}\leq C\|u\|_{W^{3,2}(K)}$$

for any K compact inside Ω. Next

$$
\begin{aligned}
&\int_\Omega \left[\sqrt{g}g^{ij}\right]^\varepsilon \left([\theta]_i^\varepsilon - \theta_i\right)\eta_j \, dx \\
&= \int_\Omega \left[\sqrt{g}g^{ij}\right]^\varepsilon \left(\left[g^{ab}\right]^\varepsilon u^\varepsilon_{abi} - \left[g^{ab}\right]^\varepsilon u_{abi} + \left[g^{ab}\right]^\varepsilon u_{abi} - g^{ab}u_{abi}\right)\eta_j \, dx \\
&\le C(\|u\|_{C^{1,1}(\Omega)}) \, \|D\eta\|_{L^\infty} \left\{ \left\| \left[g^{-1}\right]^\varepsilon \right\|_{L^2} \left\|D^3u^\varepsilon - D^3u\right\|_{L^2} + \left\| \left[g^{-1}\right]^\varepsilon - g^{-1} \right\|_{L^2} \left\|D^3 u\right\|_{L^2} \right\}
\end{aligned}
$$

by noticing that $|D^2u^\varepsilon|$ is bounded by $\|u\|_{C^{1,1}}$ for the chosen mollifiers ρ_ε. Because $u^\varepsilon \to u$ in $W^{3,2}_{loc}$, these terms go to zero.

We conclude that

$$G(\eta) = \int_\Omega \sqrt{g}g^{ij}\theta_i\eta_j \, dx = 0$$

for all test functions η. It follows that θ is a weak solution of the uniformly elliptic equation (4.7). $\square$

4.2. Lewy-Yuan rotations

In this section we discuss and motivate the Lewy-Yuan rotation. We risk giving extra descriptions here in order to give a clear motivation as to what the rotation is useful for. We also rigorously justify low regularity versions of the rotation.

In the special Lagrangian setting, Yuan [**37**] used the following unitary change of coordinates

$$
\begin{aligned}
&U : \mathbb{C}^n \to \mathbb{C}^n \\
&U(x + \sqrt{-1}y) = e^{-\sqrt{-1}\pi/4}\left(x + \sqrt{-1}y\right).
\end{aligned} \tag{4.24}
$$

In this case, a surface Γ that was the gradient graph of a convex function u over the original $\mathbb{R}^n$-plane, is now represented as a gradient graph of a new function $\bar{u}$ over the new $\mathbb{R}^n$-plane, but this time with

$$-I_n \le D^2\bar{u} \le I_n.$$

We call this a downward rotation by angle $\pi/4$: The word 'downward' refers to the fact that the argument of the complex number $e^{-\sqrt{-1}\pi/4}$ (4.24) is negative. Any surface Γ that is the gradient graph of a semi-convex function u can be rotated downward ([**38**]). If for $\beta \in (0, \pi/2)$ we have

$$D^2u \ge -\tan\beta \, I_n$$

then we can rotate the graph downward by any positive angle $\alpha < \pi/2 - \beta$. More precisely, given

$$\Gamma = \{(x, Du(x)), x \in \Omega\} \subset \mathbb{R}^n + \sqrt{-1}\mathbb{R}^n$$

over Ω, let

$$\bar{\Gamma} = U_\alpha \Gamma \tag{4.25}$$

where

$$U_\alpha = \begin{pmatrix} e^{-\sqrt{-1}\alpha} & & \\ & \ddots & \\ & & e^{-\sqrt{-1}\alpha} \end{pmatrix}. \tag{4.26}$$

Clearly, $\bar{\Gamma}$ is isometric to Γ via the unitary rotation. In coordinates, this is equivalent to the following map:

$$\begin{aligned} \bar{x} &= \cos(\alpha)x + \sin(\alpha)Du(x) \\ \bar{y} &= -\sin(\alpha)x + \cos(\alpha)Du(x). \end{aligned} \tag{4.27}$$

Here $\bar{x}$ and $\bar{y}$ are simply the projections onto $\mathbb{R}^n$ and $\sqrt{-1}\mathbb{R}^n$ of $\bar{\Gamma}$, respectively.

Considering the functions $\bar{x}(x), \bar{y}(x)$ we may compute the differential form

$$\begin{aligned} \sum_i \bar{y}^i d\bar{x}^i &= \sum_i \left(-\sin(\alpha)x^i + \cos(\alpha)u_i(x)\right)\left(\cos(\alpha)dx^i + \sin(\alpha)u_{ij}(x)dx^j\right) \\ &= \sum_i \begin{pmatrix} -\sin(\alpha)\cos(\alpha)x^i dx^i + \cos^2(\alpha)u_i(x)dx^i \\ -\sin^2(\alpha)x^i u_{ij}(x)dx^j + \cos(\alpha)\sin(\alpha)u_i(x)u_{ij}(x)dx^j \end{pmatrix} \\ &= -\sin(\alpha)\cos(\alpha)D\frac{|x|^2}{2} + \cos^2(\alpha)Du(x) \\ &\quad -\sin^2(\alpha)\left(D(x\cdot Du(x)) - Du(x)\right) + \cos(\alpha)\sin(\alpha)D\frac{|Du(x)|^2}{2} \\ &= Du + \sin(\alpha)\cos(\alpha)D\frac{|Du(x)|^2 - |x|^2}{2} - \sin^2(\alpha)\left(D(x\cdot Du)\right) \\ &= D\left(u(x) + \sin(\alpha)\cos(\alpha)\frac{|Du(x)|^2 - |x|^2}{2} - \sin^2(\alpha)\left((x\cdot Du(x))\right)\right). \end{aligned}$$

We see that the 1-form $\sum_i \bar{y}^i d\bar{x}^i$ is exact (regardless of cohomological conditions) as we can exhibit $\bar{u}(\bar{x}) = \bar{u}(\bar{x}(x))$ solving $D_{\bar{x}}\bar{u} = \bar{y}d\bar{x}^i$. It follows that

$$(\bar{x}, \bar{y}) = (\bar{x}, D_{\bar{x}}\bar{u}(\bar{x}))$$

for some function $\bar{u}(\bar{x})$. The potential $\bar{u}$ is given explicitly, however, the explicit formula is only given in terms of the x coordinates. Fortunately, $\bar{x}(x)$ is a change of coordinates (this follows from the semi-convexity, see Proposition 4.2.1 below) and is invertible.

To summarize, we have exhibited $\bar{\Gamma}$ both as the gradient graph of a function $\bar{u}$ and as an isometric image of Γ. The result will be a new graph with a potential whose Hessian satisfies (see [**35**, (1.5) and (1.6)])

$$-\tan(\beta+\alpha)I_n \leq D^2\bar{u} \leq \tan(\pi/2 - \alpha)I_n.$$

The takeaway is that any semi-convexity guarantees that the graph has a representation of bounded geometry. Also note that there is nothing sacred about downward rotations: A function with a Hessian upper bound may always be rotated upwards to obtain a representation with a Hessian lower bound as well.

4.2.1. When Γ is not smooth. In the above computation, we referenced the second derivatives of u, despite the fact that the rotation itself is actually a map on first derivatives. Our goal in this section is to rigorously show that the Lewy-Yuan rotation can be performed in some low regularity settings where the second derivatives need not exist everywhere, as long as some semi-convexity is satisfied.

For a constant $K \in \mathbb{R}$, we say that u is K-convex on Ω if

$$u(x) - K\frac{|x|^2}{2} \text{ is convex.}$$

For $u \in C^1$ this is equivalent to the condition that, for all $x_0, x_1 \in \Omega$

$$\langle Du(x_1) - Du(x_0), x_1 - x_0 \rangle \geq K \left| x_1 - x_0 \right|^2. \tag{4.28}$$

PROPOSITION 4.2.1. Suppose that $\Gamma = (x, Du(x))$ is a Lagrangian graph in $\Omega + \sqrt{-1}\mathbb{R}^n \subset \mathbb{C}^n$ with Du continuous. Suppose that

$$u + (\cot(\sigma) - \varepsilon) \frac{|x|^2}{2} \quad \text{is convex} \tag{4.29}$$

for some $\varepsilon > 0, \sigma > 0$. Consider the function

$$\bar{u}(x) = u(x) + \sin(\sigma)\cos(\sigma) \frac{|Du(x)|^2 - |x|^2}{2} - \sin^2(\sigma)\, Du(x) \cdot x$$

and the function $\bar{x} : \Omega \to \bar{\Omega} \subset \mathbb{R}^n$ given by

$$\bar{x}(x) = \cos(\sigma)\, x + \sin(\sigma)\, Du(x). \tag{4.30}$$

Then

(1) The coordinate change (4.30) is invertible with Lipschitz continuous inverse,
(2) The derivative of $\bar{u}$ in $\bar{x}$ coordinates $\frac{D\bar{u}}{d\bar{x}}$ exists everywhere, and
(3) The gradient graph $\bar{\Gamma} = (\bar{x}, D\bar{u}(\bar{x})) \subset \bar{\Omega} + \sqrt{-1}\mathbb{R}^n \subset \mathbb{C}^n$ is the isometric image of Γ under the rotation through σ as in (4.25).

PROOF. Note that the convexity condition can be written as, for any two points $x_0, x_1 \in \Omega$,

$$\langle Du(x_1) - Du(x_0) + (\cot(\sigma) - \varepsilon)(x_1 - x_0), x_1 - x_0 \rangle \geq 0.$$

This leads to

$$\left\langle \frac{Du(x_1) - Du(x_0)}{|x_1 - x_0|}, \frac{x_1 - x_0}{|x_1 - x_0|} \right\rangle \geq -\cot(\sigma) + \varepsilon. \tag{4.31}$$

It then follows, for $x_1 \neq x_0$, that

$$\begin{aligned}
\left| \frac{\bar{x}(x_1) - \bar{x}(x_0)}{|x_1 - x_0|} \right| &\geq \left\langle \frac{\bar{x}(x_1) - \bar{x}(x_0)}{|x_1 - x_0|}, \frac{x_1 - x_0}{|x_1 - x_0|} \right\rangle \\
&= \left\langle \frac{\cos(\sigma)(x_1 - x_0) + \sin(\sigma)(Du(x_1) - Du(x_0))}{|x_1 - x_0|}, \frac{x_1 - x_0}{|x_1 - x_0|} \right\rangle \\
&= \cos(\sigma) + \sin(\sigma) \left\langle \frac{Du(x_1) - Du(x_0)}{|x_1 - x_0|}, \frac{x_1 - x_0}{|x_1 - x_0|} \right\rangle \\
&\geq \cos(\sigma) - \cot(\sigma)\sin(\sigma) + \sin(\sigma)\,\varepsilon \\
&= \sin(\sigma)\,\varepsilon
\end{aligned} \tag{4.32}$$

using (4.31). Therefore the continuous map $\bar{x}$ is invertible and its inverse is Lipschitz continuous with a Lipschitz constant $1/(\sin(\sigma)\varepsilon)$.

Next, for the gradient of $\bar{u}$ in terms of $\bar{x}$, we shall compute a difference quotient

$$\bar{u}_{\bar{j}}(\bar{x}_0) = \lim_{h \to 0} \frac{\bar{u}(\bar{x}_0 + h\bar{e}_j) - \bar{u}(\bar{x}_0)}{h}.$$

Since $\bar{x}$ is invertible, for $\bar{x}_0 \in \bar{\Omega}$ we may solve, for small fixed h

$$\bar{x}(x_0) = \bar{x}_0$$
$$\bar{x}(x_h) = \bar{x}_0 + h\bar{e}_j$$

that is

$$\cos(\sigma)\, x_0 + \sin(\sigma)\, Du(x_0) = \bar{x}_0$$
$$\cos(\sigma)\, x_h + \sin(\sigma)\, Du(x_h) = \bar{x}_h = \bar{x}_0 + h\bar{e}_j.$$

Let

$$\vec{v} = x_h - x_0.$$

Then $\vec{v}$ will satisfy

$$\cos(\sigma)\, \vec{v} + \sin(\sigma)\, [Du(x_h) - Du(x_0)] = h\bar{e}_j. \tag{4.33}$$

Since $\vec{v} \neq 0$ for $h \neq 0$, there is a unique $\vec{V}$ with

$$\vec{v} = h\vec{V}$$

while the vector $\vec{V}$ depends on h, we suppress this dependence. Observe that

$$\left|\vec{V}\right| = \frac{|\vec{v}|}{h} = \frac{|x_h - x_0|}{|\bar{x}(x_h) - \bar{x}(x_0)|} \leq \frac{1}{\varepsilon \sin\sigma}$$

by (4.32). In particular, $\vec{V}$ is a bounded vector. The function $\bar{u}$ is given in term of x coordinates, so in order to evaluate it, we have to use the change of coordinates, that is

$$\bar{u}\,(\bar{x}_0) = \bar{u}(\bar{x}^{-1}(\bar{x}_0)) = \bar{u}(x_0).$$

So we may compute the difference quotient of $\bar{u}$ in terms of x

$$\begin{aligned}
&\frac{\bar{u}\,(\bar{x}_h) - \bar{u}\,(\bar{x}_0)}{h} = \frac{\bar{u}(\bar{x}^{-1}(\bar{x}_h)) - \bar{u}(\bar{x}^{-1}(\bar{x}_0))}{h} \\
&= \frac{u(x_h) - u(x_0)}{h} + \sin(\sigma)\cos(\sigma)\, \frac{|Du(x_h)|^2 - |Du(x_0)|^2 - |x_h|^2 + |x_0|^2}{2h} \\
&\quad - \frac{1}{h}\sin^2(\sigma)\, (Du(x_h) - Du(x_0)) \cdot \left(x_0 + h\vec{V}\right) - \frac{1}{h}\sin^2(\sigma)\, Du(x_0) \cdot \left(\left(x_0 + h\vec{V}\right) - x_0\right) \\
&= \frac{u(x_0 + h\vec{V}) - u(x_0)}{h} - \sin^2(\sigma)\, Du(x_0) \cdot \vec{V} \\
&\quad + \cos(\sigma)\, \frac{\left[\sin(\sigma)\left(Du(x_0 + h\vec{V}) - Du(x_0)\right)\right]\left[Du(x_0 + h\vec{V}) + Du(x_0)\right]}{2h} \\
&\quad - \sin(\sigma)\cos(\sigma)\left(x_0 \cdot \vec{V} + \frac{h}{2}\left|\vec{V}\right|^2\right) \\
&\quad - \frac{1}{h}\sin(\sigma)\left[\sin(\sigma)\left(Du(x_0 + h\vec{V}) - Du(x_0)\right)\right] \cdot \left(x_0 + h\vec{V}\right).
\end{aligned}$$

Rewriting (4.33) as

$$\sin(\sigma)\, [Du(x_h) - Du(x_0)] = h\bar{e}_j - \cos(\sigma)\, h\vec{V} \tag{4.34}$$

we see

$$
\begin{aligned}
\frac{\bar{u}\left(\bar{x}_h\right)-\bar{u}\left(\bar{x}_0\right)}{h} &= \frac{u(x_0+h\vec{V})-u(x_0)}{h}-\sin^2(\sigma)\, Du(x_0)\cdot\vec{V} \\
&+\cos(\sigma)\,\frac{\left[h\bar{e}_j-\cos(\sigma)\, h\vec{V}\right]\left[Du(x_0+h\vec{V})+Du(x_0)\right]}{2h} \\
&-\sin(\sigma)\cos(\sigma)\left(x_0\cdot\vec{V}+\frac{h}{2}\left|\vec{V}\right|^2\right)-\frac{1}{h}\sin(\sigma)\left[h\bar{e}_j-\cos(\sigma)\, h\vec{V}\right]\cdot\left(x_0+h\vec{V}\right) \\
&= \frac{u(x_0+h\vec{V})-u(x_0)}{h}-\sin^2(\sigma)\, Du(x_0)\cdot\vec{V} \\
&+\cos(\sigma)\,\frac{1}{2}\left[\bar{e}_j-\cos(\sigma)\,\vec{V}\right]\left[2Du(x_0)+\frac{h\bar{e}_j-\cos(\sigma)\, h\vec{V}}{\sin(\sigma)}\right] \\
&-\sin(\sigma)\cos(\sigma)\left(x_0\cdot\vec{V}+\frac{h}{2}\left|\vec{V}\right|^2\right)-\sin(\sigma)\left[\bar{e}_j-\cos(\sigma)\,\vec{V}\right]\cdot\left(x_0+h\vec{V}\right) \\
&= \frac{u(x_0+h\vec{V})-u(x_0)}{h}-\sin^2(\sigma)\, Du(x_0)\cdot\vec{V} \\
&+\cos(\sigma)\left[\bar{e}_j-\cos(\sigma)\,\vec{V}\right]\cdot Du(x_0)+\frac{h}{2}\frac{\cos(\sigma)}{\sin(\sigma)}\left|\bar{e}_j-\cos(\sigma)\,\vec{V}\right|^2 \\
&-\sin(\sigma)\cos(\sigma)\, x_0\cdot\vec{V}-\sin(\sigma)\cos(\sigma)\,\frac{h}{2}\left|\vec{V}\right|^2-\sin(\sigma)\,\bar{e}_j\cdot x_0-h\sin(\sigma)\,\bar{e}_j\cdot\vec{V} \\
&+\sin(\sigma)\cos(\sigma)\, x_0\cdot\vec{V}+h\sin(\sigma)\cos(\sigma)\left|\vec{V}\right|^2 \\
&= \frac{u(x_0+h\vec{V})-u(x_0)}{h}-\sin^2(\sigma)\, Du(x_0)\cdot\vec{V} \\
&+\cos(\sigma)\,\bar{e}_j\cdot Du(x_0)-\cos^2(\sigma)\, Du(x_0)\cdot\vec{V}-\sin(\sigma)\,\bar{e}_j\cdot x_0 \\
&+h\left[\begin{array}{c}\frac{\cos(\sigma)}{\sin(\sigma)}\frac{1}{2}\left|\bar{e}_j-\cos(\sigma)\,\vec{V}\right|^2-\sin(\sigma)\cos(\sigma)\,\frac{1}{2}\left|\vec{V}\right|^2 \\ -\sin(\sigma)\,\bar{e}_j\cdot\vec{V}+\sin(\sigma)\cos(\sigma)\left|\vec{V}\right|^2\end{array}\right] \\
&= Du(x^*)\cdot V-Du(x_0)\cdot\vec{V}+\cos(\sigma)\,\bar{e}_j\cdot Du(x)-\sin(\sigma)\,\bar{e}_j\cdot x_0 \\
&+h\left[\begin{array}{c}\frac{\cos(\sigma)}{\sin(\sigma)}\frac{1}{2}\left|\bar{e}_j-\cos(\sigma)\,\vec{V}\right|^2-\sin(\sigma)\cos(\sigma)\,\frac{1}{2}\left|\vec{V}\right|^2 \\ -\sin(\sigma)\,\bar{e}_j\cdot\vec{V}+\sin(\sigma)\cos(\sigma)\left|\vec{V}\right|^2\end{array}\right]
\end{aligned}
$$

where x^* is some value between $x_0+h\vec{V}$ and x_0 obtained by the mean value theorem. Now we may take a limit with h vanishing. Because $\vec{V}$ (which a priori can point in many directions) is bounded, the h-term vanishes in the limit. Because Du is continuous, and $x(\bar{x})$ is Lipschitz, we also have that

$$\lim_{h\to 0}\left|(Du(x^*)-Du(x_0))\cdot V\right|\leq\lim_{h\to 0}\sup\left|Du(x^*)-Du(x_0)\right|\left|V\right|=0.$$

We are left with

$$\lim_{h\to 0}\frac{\bar{u}(\bar{x}_0+h\bar{e}_j)-\bar{u}(\bar{x}_0)}{h}=\cos(\sigma)\, u_j(x_0)-\sin(\sigma)\, x_0^j. \tag{4.35}$$

This is precisely the $\bar{y}$-component of the image of the rotation (4.27). It follows that the gradient graph of $\bar{u}$ exists everywhere and is isometric to the gradient graph of u. □

Corollary 4.2.1. An analogous result holds when u is semi-concave, and σ is negative. The rotations through σ and $-\sigma$ are inverse operations where they are defined, up to an additive constant in the potential function.

Proof. While we could claim a proof that is formally the same as the proof of Proposition 4.2.1, we offer an alternative argument based on the fact that, whenever u is semi-concave, $-u$ must be semi-convex. Starting with a semi-convex $-u$, we may rotate the graph Γ_{-u} by a downward rotation through $-\sigma$, applying Proposition 4.2.1, and then take the complex conjugate of the result in $\mathbb{C}^n$. This follows from the fact that, as operators on $\mathbb{C}^n$ ($\mathbb{R}$-linear on $\mathbb{R}^{2n}$) for any diagonal unitary matrix U we have

$$c \circ U \circ c = U^{-1} = U^*$$

where c is the $\mathbb{R}$-linear complex conjugation map on $\mathbb{R}^{2n}$, that is

$$c(x + \sqrt{-1}y) = x - \sqrt{-1}y.$$

In particular, taking $-\overline{(-u)}$ via rotation of $-u$ (not complex conjugation), we obtain the potential $\bar{u}$ for the graph rotated through a negative angle $-\sigma$. □

The following technical result is useful when we approximate u while keeping K-convexity.

Lemma 4.2.1. Let u^ε be a standard mollification of u. If u is K-convex on Ω, then so is u^ε on

$$\Omega^\varepsilon = \{x : d(x, \partial\Omega) > \varepsilon\}. \tag{4.36}$$

Proof. Consider a mollifier ϕ that is radial, supported in $B_\varepsilon(0)$ and has unit integral. Given a point $x \in \Omega^\varepsilon$,

$$\begin{aligned} u^\varepsilon(x) &= \int_\Omega \phi(x-y)u(y)dy \\ &= \int_{B_\varepsilon(x)} \phi(x-y)u(y)dy \\ &= \int_{B_\varepsilon(0)} \phi(z)u(x+z)dz \end{aligned}$$

so we have

$$Du^\varepsilon(x) = \int_{B_\varepsilon(0)} \phi(z)Du(x+z)dz$$

Now consider, for $x_1, x_0 \in \Omega^\varepsilon$, the expression

$$\begin{aligned}
&\langle Du^\varepsilon(x_1) - Du^\varepsilon(x_0), x_1 - x_0\rangle \\
&= \left\langle \int_{B_\varepsilon(0)} \phi(z) Du(x_1+z)dz - \int_{B_\varepsilon(0)} \phi(z) Du(x_0+z)dz, x_1 - x_0 \right\rangle \\
&= \int_{B_\varepsilon(0)} \langle \phi(z)\left(Du(x_1+z) - Du(x_0+z)\right), x_1 - x_0\rangle \, dz \\
&= \int_{B_\varepsilon(0)} \phi(z) \langle Du(x_1+z) - Du(x_0+z), (x_1+z) - (x_0+z)\rangle \, dz \\
&\geq \int_{B_\varepsilon(0)} \phi(z) K \left|x_1 - x_0\right|^2 dz \\
&= K\left|x_1 - x_0\right|^2.
\end{aligned}$$

□

Proposition 4.2.2. Suppose that u is $\tan(\kappa)$-convex and C^1 and $\bar{u}$ is obtained as in Proposition 4.2.1. If $\kappa, \sigma, \kappa - \sigma \in (-\pi/2, \pi/2)$, then $\bar{u}$ is $\tan(\kappa-\sigma)$-convex.

Proof. We define the following functions

$$\begin{aligned}
\bar{x}_\varepsilon &= \cos(\sigma)x + \sin(\sigma)\, Du^\varepsilon(x) \\
\bar{y}_\varepsilon &= -\sin(\sigma)\, x + \cos(\sigma) Du^\varepsilon(x).
\end{aligned}$$

Note that, as before, the set

$$\bar{\Gamma}_\varepsilon = \{(\bar{x}_\varepsilon(x), \bar{y}_\varepsilon(x)) : x \in \Omega\}$$

is the rotation of the gradient graph of u^ε through angle σ. (To be clear, we are not taking the gradient graph of the mollified rotated function, rather we are rotating the gradient graph of the mollified function.)

Now Du is continuous, so the mollified derivatives Du^ε will converge locally uniformly to Du as $\varepsilon \to 0$ (cf. [**15**, Appendix C, Theorem 6]). It follows that the functions $\bar{x}_\varepsilon$ and $\bar{y}_\varepsilon$ will also converge locally uniformly, to $\bar{x}$ and $\bar{y}$ respectively, as functions of x, where

$$\begin{aligned}
\bar{x} &= \cos(\sigma)x + \sin(\sigma)\, Du(x) \\
\bar{y} &= -\sin(\sigma)\, x + \cos(\sigma) Du(x).
\end{aligned}$$

We have seen in Proposition 4.2.1 that

$$\bar{\Gamma} = \{(\bar{x}(x), \bar{y}(x)) : x \in \Omega\}$$

is precisely the gradient graph of the function $\bar{u}$ over $\bar{\Omega}$. The semi-convexity condition (4.28) on $\bar{u}$ that we are trying to show is

$$\langle \bar{y}(x_1) - \bar{y}(x_0), \bar{x}(x_1) - \bar{x}(x_0)\rangle \geq \tan(\kappa - \sigma)\left|\bar{x}(x_1) - \bar{x}(x_0)\right|^2.$$

We claim that

$$\langle \bar{y}_\varepsilon(x_1) - \bar{y}(x_0), \bar{x}_\varepsilon(x_1) - \bar{x}(x_0)\rangle \geq \tan(\kappa - \sigma)\left|\bar{x}_\varepsilon(x_1) - \bar{x}(x_0)\right|^2 \tag{4.37}$$

for all $\varepsilon > 0$. The local uniform convergence of $\bar{x}_\varepsilon$ and $\bar{y}_\varepsilon$ will then give us the result. To show (4.37), we start by computing the Jacobian of the map $\bar{x}_\varepsilon$:

Since u^ε is smooth

$$\frac{d\bar{x}_\varepsilon}{dx} = \cos(\sigma) I_n + \sin(\sigma)\, D^2 u^\varepsilon(x).$$

By assumption, u is $\tan(\kappa)$-convex, and hence so is u^{ε}, by Lemma 4.2.1, at least on Ω^{ε} (recall (4.36)). It follows that

$$D^2 u^{\varepsilon}(x) \geq \tan(\kappa)\, I_n.$$

So

$$\begin{aligned}\frac{d\bar{x}_{\varepsilon}}{dx} &\geq \cos(\sigma) I_n + \sin(\sigma)\tan(\kappa)\, I_n \\ &= \frac{\cos(\sigma-\kappa)}{\cos(\kappa)} I_n > 0\end{aligned}$$

since κ and $\sigma - k \in (-\pi/2, \pi/2)$. The coordinate change is invertible and the Jacobian can be computed

$$\frac{dx}{d\bar{x}_{\varepsilon}} = \left(\cos(\sigma) I_n + \sin(\sigma)\, D^2 u^{\varepsilon}(x)\right)^{-1}.$$

Next

$$D\bar{y}_{\varepsilon} = \left(-\sin(\sigma)\, I_n + \cos(\sigma) D^2 u^{\varepsilon}(x)\right).$$

Now each $\bar{\Gamma}_{\varepsilon}$ is the gradient graph of a function $\bar{u}_{\varepsilon}(\bar{x}_{\varepsilon})$ on the region $\bar{x}_{\varepsilon}(\Omega)$. In order to compute the Hessian of $\bar{u}_{\varepsilon}$ in terms of $\bar{x}_{\varepsilon}$, we compute

$$\begin{aligned}D^2_{\bar{x}_{\varepsilon}} \bar{u}_{\varepsilon} &= D_x \bar{y}_{\varepsilon} \cdot \frac{dx}{d\bar{x}_{\varepsilon}} = D_{\bar{x}_{\varepsilon}} \bar{y}_{\varepsilon} \\ &= \left(-\sin(\sigma)\, I_n + \cos(\sigma) D^2 u^{\varepsilon}(x)\right)\left(\cos(\sigma) I_n + \sin(\sigma)\, D^2 u^{\varepsilon}(x)\right)^{-1}.\end{aligned}$$

At any point, we may diagonalize the expression for $D^2_{\bar{x}^{\varepsilon}} \bar{u}_{\varepsilon}(\bar{x})$ by diagonalizing $D^2 u^{\varepsilon}(x(\bar{x}))$:

$$D^2_{\bar{x}^{\varepsilon}} \bar{u}_{\varepsilon} = \begin{pmatrix} \frac{-\sin(\sigma)+\cos(\sigma)\lambda_1}{\cos(\sigma)+\sin(\sigma)\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{-\sin(\sigma)+\cos(\sigma)\lambda_n}{\cos(\sigma)+\sin(\sigma)\lambda_n} \end{pmatrix} = \begin{pmatrix} \bar{\lambda}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \bar{\lambda}_n \end{pmatrix}.$$

Now

$$\bar{\lambda}_j = \frac{-\sin(\sigma) + \cos(\sigma)\,\lambda_j}{\cos(\sigma) + \sin(\sigma)\,\lambda_j} = \frac{-\frac{\sin(\sigma)}{\cos(\sigma)} + \lambda_j}{1 + \frac{\sin(\sigma)}{\cos(\sigma)}\lambda_j} = \tan(-\sigma + \arctan(\lambda_j)).$$

Because

$$\arctan(\lambda_j) \geq \kappa$$

we conclude that

$$\bar{\lambda}_j \geq \tan(-\sigma + \kappa)$$

and $D^2_{\bar{x}^{\varepsilon}} \bar{u}_{\varepsilon}$ is $\tan(-\sigma+\kappa)$-convex, that is

$$\langle D_{\bar{x}_{\varepsilon}} \bar{u}_{\varepsilon}(x_1) - D_{\bar{x}_{\varepsilon}} \bar{u}_{\varepsilon}(x_0), \bar{x}_{\varepsilon}(x_1) - \bar{x}_{\varepsilon}(x_0)\rangle \geq \tan(-\sigma+\kappa)\, |\bar{x}_{\varepsilon}(x_1) - \bar{x}_{\varepsilon}(x_0)|^2 \tag{4.38}$$

or

$$\langle \bar{y}_{\varepsilon}(x_1) - \bar{y}_{\varepsilon}(x_0), \bar{x}_{\varepsilon}(x_1) - \bar{x}_{\varepsilon}(x_0)\rangle \geq \tan(-\sigma+\kappa)\, |\bar{x}_{\varepsilon}(x_1) - \bar{x}_{\varepsilon}(x_0)|^2 \tag{4.39}$$

provided that x_1 and x_0 are at least ε away from the boundary of Ω. By the local uniform convergence, we conclude that

$$\langle \bar{y}(x_1) - \bar{y}(x_0), \bar{x}(x_1) - \bar{x}(x_1)\rangle \geq \tan(-\sigma+\kappa)\, |\bar{x}(x_1) - \bar{x}(x_1)|^2 \tag{4.40}$$

that is, $\bar{u}$ is $\tan(\kappa - \sigma)$-convex. $\square$

The following is an observation on how semi-convexity can lead to bounded geometry, even when the potential is not given as being twice differentiable.

COROLLARY 4.2.2. Suppose that $u \in C^1$ and is semi-convex. Then the gradient graph of u is isometric to the gradient graph of a $C^{1,1}$ function.

PROOF. Choose $\sigma \in (0, \pi/2)$ and $\varepsilon > 0$ for which (4.29) is satisfied. Now to control the $C^{1,1}$ norm of $\bar{u}$ we note that

$$\begin{aligned}\|\bar{u}\|_{C^{1,1}(\bar{\Omega})} &= \sup_{\bar{x}_0, \bar{x}_1 \in \bar{\Omega}} \frac{|D\bar{u}(\bar{x}_1) - D\bar{u}(\bar{x}_0)|}{|\bar{x}_1 - \bar{x}_0|} \\ &= \sup_{x_0, x_1 \in \Omega} \frac{|\bar{y}(x_1) - \bar{y}(x_0)|}{|\bar{x}(x_1) - \bar{x}(x_0)|}.\end{aligned}$$

So for any pair $x_0, x_1 \in \Omega$

$$\begin{aligned}\frac{|\bar{y}(x_1) - \bar{y}(x_0)|}{|\bar{x}(x_1) - \bar{x}(x_0)|} &= \frac{|\cos(\sigma)\, Du(x_1) - \sin(\sigma)\, x_1 - \cos(\sigma)\, Du(x_0) + \sin(\sigma)\, x_0|}{|\cos(\sigma)\, x_1 + \sin(\sigma)\, Du(x_1) - \cos(\sigma)\, x_0 + \sin(\sigma)\, Du(x_0)|} \\ &= \frac{|\cos(\sigma)\,(Du(x_1) - Du(x_0)) - \sin(\sigma)\,(x_1 - x_0)|}{|\cos(\sigma)\,(x_1 - x_0) + \sin(\sigma)\,(Du(x_1) - Du(x_0))|}.\end{aligned}$$

To show this is bounded, we explore two cases. Let $A = 2\cot(\sigma) > 0$. The first case is when

$$|Du(x_1) - Du(x_0)| \leq A\,|x_1 - x_0|\,. \tag{4.41}$$

Recall $\sigma \in (0, \pi/2)$, we have

$$\frac{|\cos(\sigma)\,(Du(x_1) - Du(x_0)) - \sin(\sigma)\,(x_1 - x_0)|}{|\cos(\sigma)\,(x_1 - x_0) + \sin(\sigma)\,(Du(x_1) - Du(x_0))|} \leq \frac{|\cos(\sigma)\, A\,|x_1 - x_0| + \sin(\sigma)\,|x_1 - x_0||}{|\cos(\sigma)\,(x_1 - x_0) + \sin(\sigma)\,(Du(x_1) - Du(x_0))|}$$

and

$$\begin{aligned}&\left\langle \cos(\sigma)\,(x_1 - x_0) + \sin(\sigma)\,(Du(x_1) - Du(x_0))\,, \frac{x_1 - x_0}{|x_1 - x_0|} \right\rangle \\ &= \cos(\sigma)\,|x_1 - x_0| + \left\langle \sin(\sigma)\,(Du(x_1) - Du(x_0))\,, \frac{x_1 - x_0}{|x_1 - x_0|} \right\rangle \\ &\geq \cos(\sigma)\,|x_1 - x_0| + \sin(\sigma)\,|x_1 - x_0|\,(-\cot(\sigma) + \varepsilon) \\ &= \sin(\sigma)\,|x_1 - x_0|\,\varepsilon\end{aligned}$$

where we used (4.31) in the second line. Thus (4.41) leads to

$$\frac{|\bar{y}(x_1) - \bar{y}(x_0)|}{|\bar{x}(x_1) - \bar{x}(x_0)|} \leq \left| \frac{\cos(\sigma)\, A + \sin(\sigma)}{\sin(\sigma)\,\varepsilon} \right| = \frac{\cos^2(\sigma) + 1}{\sin^2(\sigma)} \frac{1}{\varepsilon}.$$

The next case is when

$$|Du(x_1) - Du(x_0)| \geq A\,|x_1 - x_0|\,. \tag{4.42}$$

Then by the triangle inequality and (4.42)

$$\begin{aligned}|\cos(\sigma)\,(x_1 - x_0) + \sin(\sigma)\,(Du(x_1) - Du(x_0))| &\geq \sin(\sigma)|Du(x_1) - Du(x_0)| - \cos(\sigma)|x_1 - x_0| \\ &\geq \left(\sin(\sigma) - \frac{\cos(\sigma)}{A}\right) |Du(x_1) - Du(x_0)| \\ &= \frac{1}{2}\sin(\sigma)\,|Du(x_1) - Du(x_0)|\end{aligned}$$

and

$$\begin{aligned}
&\frac{\left|\cos\left(\sigma\right)\left(Du(x_1)-Du(x_0)\right)-\sin\left(\sigma\right)\left(x_1-x_0\right)\right|}{\left|\cos\left(\sigma\right)\left(x_1-x_0\right)+\sin\left(\sigma\right)\left(Du(x_1)-Du(x_0)\right)\right|}\\
&\quad\le\frac{\cos\left(\sigma\right)\left|Du(x_1)-Du(x_0)\right|+\sin\left(\sigma\right)\frac{\left|Du(x_1)-Du(x_0)\right|}{A}}{\frac{1}{2}\sin\left(\sigma\right)\left|Du(x_1)-Du(x_0)\right|}\\
&\quad=\frac{\cos^2\left(\sigma\right)+1}{\sin\left(\sigma\right)\cos\left(\sigma\right)}.
\end{aligned}$$

In either case, we have

$$\frac{\left|\bar{y}(x_1)-\bar{y}(x_0)\right|}{\left|\bar{x}(x_1)-\bar{x}(x_0)\right|}\le\max\left\{\frac{\cos^2\left(\sigma\right)+1}{\sin^2\left(\sigma\right)}\frac{1}{\varepsilon},\frac{\cos^2\left(\sigma\right)+1}{\sin\left(\sigma\right)\cos\left(\sigma\right)}\right\}=C$$

and $\bar{u}$ is $C^{1,1}$. □

The following corollary is immediate from the above by applying the De Giorgi-Nash theorem.

Corollary 4.2.3. Suppose that $u\in C^1$ is a semi-convex weak solution to (4.7). Then the phase θ enjoys interior Hölder estimates (with respect to the metric distances) on Γ_u.

Finally, we show that smoothness and strong semi-concavity estimates on the rotated potential can be used to conclude smoothness on u.

Proposition 4.2.3. Suppose that u and $\bar{u}$ are as in Proposition 4.2.1 and $\bar{u}\in C^2\left(\bar{\Omega}\right)$. Suppose also that for some constant $\epsilon>0$

$$D^2_{\bar{x}}\bar{u}\le\left(\frac{\cos\left(\sigma\right)}{\sin\left(\sigma\right)}-\epsilon\right)I_n. \tag{4.43}$$

Then for any integer $k>1$

$$\left\|D^k u\right\|_{L^\infty(\Omega)}\le C\left(\sigma,\epsilon,n\right)\left(\left\|D^k\bar{u}\right\|_{L^\infty(\bar{\Omega})},\left\|D^{k-1}u\right\|_{L^\infty(\Omega)}\right).$$

Proof. The function $\bar{u}$ was obtained by a downward rotation of σ from u, so u may be obtained by the inverse rotation. In particular as $\bar{u}\in C^2\left(\bar{\Omega}\right)$, the change of variable formulae hold on $\bar{\Omega}$:

$$\begin{aligned}
x&=\cos(\sigma)\bar{x}-\sin(\sigma)D_{\bar{x}}\bar{u}(\bar{x})\\
y&=\sin(\sigma)\bar{x}+\cos(\sigma)D_{\bar{x}}\bar{u}(\bar{x}).
\end{aligned}$$

Differentiating the first formula leads to

$$\frac{dx}{d\bar{x}}=\cos\left(\sigma\right)I_n-\sin\left(\sigma\right)D^2_{\bar{x}}\bar{u}(\bar{x})$$

and noting that

$$y=D_x u(x)=D_x u(x(\bar{x}))$$

we have

$$D_x u(\bar{x})=\sin(\sigma)\bar{x}+\cos(\sigma)D_{\bar{x}}\bar{u}(\bar{x}).$$

Now

$$\begin{aligned} D_x^2 u &= D_x D_x u \\ &= D_x \left(\sin(\sigma)\bar{x} + \cos(\sigma) D_{\bar{x}} \bar{u}(\bar{x})\right) \\ &= \left(\sin(\sigma) I_n + \cos(\sigma) D_{\bar{x}}^2 \bar{u}(\bar{x})\right) \frac{d\bar{x}}{dx}. \end{aligned}$$

Noting (4.43), we may invert (4.2.1) and conclude

$$\begin{aligned} D_x^2 u\left(\bar{x}\right) &= \left(\sin(\sigma) I + \cos(\sigma) D_{\bar{x}}^2 \bar{u}(\bar{x})\right) \cdot \left(\cos\left(\sigma\right) I_n - \sin\left(\sigma\right) D_{\bar{x}}^2 \bar{u}(\bar{x})\right)^{-1} \\ &:= F_\sigma(D_{\bar{x}}^2 \bar{u}(\bar{x}(x))). \end{aligned} \tag{4.44}$$

First, we shall show that if $D_{\bar{x}}^3 \bar{u}$ exists, then so will $D_x^3 u(x)$. To do this we differentiate (4.44) in x, obtaining

$$\begin{aligned} D_x D_x^2 u(x) &= D_x F_\sigma(D_{\bar{x}}^2 \bar{u}(\bar{x}(x))) \\ &= \frac{dF_\sigma}{dD_{\bar{x}}^2 \bar{u}} \cdot \frac{dD_{\bar{x}}^2 \bar{u}}{d\bar{x}} \cdot \frac{d\bar{x}}{dx}. \end{aligned}$$

Combining (4.43), the assumption that $D_{\bar{x}}^3 \bar{u}$ exists, and the fact that all of these factors are well-defined and bounded, we conclude that $D_x^3 u$ exists and is controlled in terms of $D_{\bar{x}}^3 \bar{u}$.

Higher order estimates follow in the same way inductively. $\square$

4.3. Proof of Theorem 4.1.2

Proof. We are assuming that the function θ is a weak solution to a divergence type equation (4.7) on the set $\mathbb{B}_1(0)\backslash Q$. Because the conditions (4.13), (4.14) and (4.15) each guarantee uniform ellipticity of the Laplace equation, we may immediately apply Theorem 4.1 and conclude that θ is a weak solution over the whole ball $\mathbb{B}_1(0)$.

Recall that

$$F(D^2 u) = F(\lambda_1, \cdots, \lambda_n) = \sum_{i=1}^n \arctan \lambda_i.$$

To begin, we claim that if either of the conditions (4.13) or (4.14) holds, then for u satisfying

$$F(D^2 u) = \theta$$

it follows that u is a solution to a concave equation.

For the case $\theta \geq \delta + \frac{\pi}{2}(n-2)$, we recall that by [**38**, Lemma 2.1] (see also [**?**, section 8]) the level sets of F, at any level c with $|c| \geq \frac{\pi}{2}(n-2)$, are convex. We have a uniform bound $\left|D^2 u\right| \leq C_0$ wherever the Hessian exists, so we may find a compact set $\mathcal{K} \subset S^{n\times n}$, where $S^{n\times n}$ is the space of symmetric $n \times n$ real matrices, such that

$$\begin{aligned} &D^2 u(\mathbb{B}_1(0)) \subset \mathcal{K} \\ &F(M) > \frac{\delta}{2} + \frac{\pi}{2}(n-2) \text{ for all } M \in \mathcal{K}. \end{aligned}$$

We may smoothly modify F on $\mathcal{K}$,

$$\tilde{F} = f(F)$$

so that $\tilde{F}$ is a uniformly concave function and has the same level sets as F on $\mathcal{K}$. (For a recent detailed proof of this fact, see [**?**, Lemma 2.2] .) In this case

$$\tilde{F}(D^2u) = \tilde{\theta}$$

for some smoothly modified $\tilde{\theta}$, constructed from f such that

$$\left\|\tilde{\theta}\right\|_{C^\alpha} \leq C\left\|\theta\right\|_{C^\alpha}.$$

For the second case, (4.14), u is uniformly convex, and the function F is clearly concave in the eigenvalues. So by taking $\tilde{F} = F$ (see [**?**, section 3]) we already have that

$$\tilde{F}(D^2u) = \theta$$

for some concave $\tilde{F}$. Again, because $\left|D^2u\right| \leq C_0$ where it exists, we can find a compact set $\mathcal{K}$ (still using the same notation as above for simplicity) such that $D^2u(\mathbb{B}_1(0)) \subset \mathcal{K}$ and F is uniformly concave on $\mathcal{K}$.

In either case, (4.13) or (4.14), we may extend $\tilde{F}$ beyond $\mathcal{K}$ to a global function $\bar{F}$ on $S^{n\times n}$ to obtain a uniformly elliptic $\bar{F}$, satisfying $\bar{F}(M) = \tilde{F}(M)$ for $M \in \mathcal{K}$, $\bar{F}$ is uniformly elliptic, $\bar{F}$ is concave, and $\bar{F}$ is continuous on $S^{n\times n}$ and still smooth on the interior of $\mathcal{K}$. (For example, see [**34**, Lemma 2.2].)

Now we apply [**?**, Theorem 8.1 and Remarks 1 and 3 following, see also Remark 1 in 6.2], which is Schauder theory for uniformly elliptic concave equations. Note that [**?**, p. 54] only requires the function $\bar{F}$ to be concave and continuous. First note that by De Giorgi-Nash, when $u \in C^{1,1}$ the equation (4.7) is uniformly elliptic, so the function θ enjoys Hölder estimates. Thus we also have Hölder estimates on the modified $\tilde{\theta}$. Now our definition of weak solution is that $F(D^2u) = \theta$, almost everywhere, so also, $\bar{F}(D^2u) = \tilde{\theta}$ almost everywhere, and we may apply [**?**, Corollary 3] to conclude that u is also a viscosity solution to $\bar{F}(D^2u) = \tilde{\theta}$. Because the modification of F was either smooth or away from a compact set containing the image of D^2u, we still have

$$\left\|\bar{F}(D^2u)\right\|_{C^\alpha\left(\mathbb{B}_{4/5}(0)\right)} \leq C_1$$

for some C_1 depending on the ellipticity constants obtained in our application of De Giorgi-Nash, noting that $\left\|\theta\right\|_{L^\infty} \leq n\pi/2$. We conclude from [**?**] that

$$\left\|D^2u\right\|_{C^\alpha\left(\mathbb{B}_{3/4}(0)\right)} \leq C_2$$

for C_2 depending on the ellipticity constants, C_1, and the oscillation of u.

Now with interior $C^{2,\alpha}$ estimates in hand, we return to θ, which is a solution to a divergence type equation with C^α coefficients, so we may apply [**19**, Theorem 3.13] to conclude that

$$\left\|\theta\right\|_{C^{1,\alpha}\left(\mathbb{B}_{2/3}(0)\right)} \leq C_3.$$

Now for e_k , consider the function

$$\theta^{(h_k)}(x) = \frac{\theta(x + he_k\ ) - \theta(x)}{h}$$

defined on some interior region, for small $h > 0$. Because $\theta \in C^{1,\alpha}\left(\mathbb{B}_{2/3}(0)\right)$ we have

$$\left\|\theta^{(h_k)}\right\|_{C^\alpha\left(\mathbb{B}_{2/3-h}(0)\right)} \leq C_3.$$

Now

$$\begin{aligned}
\theta^{(h_k)}(x) &= \frac{1}{h}\int_0^1 \frac{d}{dt}F(D^2u(x+he_k\ )t+(1-t)D^2u(x))dt \\
&= \frac{1}{h}\int_0^1 g^{ij}\left(D^2u(x+he_k\ )t+(1-t)D^2u(x)\right)\left(u(x+he_k\ )_{ij}-u_{ij}(x)\right)dt \\
&= \int_0^1 g^{ij}\left(D^2u(x+he_k\ )t+(1-t)D^2u(x)\right)\left(\frac{u(x+he_k\ )_{ij}-u_{ij}(x)}{h}\right)dt \\
&= G^{ij}u_{ij}^{(h_k)}(x) \\
&:= Lu^{(h_k)}(x)
\end{aligned}$$

for some uniformly elliptic $L = G^{ij}\partial_i\partial_j$ which is an average of elliptic operators with C^α coefficients, where

$$u^{(h_k)}(x) = \frac{u(x+he_k)-u(x)}{h}.$$

Thus, each $u^{(h_k)}$ satisfies an uniformly elliptic equation of non-divergence type, that is

$$Lu^{(h_k)} = \theta^{(h_k)} \in C^\alpha\left(\mathbb{B}_{2/3-h}(0)\right)$$

with Hölder estimate uniform in h. Noting that each $u^{(h_k)} \in C^{2,\alpha}$ we may apply the non-divergence Schauder theory [**17**, Theorem 6.6] to conclude a uniform $C^{2,\alpha}$ estimate as $h \to 0$. Thus, for each $u_k = \lim_{h\to 0} u^{(h_k)}$, where $k \in 1, ..., n$, we have

$$\|u_k\|_{C^{2,\alpha}\left(\mathbb{B}_{1/2}(0)\right)} \leq C_4$$

that is

$$\begin{aligned}
u &\in C^{3,\alpha}\left(\mathbb{B}_{1/2}(0)\right) \\
g &\in C^{1,\alpha}\left(\mathbb{B}_{1/2}(0)\right)
\end{aligned}$$

with estimates.

Now from $\Delta_g\theta = 0$ we get

$$\sqrt{g}g^{ij}\theta_{ij} = -\partial_i\left(\sqrt{g}g^{ij}\right)\theta_i \in C^\alpha\left(\mathbb{B}_{1/2}(0)\right)$$

thus θ satisfies a non-divergence equation with Hölder continuous right hand side f. By Schauder theory [**17**, Theorem 6.13], θ must be $C^{2,\alpha}$. (More precisely, θ is the unique viscosity solution to the Dirichlet problem $\sqrt{g}g^{ij}\varphi_{ij} = f$ on $\mathbb{B}_{1/2}(0)$ and $\varphi = \theta$ on $\partial\mathbb{B}_{1/2}(0)$.) Iterating the previous two steps, we may obtain all higher order estimates for any region further in the interior.

Next we assume that (4.15) holds. Suppose that a function u satisfies (4.15). Let

$$\kappa = \arctan(1-\delta) < \frac{\pi}{4}.$$

Condition (4.15) gives us that u is $-\tan(\kappa)$-convex. Perform a downward rotation of the graph of u with $\sigma = \frac{\pi}{4}$. Proposition 4.2.1 implies that the corresponding coordinate change $\bar{x}(x)$ defined by (4.30) is bi-Lipschitz. It will follow that any interior region of $\bar{\Omega}^\varepsilon$ (recall (4.36)) will be the homeomorphic image of an interior region Ω' with

$$\Omega^{\varepsilon_2} \subset \Omega' \subset \Omega^{\varepsilon_1}$$

with $\varepsilon_1/\varepsilon$ and $\varepsilon_2/\varepsilon$ bounded above and away from 0. It follows that interior estimates for $\bar{u}$ on $\bar{\Omega}$ will correspond to interior estimates for u on Ω.

Now by Proposition 4.2.2, $\bar{u}$ is β_0-convex for

$$\beta_0 = \tan\left(\arctan(\delta - 1) - \frac{\pi}{4}\right) = \frac{\delta - 2}{\delta}.$$

Now letting $v = -u$, we may also rotate *upward* by $\sigma = \frac{\pi}{4}$, to obtain a function $\bar{v}$ that is β_1-convex for

$$\beta_1 = \tan\left(\arctan(\delta - 1) + \frac{\pi}{4}\right) = \frac{\delta}{2 - \delta}$$

by Proposition 4.2.2. From the discussion in the proof of Corollary 4.2.1, we have that $\bar{v} = -\bar{u}$. In particular, $-\bar{u}$ is $C^{1,1}$, uniformly convex, and clearly is also a weak solution of (4.7), as the quantity θ is odd in D^2u. We are then back to the case (4.14) , and may conclude interior estimates on the derivatives of $-\bar{u}$ for any order, and hence also for derivatives of $\bar{u}$. Now certainly (4.43) holds for $\epsilon = 1$, so we may apply Proposition 4.2.3 and get interior derivative estimates on u. □

4.3.1. Proof of Theorem 4.1.3.

PROOF. Let u be a $W^{2,n}(\Omega)$ solution to (4.6). Let $\Gamma_u = \{(x, Du(x)) : x \in \Omega\}$. First note that the Grassmannian geometry (in particular, the distance function) is invariant under unitary actions on $\mathbb{C}^n$. Observe also that for small enough $c_0(n)$, all Lagrangian planes within distance $c_0(n)$ from each other must be graphical over each other. Thus at any point p where D^2u exists, the tangent space to Γ is well-defined, and we can locally take Γ to be a graph over T_pL. By taking a unitary map sending $T_p\Gamma$ to $\mathbb{R}^n \times \{0\}$, we may express the isometric image $\bar{\Gamma}$ locally as a gradient graph of some function $\bar{u}$ over a region $\bar{\Omega} \subset \mathbb{R}^n$, with $D^2\bar{u}(p) = 0$. For Lagrangian tangent planes near $\mathbb{R}^n \times \{0\}$, the topology on the Lagrangian Grassmannian is equivalent to the topology on Hessian space, so by choosing $c_0(n)$ small we have also guaranteed that

$$\|u\|_{C^{1,1}(\Omega)} \leq c(n) < 1$$

where $c(n)$ is from Theorem 4.1.1. Applying Theorem 4.1.1, we may conclude that u is a weak solution to (4.7). By Theorem 4.1.2, $\bar{u}$ is smooth inside $\bar{\Omega}$. So $\bar{\Gamma}$ is the gradient graph of a smooth function over $\bar{\Omega}$, hence it is a smooth submanifold of $\mathbb{R}^{2n}$. □

Our result allows for the Hessian of the potential function u to be just continuous or even have mild discontinuities provided that $\|u\|_{C^{1,1}} \leq c(n)$. The following result is obtained by Schoen and Wolfson [**?**, Proposition 4.6], for Lagrangian stationary surfaces (when the potential functions are locally in $C^{2,\alpha}$) in general Kählerian ambient manifolds.

COROLLARY 4.3.1. Suppose that $u \in C^2$ is a weak solution to (4.6). Then u is smooth.

PROOF. Let $\Gamma = \{(x, Du(x)) : x \in \Omega\}$. Near any point $x_0 \in \Gamma$, we may write Γ locally as as gradient graph of a function v over its tangent plane $T_{x_0}\Gamma$. Necessarily, this choice gives us $D^2v(0) = 0$. Now v is also stationary for compactly supported variations near x_0, so v must satisfy (4.6) as well. Because $D^2u \in C^0$, the tangent planes change continuously. It follows that also $D^2v \in C^0$, and because we have chosen $D^2v(0) = 0$, we may find a small neighborhood for which

$$\left\|D^2v\right\|_{C^0} \leq c(n).$$

Applying Theorem 4.1.3, v is smooth near x. It follows that Γ is smooth near x. Now because D^2u was bounded, we may project the smooth object Γ back to the original coordinates Ω, and the Jacobian does not vanish. Thus we conclude that u is a smooth function on Ω. $\square$

4.4. Removable singularities

4.4.1. Graphical case. To extend solutions across a small set in Theorem 4.1.1. we shall need the following theorem of Serrin (Theorem 2 in [**?**]).

THEOREM 4.1. *(Serrin) Suppose $n \geq 2$ and that f is a bounded continuous weak solution to a uniformly elliptic second order divergence equation with bounded measurable coefficients on $\Omega - Q$, for an open domain Ω and Q a compact subset. If Q has capacity zero, then f may be extended to a weak solution across the domain Ω.*

4.4.2. Non-graphical case. The following volume upper estimate is a direct consequence of the standard monotonicity formula for volumes. It will be used in the proof of Theorem 1.1.

PROPOSITION 4.4.1. Let L be an integral n-rectifiable varifold in $\mathbb{R}^{n+l}$, with generalized mean curvature $\mathcal{H}$ locally in $L^n(L,\mu)$ where μ is the Radon measure associated with L. Given any $x \in \mathbb{R}^{n+l}$ and any fixed $\rho_0 > 0$, there exists a C such that

$$\mu(B_\rho(x)) \leq C\left(|\ln \rho| + 1\right)^n \rho^n \tag{4.45}$$

for all $0 < \rho < \rho_0$ with C depending on ρ_0, $\mu(B_{\rho_0}(x))$ and the L^n norm of $\mathcal{H}$ over $B_{\rho_0}(x)$.

In particular if $\mathcal{H}$ is $L^n(L,\mu)$, μ is finite, and $n \geq 2$, then for any $0 \leq k \leq n-2$ it holds for small ρ

$$\mu(B_\rho(x)) \leq C\rho^{k+\frac{n}{n-1}} \tag{4.46}$$

for a constant C not depending on x.

PROOF. Recall the monotonicity formula [**30**, 17.3 p. 84]

$$\begin{aligned} \frac{d}{d\rho}\left(\rho^{-n}\mu(B_\rho(x))\right) &= \frac{d}{d\rho}\int_{B_\rho(x)} \frac{|\mathrm{D}^\perp r|^2}{r^n}\mathrm{d}\mu + \rho^{-1-n}\int_{B_\rho(x)} \langle y-x, \mathcal{H}\rangle \mathrm{d}\mu \\ &\geq \rho^{-1-n}\int_{B_\rho(x)} \langle y-x, \mathcal{H}\rangle \mathrm{d}\mu \\ &\geq -\rho^{-1-n}\int_{B_\rho(x)} \rho\,|\mathcal{H}|\,\mathrm{d}\mu \\ &\geq -\rho^{-n}\left(\int_{B_\rho(x)} |\mathcal{H}|^n\,\mathrm{d}\mu\right)^{1/n} \mu(B_r(x))^{\frac{n-1}{n}}. \end{aligned} \tag{4.47}$$

Now let

$$w(\rho) = \frac{\mu(B_\rho(x))^{1/n}}{\rho}$$

in which case we have

$$\frac{d}{d\rho}\left[w(\rho)\right]^n \geq -\frac{1}{\rho}\left(\int_{B_\rho(x)} |\mathcal{H}|^n\,\mathrm{d}\mu\right)^{1/n} w^{n-1}$$

and

$$nw^{n-1}\frac{d}{d\rho}w \geq -\frac{1}{\rho}\left(\int_{B_\rho(x)}|\mathcal{H}|^n\,\mathrm{d}\mu\right)^{1/n} w^{n-1}$$

$$\frac{d}{d\rho}w \geq -\frac{1}{\rho n}\left(\int_{B_\rho(x)}|\mathcal{H}|^n\,\mathrm{d}\mu\right)^{1/n}.$$

Integrating over (ρ,ρ_0),

$$w(\rho_0)-w(\rho) \geq \left(\int_{B_\rho(x)}|\mathcal{H}|^n\,\mathrm{d}\mu\right)^{1/n}\frac{1}{n}\left[\ln\rho-\ln\rho_0\right]$$

that is

$$w(\rho) \leq w(\rho_0)+\left(\int_{B_\rho(x)}|\mathcal{H}|^n\,\mathrm{d}\mu\right)^{1/n}\frac{1}{n}\left(-\ln\rho+\ln\rho_0\right)$$

or

$$\frac{\mu(B_\rho(x))}{\rho^n} \leq \left\{\mu(B_{\rho_0}(x))+\left(\int_{B_\rho(x)}|\mathcal{H}|^n\,\mathrm{d}\mu\right)^{1/n}\frac{1}{n}\left(|\ln\rho|+\ln\rho_0\right)\right\}^n$$

and finally

$$\mu(B_\rho(x)) \leq \rho^n\left\{\mu(B_{\rho_0}(x))+\left(\int_{B_\rho(x)}|\mathcal{H}|^n\,\mathrm{d}\mu\right)^{1/n}\frac{1}{n}\left(|\ln\rho|+\ln\rho_0\right)\right\}^n.$$

The estimate (4.45) is immediate. To see (4.46), we have

$$\rho^{-k-\frac{n}{n-1}}\mu(B_\rho(x)) \leq \rho^{\frac{n(n-2)}{n-1}-k}\left\{\mu(B_{\rho_0}(x))+\left(\int_{B_\rho(x)}|\mathcal{H}|^n\,\mathrm{d}\mu\right)^{1/n}\frac{1}{n}\left(|\ln\rho|+\ln\rho_0\right)\right\}^n$$

and the term on the right hand side tends to zero as $\rho\to 0$ when $n>2$, as $k\leq n-2$ by assumption; however, when $n=2$, this term becomes unbounded.

For $n=2$, k must be 0, and the desired result follows from [**?**, (A.6)] (cf. [**?**]): for any $0<\rho<\rho_0$,

$$\rho^{-2}\mu(B_\rho(x)) \leq C\rho_0^{-2}\mu(B_{\rho_0}(x))+C\int_{B_{\rho_0}(x)}|\mathcal{H}|^2\mathrm{d}\mu<\infty.$$

□

REMARK 4.2. Unlike the standard monotonicity formula for rectifiable varifolds with $H\in L^p, p>n$ (in particular for stationary ones), our assumption $H\in L^n$ yields a weaker conclusion as the domination of volume ratio involves a logarithmic term, instead of a pure constant.

THEOREM 4.3. *Let $N=\cup_{\alpha=1}^{\alpha_0}N_\alpha$ be a finite union of compact sets N_α in a domain $\Omega\subset\mathbb{C}^n$ where each N_α has finite k_α-dimensional Hausdorff measure with $k_\alpha\leq n-2$ and satisfies the local k_α-noncollapsing property*

$$\inf_{x\in N_\alpha}\mathcal{H}^{k_\alpha}(N_\alpha\cap B_\varepsilon(x))\geq C_3\varepsilon^{k_\alpha} \tag{4.48}$$

for all $\varepsilon\in(0,\delta)$ for some δ and a constant $C_3>0$ independent of ε. Let L be an immersed Lagrangian submanifold in $\Omega\backslash N$ with $\overline{L}\backslash L\subseteq N$ such that (L,μ_L) is

Hamiltonian stationary in $\Omega\backslash N$, where $\mu_L = \mathcal{H}^n \llcorner \beta$ is the measure on L and β is an $\mathbb{N}$-valued $\mathcal{H}^n$-integrable function on L. Assume

(i) $\displaystyle\int_\Omega |H|^n \, d\mu_L < C_1$, *where H is the generalized mean curvature vector of (L, μ_L) in $\Omega\backslash N$;*

(ii) *There exists a positive constant C_4 such that for any open set $E \subseteq L$*
$$\mu_L(E) \leq C_4 \mathcal{H}^n(E);$$

(iii) *For any $x \in N$, (4.46) holds over $B_\rho(x)$ for all $\rho \leq \rho_0$ where ρ_0 is a constant independent of x.*

Then the closure $\overline{L}$ of L is Hamiltonian stationary in Ω: $\overline{L}$ admits a generalized mean curvature $\mathcal{H}$ in Ω such that for any $f \in C_0^\infty(\Omega)$ it holds
$$\int_\Omega \langle JDf, \mathcal{H} \rangle \; d\mu_{\overline{L}} = 0.$$

Proof. Define the ε-neighborhood of the compact set N_α by
$$U_\varepsilon^\alpha = \{x \in \mathbb{R}^{2n} : \min_{y \in N_\alpha} |x - y| < \varepsilon\}.$$
Then
$$U_\varepsilon = \bigcup_{\alpha=1}^{\alpha_0} U_\varepsilon^\alpha$$
is the ε-neighborhood of N. Since N is compact, we may assume U_ε is contained in the open domain Ω by choosing ε small. For simplicity of notations, we will assume (4.46) holds for $3\varepsilon_i$'s, where $\{\varepsilon_i\}$ will be a sequence of radii descending towards 0.

Step 1. *Volume estimate of $L \cap U_{\varepsilon_j}$.*

For any fixed large j, let $\{B_{\varepsilon_j}(x_1^\alpha), ..., B_{\varepsilon_j}(x_{\ell(\varepsilon_j)}^\alpha)\}$ be the maximal family of disjoint balls in $\Omega \subset \mathbb{R}^{2n}$ centered at $x_i^\alpha \in N_\alpha$ of radius ε_j. Compactness of N_α ensures the number $\ell_\alpha(\varepsilon_j)$ well defined. The maximality assumption then implies
$$N_\alpha \subseteq \bigcup_{i=1}^{\ell_\alpha(\varepsilon_j)} B_{2\varepsilon_j}(x_i^\alpha).$$
To estimate $\ell_\alpha(\varepsilon_j)$, summing the k_α-dimensional Hausdorff measures over the disjoint balls and using the local k_α-noncollapsing assumption (4.48), we have
$$\ell_\alpha(\varepsilon_j) C_3 \varepsilon_j^{k_\alpha} \leq \sum_{i=1}^{\ell_\alpha(\varepsilon_j)} \mathcal{H}^{k_\alpha}(N_\alpha \cap B_{\varepsilon_j}(x_i^\alpha)) \leq \mathcal{H}^{k_\alpha}(N_\alpha)$$
Therefore
$$\ell_\alpha(\varepsilon_j) \leq \frac{\mathcal{H}^{k_\alpha}(N_\alpha)}{C_3 \, \varepsilon_j^{k_\alpha}}.$$

Next, we claim
$$U_{\varepsilon_j}^\alpha \subset \bigcup_{i=1}^{\ell_\alpha(\varepsilon_j)} B_{3\varepsilon_j}(x_i^\alpha).$$

This can be seen from that for any point $p \in U^\alpha_{\varepsilon_j}$ there is a point $q \in N_\alpha$ with $|p-q| \leq \varepsilon_j$ and $q \in B_{2\varepsilon_j}(x_i^\alpha)$ for some i, and it follows $p \in B_{3\varepsilon_j}(x_i^\alpha)$. Now by the assumptions (ii) and (iii),

$$\begin{aligned}\int_{U_{\varepsilon_j}} \mathrm{d}\mu_L &\leq \sum_{\alpha=1}^{\alpha_0} \int_{U^\alpha_{\varepsilon_j}} \mathrm{d}\mu_L \\ &\leq \sum_{\alpha=1}^{\alpha_0} \sum_{i=1}^{\ell_\alpha(\varepsilon_j)} \int_{B_{3\varepsilon_j}(x_i^\alpha)} \mathrm{d}\mu_L \\ &\leq \sum_{\alpha=1}^{\alpha_0} \ell_\alpha(\varepsilon_j)\, C_4 C_2\, (3\varepsilon_j)^{k_\alpha + \frac{n}{n-1}} \\ &\leq \sum_{\alpha=1}^{\alpha_0} \frac{\mathcal{H}^{k_\alpha}(N_\alpha)}{C_3}\, C_4 C_2\, 3^{k_\alpha+\frac{n}{n-1}}\, \varepsilon_j^{\frac{n}{n-1}} \\ &= C_5(N)\, \varepsilon_j^{\frac{n}{n-1}}.\end{aligned} \tag{4.49}$$

Step 2. *Existence of the generalized mean curvature $\mathcal{H}$ of $\overline{L}$ in Ω.*

Let X be an arbitrary C^1 vector field on Ω with compact support. Our goal is to verify [**30**, Definition 16.5]

$$\int_\Omega \mathrm{div}_{\overline{L}} X \, \mathrm{d}\mu_{\overline{L}} = -\int_\Omega \langle \mathcal{H}, X\rangle \, \mathrm{d}\mu_{\overline{L}} \tag{4.50}$$

for some locally $\mu_{\overline{L}}$ -integrable $\mathbb{R}^{2n}$-valued function $\mathcal{H}$ on $\overline{L}$.

Let ϕ_{ε_j} be a cut-off function satisfying

$$\begin{cases} \phi_{\varepsilon_j} = 0 \ \text{ on } U_{\varepsilon_j/2} \\ \phi_{\varepsilon_j} = 1 \ \text{ on } \Omega \backslash U_{\varepsilon_j} \\ 0 \leq \phi_{\varepsilon_j} \leq 1 \\ |D\phi_{\varepsilon_j}| < C/\varepsilon_j. \end{cases}$$

The existence of such ϕ_{ε_j} is given, for example, in Lemma 2.2 in [**?**] and is also due to Bochner [**?**]. Then $\phi_{\varepsilon_j} X$ is a C^1 vector field which vanishes on $U_{\varepsilon_j/2}$. By the standard first variation formula, we have

$$\begin{aligned}\int_\Omega \langle H, \phi_{\varepsilon_j} X\rangle \ \mathrm{d}\mu_L &= -\int_\Omega \mathrm{div}_L(\phi_{\varepsilon_j} X) \ \mathrm{d}\mu_L \\ &= -\int_\Omega \left\{ \langle \nabla\phi_{\varepsilon_j}, X\rangle + \phi_{\varepsilon_j} \mathrm{div}_L X \right\} \ \mathrm{d}\mu_L.\end{aligned} \tag{4.51}$$

From the volume estimate (4.49),

$$\left| \int_\Omega \langle \nabla\phi_{\varepsilon_j}, X\rangle \ \mathrm{d}\mu_L \right| \leq C(X)\, {\varepsilon_j}^{-1} \int_{U_{\varepsilon_j}\backslash U_{\varepsilon_j/2}} \mathrm{d}\mu_L \to 0.$$

Now letting $\varepsilon_j \to 0$ in (4.51)

$$\int_\Omega \langle H, X\rangle \ \mathrm{d}\mu_L = -\int_\Omega \mathrm{div}_L X \ \mathrm{d}\mu_L. \tag{4.52}$$

By assumption, $\overline{L}\backslash L \subseteq N$ and $\mathcal{H}^k(N) < +\infty$ and $k \leq n-2$, we have $\mathcal{H}^n\left(\overline{L}\backslash L\right) = 0$. So $\overline{L} = L \cup (\overline{L}\backslash L)$ is a rectifiable n-varifold. The divergence operator $\mathrm{div}_{\overline{L}}$ is defined as div_L, by noting that $\overline{L}\backslash L$ has zero measure (cf. [**30**, 16.2]). Then by (4.52)

$$\begin{aligned}\int_\Omega \mathrm{div}_{\overline{L}} X \; \mathrm{d}\mu_{\overline{L}} &= \int_\Omega \mathrm{div}\, X \; \mathrm{d}\mu_L \\ &= -\int_\Omega \langle H, X\rangle \; \mathrm{d}\mu_L \\ &= -\int_\Omega \langle \mathcal{H}, X\rangle \; \mathrm{d}\mu_{\overline{L}}\end{aligned} \tag{4.53}$$

where $\mathcal{H}$ equals H on L and zero on $\overline{L}\backslash L$, so it is locally μ_L-integrable on $\overline{L}$, in turn $\mathcal{H}$ is the generalized mean curvature of $\overline{L}$ in Ω since X is arbitrary.

Step 3. *$\overline{L}$ is Hamiltonian stationary in Ω.*

Our goal is to show that

$$\int_\Omega \langle J\mathrm{D}f, \mathcal{H}\rangle \, \mathrm{d}\mu_{\overline{L}} = 0 \tag{4.54}$$

for all $f \in C_0^\infty(\Omega)$. For any smooth function f with compact support in Ω, $J\mathrm{D}(\phi_{\varepsilon_j} f)$ is a Hamiltonian vector field on Ω with compact support, in particular it vanishes on $U_{\varepsilon_j/2}$ containing N. Applying (4.53) with $X = J\nabla f$, we see

$$\begin{aligned}\int_\Omega \langle J\mathrm{D}f, \mathcal{H}\rangle \, \mathrm{d}\mu_{\overline{L}} &= \int_L \langle J\nabla f, H\rangle \, \mathrm{d}\mu_L \\ &= \int_{L\cap U_{\varepsilon_j}} \langle J\nabla f, H\rangle \, \mathrm{d}\mu_L + \int_{L\backslash U_{\varepsilon_j}} \langle J\nabla f, H\rangle \, \mathrm{d}\mu_L.\end{aligned} \tag{4.55}$$

Since L is Hamiltonian stationary in $\Omega\backslash N$, we have

$$\begin{aligned}\left|\int_{L\backslash U_{\varepsilon_j}} \langle J\nabla f, H\rangle \, \mathrm{d}\mu_L\right| &= \left|\int_L \langle J\nabla(\phi_{\varepsilon_j} f), H\rangle \, \mathrm{d}\mu_L - \int_{L\cap U_{\varepsilon_j}} \langle J\nabla(\phi_{\varepsilon_j} f), H\rangle \, \mathrm{d}\mu_L\right| \\ &= \left|0 - \int_{L\cap(U_{\varepsilon_j}\backslash U_{\varepsilon_j/2})} \big(\langle \phi_{\varepsilon_j} J\nabla f, H\rangle + \langle f J\nabla\phi_{\varepsilon_j}, H\rangle\big) \, \mathrm{d}\mu_L\right| \\ &\le C(f)(1+{\varepsilon_j}^{-1}) \int_{L\cap(U_{\varepsilon_j}\backslash U_{\varepsilon_j/2})} |H| \, \mathrm{d}\mu_L \\ &\le C(f)(1+{\varepsilon_j}^{-1}) \left(\int_{L\cap(U_{\varepsilon_j}\backslash U_{\varepsilon_j/2})} |H|^n \, \mathrm{d}\mu_L\right)^{\frac{1}{n}} \left(\int_{U_{\varepsilon_j}\backslash U_{\varepsilon_j/2}} \mathrm{d}\mu_L\right)^{\frac{n-1}{n}}\end{aligned} \tag{4.56}$$

by Hölder's inequality, where $C(f)$ depends on f and $|\mathrm{D}f|$ as ∇f is the tangential projection of $\mathrm{D}f$ along L so

$$|J\nabla f| = |\nabla f| \le |\mathrm{D}f|.$$

Similarly

$$\left|\int_{L\cap U_{\varepsilon_j}} \langle J\nabla f, H\rangle \, \mathrm{d}\mu_L\right| \le C(f) \left(\int_{L\cap U_{\varepsilon_j}} |H|^n \, \mathrm{d}\mu_L\right)^{\frac{1}{n}} \left(\int_{U_{\varepsilon_j}} \mathrm{d}\mu_L\right)^{\frac{n-1}{n}} \tag{4.57}$$

It then follows from the assumption (i), and the volume estimate (4.49) that both terms (4.56) and (4.57) vanish as $\varepsilon_j \to 0$. Combining with (4.55) we conclude (4.54). $\square$

The local k-noncollapsing property is automatically satisfied if N is a compact manifold of dimension no larger than $n-2$.

COROLLARY 4.4.1. Let N be a compact submanifold in a domain $\Omega \subset \mathbb{R}^{2n}$ of dimension $k \leq n-2$. Let L be Hamiltonian stationary in $\Omega \backslash N$ as in Theorem 4.3 with (i), (ii) and (iii) therein. Then $\overline{L}$ is Hamiltonian stationary in Ω.

COROLLARY 4.4.2. With the assumptions on N and (i), (ii), (iii) as in Theorem 4.3, let $\iota : M \to \Omega \backslash N$ be a proper immersion of an n-dimensional manifold M in $\Omega \backslash N$ and $L = \iota(M)$ is Hamiltonian stationary Lagrangian in $\Omega \backslash N$. Then $\overline{L}$ is Hamiltonian stationary Lagrangian in Ω.

PROOF. In light of Theorem 4.3, the only thing to verify is: $\overline{L} \backslash L \subseteq N$. For any $y \in \overline{L} \backslash L$, if $y \notin N$ then by compactness of N there will be a neighborhood W of y such that $\overline{W} \cap N = \emptyset$; then there exists a sequence $y_j \in \overline{W} \cap L \to y$. By properness of ι, it follows that $\iota^{-1}(\{y_j : j \in \mathbb{N}\})$ contains a converging subsequence in M since $\iota^{-1}(\overline{W})$ is compact in M; then y is the image of the limit point which is in L, and we have a contradiction. $\square$

4.5. Regularity of HSL submanifolds in a symplectic manifold

In this section, we provide a different method to the regularity of HSL manifolds by develop a theory for the regularity of a class of nonlinear fourth order of double divergence form. This allows to deal with general symplectic ambient space.

We consider

$$\partial_{x_l}\partial_{x_j} F^{jl}(x, Du, D^2u) = \partial_{x_k} a^k(x, Du, D^2u) - b(x, Du, D^2u). \tag{4.58}$$

The coefficient functions F^{jl}, a^k, b are smooth in the entries (x, Du, D^2u) over a convex region $U \subset \mathbb{R}^n \times \mathbb{R}^n \times S^{n\times n}$, and the Legendre ellipticity condition holds: for a constant $\Lambda > 0$

$$\frac{\partial F^{jl}}{\partial u_{ik}}(\xi)\sigma_{ij}\sigma_{kl} \geq \Lambda \|\sigma\|^2, \ \forall\, \sigma \in S^{n\times n} \text{ and } \xi \in U. \tag{4.59}$$

A function $u \in W^{2,\infty}$ is said to be a weak solution to the double divergence equation (4.58) if each of the derivatives ∂_{x_i} presented in (4.58) are taken in a distributional sense, as in (4.61). For non-classical solutions to nonlinear partial differential equations, especially of order beyond two, attention needs to be paid even for the meaning of solutions, due to the fact that no uniform theory exists. In our case, the double divergence structure on the matrix-valued operator F, which involves D^2u itself, permits us to define solutions, possibly in the weakest form, by flipping derivatives on F and the lower order terms, to test functions via integration by parts as traditionally done for distributional solutions, but now only for half of the total order.

Equations in divergence form occupy an important place in the second order PDE theory. In fourth order, the most natural counterpart is an equation, linear or nonlinear, with a double divergence structure. Many well-known equations enjoy the structure such as for the bi-harmonic functions, extremal Kähler metrics, the

Willmore surface, and the Hamiltonian stationary Lagrangian equations which are closely linked to elastic mechanics. We find that the double divergence structure, a less explored area, shares similar features, as second order equations in divergence form, toward a regularity theory. We demonstrate that when (4.59) holds, any weak solution u to (4.58) is smooth, provided that the oscillation of $D_qF(x, Du, D^2u)$ can be bounded locally (in x) by a small positive constant.

The above fourth order nonlinear elliptic equation originates in the variational problem for volume of Lagrangian submanifolds under Hamiltonian variations in a symplectic manifold (M, ω) with a Riemannian metric g compatible with ω in the sense that $\omega(X, Y) = g(JX, Y)$ for an almost complex structure J on M.

In $\mathbb{C}^n$ with the standard Kähler structure, a particular expression for Θ is available, namely, it is a sum of arctan of the eigenvalues of the Hessian of the potential function u for a local graphical representation $L = (x, Du)$. This decomposition feature of the fourth order operator into two second order elliptic operators is essential in the work of Chen-Warren [**?**] in which it is shown that a C^1-regular Hamiltonian stationary Lagrangian submanifold in $\mathbb{C}^n$ is real analytic. However, the same strategy for a Calabi-Yau other than $\mathbb{C}^n$ encounters difficulties for the reason that Θ, still well-defined by Ω at least locally, now is no longer written in a clean form as sum of arctan functions, when representing L as a gradient graph in a Darboux coordinate chart.

To overcome the obstacle presented above in the Calabi-Yau case, we find that, in a more general standpoint, the Riemannian picture without referring to a symplectic structure is helpful: dealing directly with the stationary point of the volume of $L = (x, Du)$ in an open ball $B \subset \mathbb{R}^{2n}$ equipped with a Riemannian metric among nearby competing gradient graphs $L_t = (x, Du + tD\eta)$ for compactly supported smooth functions η. This leads us to study the fourth order nonlinear equation (4.58) with (4.59).

We now outline our approach to the regularity problem. Given a $W^{2,\infty}$ weak solution u of (4.58) that satisfies the Legendre ellipticity condition (4.59), we show, in Proposition 4.6.1, that the difference quotient $[u(x) - u(x-h)]/|h|$ can be bounded in $W^{2,2}$ uniformly in h. Letting $h \to 0$ asserts $u \in W^{3,2}$ with estimates controlled by $\|u\|_{W^{2,\infty}}$. This boosted regularity is then used to bound the $C^{1,\alpha}$ norm of the difference quotient uniformly in h in Proposition 4.6.2, leading to a $C^{2,\alpha}$ bound on u. The key ingredient for this step is a closeness assumption, given by (4.73): this ensures that the operator is in fact close to a constant coefficient operator, given by its linearization at the origin, that leads to a uniform $C^{1,\alpha}$ bound on the difference quotient. Note that reaching $C^{2,\alpha}$ is a crucial step in proving smoothness since once $C^{2,\alpha}$ is achieved the functions $\frac{\partial F^{jl}}{\partial u_{ik}}, \frac{\partial F^{jl}}{\partial u_k}, \frac{\partial F^{jl}}{\partial x_p}$, which were barely measurable, are now all Hölder continuous in x, and this is sufficient to prove higher regularity for the equation satisfied by the difference quotient. The enhanced regularity alone improves the bound on the difference between the actual operator and its linearization by a factor of a power of r, which in turn ultimately leads to $u \in C^{3,\alpha}$. Moving from $C^{3,\alpha}$ to C^∞ involves a similar bootstrapping procedure employed in [**4**] by considering the difference quotient.

For the general fourth order nonlinear equation, our main result is the following.

THEOREM 4.4. *Suppose that $u \in W^{2,\infty}(B_1)$ is a weak solution of* (4.58) *that satisfies condition* (4.59) *on the unit ball B_1 in $\mathbb{R}^n$. There is an $\varepsilon_0(\Lambda, n) > 0$ such* *that if*

$$\left|\frac{\partial F^{jl}}{\partial u_{ik}}(x, Du, D^2u) - \frac{\partial F^{jl}}{\partial u_{ik}}(\xi)\right| < \varepsilon_0 \tag{4.60}$$

for some $\xi \in U$ and all $x \in B_1$, then u is smooth in B_1.

This regularity statement suffices for answering affirmatively the motivating geometric question on smoothness of a C^1-regular critical point under Hamiltonian deformations in a symplectic manifold. The transition, from the general theory in euclidean space to the specific symplectic setting, is done in a Darboux coordinate chart with estimates on the Riemannian metric within the special coordinates. This is given by [**20**, Prop. 3.2 and Prop. 3.4]. Our main result is the following.

Theorem 4.5. *Let (M, ω) be a compact symplectic manifold with a Riemannian metric g compatible with ω and some almost complex structure J on M. Let L be a Hamiltonian stationary Lagrangian C^1-regular submanifold in M with respect to ω, g. Then L is smooth.*

4.6. A fourth order elliptic theory

4.6.1. Preliminaries. We consider the following fourth order equation, written in double divergence form:

$$\int_{B_1} \left[F^{jl}(x, Du, D^2u)\eta_{jl} + a^k(x, Du, D^2u)\eta_k + b(x, Du, D^2u)\eta\right] dx = 0 \tag{4.61}$$

for all $\eta \in C_c^\infty(B_1)$ where B_1 is the unit ball in $\mathbb{R}^n$. The coefficients are smooth in the entries (x, Du, D^2u) over a given convex region $U \subset \mathbb{R}^n \times \mathbb{R}^n \times S^{n\times n}$. Lower indices on a function stand for partial derivatives, e.g. η_{jl}, η_k, and summation convention is assumed.

We write $h_p = he_p$ and denote the difference quotient of u in the e_p direction by u^{h_p}. We start by deriving a difference quotient expression from (4.61) in the direction h_p. Fixing a compactly supported function η we can choose h small enough so the function

$$\eta^{-h_p}(x) = \frac{\eta(x - h_p) - \eta(x)}{h} \tag{4.62}$$

is a valid test function. Using a change of variables $x \to x + h_p$ on the first term of (4.62) with the first two terms of (4.61) and recombining, we get

$$\int_{B_1} \left([F^{jl}(x, Du, D^2u)]^{h_p}\eta_{jl} + a^k(x, Du, D^2u)\eta_k^{-h_p} + b(x, Du, D^2u)\eta^{-h_p}\right) dx = 0. \tag{4.63}$$

The function F^{jl} is defined on open subsets of the vector space so for any fixed x where $D^2u(x)$ is defined we can define

$$\begin{aligned}
\xi_0 &= \big(x, Du(x), D^2u(x)\big) \in \mathbb{R}^n \times \mathbb{R}^n \times S^{n\times n}\\
\xi_h &= \big(x + h_p, Du(x + h_p), D^2u(x + h_p)\big) \in \mathbb{R}^n \times \mathbb{R}^n \times S^{n\times n}\\
\vec{V} &= \xi_h - \xi_0
\end{aligned}$$

in which case we have

$$
\begin{aligned}
[F^{jl}(x, Du, D^2u)]^{h_p} &= \frac{1}{h}\{F^{jl}(\xi_0 + \vec{V}) - F^{jl}(\xi_0)\} \\
&= \frac{1}{h}\int_0^1 \frac{d}{dt}F^{jl}(\xi_0 + t\vec{V})dt \\
&= \frac{1}{h}\int_0^1 DF^{jl}|_{\xi_0+t\vec{V}} \cdot \vec{V}dt \\
&= \int_0^1 \frac{\partial F^{jl}}{\partial u_{ik}}(\xi_0 + t\vec{V}) \cdot u_{ik}^{h_p}dt + \int_0^1 \left(\frac{\partial F^{jl}}{\partial u_k}(\xi_0 + t\vec{V})u_k^{h_p} + \frac{\partial F^{jl}}{\partial x_p}(\xi_0 + t\vec{V})\right)dt \\
&= \left(\int_0^1 \frac{\partial F^{jl}}{\partial u_{ik}}(\xi_0 + t\vec{V})dt\right) \cdot u_{ik}^{h_p} + \int_0^1 \left(\frac{\partial F^{jl}}{\partial u_k}(\xi_0 + t\vec{V})u_k^{h_p} + \frac{\partial F^{jl}}{\partial x_p}(\xi_0 + t\vec{V})\right)dt \\
&= \beta^{ij,kl} \cdot u_{ik}^{h_p} + \gamma_1^{jl,k}u_k^{h_p} + \gamma_2^{jl}
\end{aligned}
$$

where we define

$$\beta^{ij,kl}(x) = \int_0^1 \frac{\partial F^{jl}}{\partial u_{ik}}(\xi_0 + t\vec{V})dt \tag{4.64}$$

and

$$\gamma_1^{jl,k}(x) = \int_0^1 \frac{\partial F^{jl}}{\partial u_k}(\xi_0 + t\vec{V})dt \tag{4.65}$$

$$\gamma_2^{jl}(x) = \int_0^1 \frac{\partial F^{jl}}{\partial x_p}(\xi_0 + t\vec{V})dt. \tag{4.66}$$

Letting $f = u^{h_p}$ and

$$\psi^k(x) = a^k(x, Du, D^2u) \tag{4.67}$$

$$\zeta(x) = b(x, Du, D^2u), \tag{4.68}$$

we arrive the following equation by plugging the above expressions into (4.63) governing the difference quotients

$$\int_{B_1}\left(\beta^{ij,kl}f_{ik}\eta_{jl} + \gamma_1^{jl,k}f_k\eta_{jl} + \gamma_2^{jl}\eta_{jl} + \psi^k\eta_k^{-h_p} + \zeta\eta^{-h_p}\right)dx = 0.$$

This linearized equation, which holds true provided $\eta \in C_c^\infty(B_{1-h})$ governs difference quotients for solutions to (4.61). Further simplifying notation we define

$$\gamma^{jl}(x) = \int_0^1 \left(\frac{\partial F^{jl}}{\partial u_k}(\xi_0 + t\vec{V})f_k + \frac{\partial F^{jl}}{\partial x_p}(\xi_0 + t\vec{V})\right)dt \tag{4.69}$$

to get

$$\int_{B_1}\left(\beta^{ij,kl}f_{ik}\eta_{jl} + \gamma^{jl}\eta_{jl} + \psi^k\eta_k^{-h_p} + \zeta\eta^{-h_p}\right)dx = 0. \tag{4.70}$$

Observe that since we do not start with a continuous Hessian, we leave the expressions for the above leading coefficients in their integral form.

Definition 4.6. We define the nonlinear fourth order equation (4.61) to be Λ-**uniform** on a convex neighborhood $U \subset \mathbb{R}^n \times \mathbb{R}^n \times S^{n\times n}$ if the standard Legendre ellipticity condition is satisfied for any $\xi \in U$

$$\frac{\partial F^{jl}}{\partial u_{ik}}(\xi)\sigma_{ij}\sigma_{kl} \geq \Lambda \|\sigma\|^2, \ \forall\, \sigma \in S^{n\times n}. \tag{4.71}$$

REMARK 4.7. While this definition is tailored to equations of the form (4.61) it is important to note that it also applies to linear equations of the form (4.70), in which case

$$F^{jl}(x) = \beta^{ij,kl}(x) f_{ik} + \gamma^{jl}(x)$$

and

$$\frac{\partial F^{jl}}{\partial u_{ik}} = \beta^{ij,kl}(x).$$

Thus when the nonlinear equation (4.61) is Λ-uniform, then so is the linearized equation (4.70).

We will use the following results to prove higher regularity in section 4.6.2. We state the results here for the convenience of the reader.

THEOREM 4.8. [**4**, Theorem 2.1]. *Suppose $w \in W^{2,2}(B_r)$ satisfies the Λ-uniform constant coefficient equation*

$$\int c_0^{ik,jl} w_{ik}\eta_{jl} dx = 0, \quad \forall \eta \in C_0^\infty(B_r).$$

Then for any $0 < \rho \leq r$ there holds

$$\int_{B_\rho} |D^2 w|^2 \leq C_1 \left(\frac{\rho}{r}\right)^n ||D^2 w||^2_{L^2(B_r)},$$

$$\int_{B_\rho} |D^2 w - (D^2 w)_\rho|^2 \leq C_2 \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |D^2 w - (D^2 w)_r|^2$$

where C_1, C_2 depend on the ellipticity constant Λ and $(D^2 w)_\rho$ is the average value of $D^2 w$ on a ball of radius ρ.

COROLLARY 4.6.1. [**4**, Corollary 2.2]. *Suppose w is as in the Theorem 4.8. Then for any* $u \in W^{2,2}(B_r)$, and *for any* $0 < \rho \leq r$, there holds

$$\int_{B_\rho} \left|D^2 u\right|^2 \leq 4C_1 \left(\frac{\rho}{r}\right)^n \left\|D^2 u\right\|^2_{L^2(B_r)} + (2 + 8C_1) \left\|D^2 (w-u)\right\|^2_{L^2(B_r)}$$

and

$$\int_{B_\rho} \left|D^2 u - (D^2 u)_\rho\right|^2 \leq 4C_2 \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} \left|D^2 u - (D^2 u)_r\right|^2 + (8 + 16C_2) \int_{B_r} \left|D^2 (u-w)\right|^2$$

where C_1, C_2 depend on the ellipticity constant Λ.

LEMMA 4.9. [**19**, Lemma 3.4]. *Let ϕ be a nonnegative and nondecreasing function on $[0, R]$. Suppose that*

$$\phi(\rho) \leq A\left[\left(\frac{\rho}{r}\right)^\alpha + \varepsilon\right]\phi(r) + Br^\beta$$

for any $0 < \rho \leq r \leq R$, *with* A, B, α, β *nonnegative constants and* $\beta < \alpha$. *Then for any* $\gamma \in (\beta, \alpha)$, *there exists a constant* $\varepsilon^* = \varepsilon^*(A, \alpha, \beta, \gamma)$ *such that if* $\varepsilon < \varepsilon^*$ *we have for all* $0 < \rho \leq r \leq R$

$$\phi(\rho) \leq c\left[\left(\frac{\rho}{r}\right)^{\gamma}\phi(r) + Br^{\beta}\right]$$

where c *is a positive constant depending on* A, α, β, γ. *In particular, we have for any* $0 < r \leq R$

$$\phi(r) \leq c\left[\frac{\phi(R)}{R^{\gamma}}r^{\gamma} + Br^{\beta}\right].$$

The following boundary value problem existence result should come as no surprise, but is included for completeness.

Lemma 4.10. *Suppose that* $g \in W^{2,2}(B_r)$, *and* $c_0^{ij,kl}$ *is as in Theorem 4.8. There exists a unique solution* $w \in W^{2,2}(B_r)$ *solving the following BVP*

$$\int_{B_r} c_0^{ij,kl} w_{ik}\eta_{jl}dx = 0, \qquad \forall \eta \in C_0^{\infty}(B_r)$$
$$w = g, \quad Dw = Dg \qquad \text{on } \partial B_r(y).$$

Proof. By [**?**, Corollary 6.48, 6.49] the boundary condition is equivalent to $w - g \in H_0^2(B_r)$. The problem will be solved if we can find a function $v = w - g \in H_0^2(B_r)$ such that

$$\int_{B_r} c_0^{ij,kl}\left(w - g\right)_{ik}\eta_{jl}dx + \int_{B_r} c_0^{ij,kl} g_{ik}\eta_{jl}dx = 0.$$

So it suffices to solve the problem

$$\int_{B_r} c_0^{ij,kl} v_{ik}\eta_{jl}dx = -\int_{B_r} c_0^{ij,kl} g_{ik}\eta_{jl}dx$$
$$v \in H_0^2(B_r).$$

First, we claim that

$$\langle \phi, \varphi \rangle = \int_{B_r} c_0^{ij,kl}\phi_{ik}\varphi_{jl}dx \tag{4.72}$$

defines a Hilbert space norm on the function space $H_0^2(B_r)$. In other words, the norm defined by (4.72) is equivalent to the $W_0^{2,2}(B_r)$ norm and the inner product is symmetric. First note that by the Legendre condition

$$\langle \phi, \phi \rangle \geq \Lambda_1 \int_{B_r} \left|D^2\phi\right|^2$$

where Λ_1 depends on Λ, n, and because $c_0^{ij,kl}$ is bounded we have

$$\langle \phi, \phi \rangle \leq \Lambda_2 \int_{B_r} \left|D^2\phi\right|^2$$

where Λ_2 depends on $n, \|c_0^{ij,kl}\|_{L^\infty}$ for $1 \leq i, j, k, l, \leq n$. Using the Poincaré inequality [**17**, (7.44)], for any $\phi \in W_0^{2,2}$ (hence $D\phi \in W_0^{1,2}$)

$$\frac{1}{C}\langle \phi, \phi \rangle \leq \|\phi\|^2_{W^{2,2}(B_r)} \leq C\langle \phi, \phi \rangle.$$

Thus the norm $\langle \phi, \phi \rangle$ is continuous with respect to the $W^{2,2}$ norm.

Next we argue symmetry of (4.72): For $\phi, \varphi \in H_0^2(B_r)$ we may take $\phi_m, \varphi_m \in C_c^\infty(B_r) \cap W^{2,2}(B_r)$, which converge respectively to ϕ, φ in $W^{2,2}$, as $m \to \infty$. We have

$$\begin{aligned}
\langle \phi, \varphi \rangle &= \lim_{m\to\infty} \langle \phi_m, \varphi_m \rangle \\
&= \lim_{m\to\infty} \int_{B_r} c_0^{ij,kl} \left(\phi_m\right)_{ik} \left(\varphi_m\right)_{jl} dx \\
&= (-1)^2 \lim_{m\to\infty} \int_{B_r} c_0^{ij,kl} \left(\phi_m\right)_{ikjl} \left(\varphi_m\right) dx \\
&= (-1)^4 \lim_{m\to\infty} \int_{B_r} c_0^{ij,kl} \left(\phi_m\right)_{jl} \left(\varphi_m\right)_{ik} dx \\
&= \lim_{m\to\infty} \langle \varphi_m, \phi_m \rangle \\
&= \langle \varphi, \phi \rangle.
\end{aligned}$$

The linear operator

$$f(\phi) = -\int_{B_r} c_0^{ij,kl} g_{ik} \phi_{jl} dx$$

on $W_0^{2,2}(B_r)$ is bounded with respect to the norm defined by (4.72). To see this, take any ϕ in $H_0^2(B_r)$, then

$$\begin{aligned}
|f(\phi)| &= \left| -\int_{B_r} c_0^{ij,kl} g_{ik} \phi_{jl} dx \right| \\
&\leq C_1 \|g\|_{W^{2,2}(B_r)} \|\phi\|_{W^{2,2}(B_r)} \\
&\leq C_1 \|g\|_{W^{2,2}(B_r)} C_2 \left(\langle \phi, \phi \rangle\right)^{1/2}.
\end{aligned}$$

By the Riesz representation theorem, there is a unique solution $v \in H_0^2(B_r)$ such that

$$f(\eta) = \langle \eta, v \rangle = \int_{B_r} c_0^{ij,kl} v_{ik} \eta_{jl} dx$$

that is

$$-\int_{B_r} c_0^{ij,kl} g_{ik} \eta_{jl} dx = \int_{B_r} c_0^{ij,kl} v_{ik} \eta_{jl} dx.$$

Thus we can let

$$w = v + g.$$

This gives the solvability of the boundary value problem in $H_0^2(B_r)$. □

4.6.2. Main regularity results. We will establish Theorem 4.4 by first proving the solution is $C^{2,\alpha}$ and then by bootstrapping for smoothness. We state our two main regularity boosting results below.

THEOREM 4.11. *Suppose that $u \in W^{2,\infty}(B_1)$ is a weak solution of the Λ-uniform equation (4.61) on B_1, such that*

$$\left\{ (x, Du(x), D^2u(x)) : x \in B_1 \right\} \subset U.$$

Fix $\alpha \in (0,1)$ and let $q = \frac{n}{2(1-\alpha)}$. There exists an $\varepsilon_0 > 0$, depending only on Λ, α and n such that if the coefficients $\beta^{ij,kl}$ given by (4.64) satisfy

$$\left| \beta^{ij,kl}(x, Du, D^2u) - a_0^{ij,kl} \right| < \varepsilon_0 \tag{4.73}$$

where $a_0^{ij,kl} = \frac{\partial F^{jl}}{\partial u_{ik}}(\xi)$ *for some* $\xi \in U$, *then* $u \in C^{2,\alpha}(B_1)$ *with*

$$||D^2 u||_{C^\alpha(B_{1/4})} \leq C(\Lambda, \alpha, ||u||_{W^{2,\infty}(B_1)}, \|DF\|_{L^\infty(U)}, \left\|a^k\right\|_{L^\infty(U)}, \|b\|_{L^\infty(U)}).$$

THEOREM 4.12. *Suppose that* $u \in C^{2,\alpha}(B_1)$ *satisfies the* Λ*-uniform equation (4.61) on* B_1*. Then* u *is smooth in* B_1*.*

REMARK 4.13. The closeness condition (4.73) is not needed to reach $W^{3,2}$ from $W^{2,\infty}$. It is used to bootstrap to $C^{2,\alpha}$ from $W^{3,2}$, and $C^{2,\alpha}$ is enough to bootstrap further.

4.6.3. Proof of Theorem 4.11. To boost up regularity, we will work with equation (4.70) on the difference quotient u^{h_p}, rather than directly on (4.61) for u. Given a solution f to (4.70), we begin with bounding its $W^{2,2}$ norm in terms of its $W^{1,\infty}$ norm in Proposition 4.6.1, then in Proposition 4.6.2, we show that the $C^{1,\alpha}$ norm of f depends on its $W^{2,2}$ norm. This follows essentially the same arguments as in [**8**, Lemma 3.1] and [**4**, Proposition 1.3].

Theorem 4.11 will then follow from Propositions 4.6.2 and 4.6.1, by taking $f = u^{h_p}$ therein.

PROPOSITION 4.6.1. Suppose that $f \in W^{2,\infty}(B_1)$ satisfies the uniformly elliptic weak double divergence equation (4.70) on B_1. Then f satisfies the following estimate:

$$||f||_{W^{2,2}(B_{1/2})} \leq C\left(\Lambda, \|f\|_{W^{1,\infty}(B_1)}, \|\psi\|_{L^2(B_1)}, \|\zeta\|_{L^2(B_1)}, \|\beta\|_{L^\infty(B_1)}\right). \tag{4.74}$$

PROOF. Assuming $f \in W^{2,\infty}(B_1)$, f will be $W^{2,2}$ and the function $\tau^4 f$ can be approximated by functions $\eta \in C_c^\infty(B_{3/4})$ in $W^{2,2}$ norm for τ smooth compactly supported on $B_{3/4}$ which is 1 on $B_{1/2}$. Thus

$$\int_{B_1} \left[\beta^{ij,kl} f_{ik} \left(\tau^4 f\right)_{jl} + \gamma^{jl} \left(\tau^4 f\right)_{jl} + \psi^k \left(\tau^4 f\right)_k^{-h_p} + \zeta \left(\tau^4 f\right)^{-h_p} \right] dx = 0.$$

Applying uniform ellipticity to the first term of the above expression, we get

$$\begin{aligned}\Lambda \int_{B_1} \tau^4 \left|D^2 f\right|^2 dx \leq & \int_{B_1} \left|\beta^{ij,kl} f_{ik} \left(\left(\tau^4\right)_{jl} f + \left(\tau^4\right)_l f_j + \left(\tau^4\right)_j f_l\right)\right| dx \\ & + \int_{B_1} \left(\left|\gamma^{jl} \left(\tau^4 f\right)_{jl}\right| + \left|\psi^k \left(\tau^4 f\right)_k^{-h_p}\right| + \left|\zeta \left(\tau^4 f\right)^{-h_p}\right|\right) dx.\end{aligned} \tag{4.75}$$

Straightforward use of inequalities gives

$$\begin{aligned}&\int_{B_1} \left|\beta^{ij,kl} f_{ik} \left(\left(\tau^4\right)_{jl} f + \left(\tau^4\right)_l f_j + \left(\tau^4\right)_j f_l\right)\right| dx \\ &\quad \leq C\left(D\tau, D^2\tau, \|f\|_{W^{1,\infty}}, \|\beta\|_{L^\infty}\right) \int_{B_1} \tau^2 \left|D^2 f\right| dx \\ &\quad \leq C\left(D\tau, D^2\tau, \|f\|_{W^{1,\infty}}, \|\beta\|_{L^\infty}\right) \left(\frac{1}{\varepsilon} + \varepsilon \int_{B_1} \tau^4 \left|D^2 f\right|^2 dx\right).\end{aligned}$$

Similarly

$$\int_{B_1} \left|\gamma^{jl} \left(\tau^4 f\right)_{jl}\right| dx \leq C\left(D\tau, D^2\tau, \|f\|_{W^{1,\infty}}, \|\beta\|_{L^\infty}\right) \left(\frac{1}{\varepsilon} + \varepsilon \int_{B_1} \tau^4 \left|D^2 f\right|^2 dx\right). \tag{4.76}$$

Now for

$$\int_{B_1} \left|\psi^k \left(\tau^4 f\right)_k^{-h_p}\right| dx \tag{4.77}$$

observe that

$$\begin{aligned}
\int_{B_1} \left|\psi^k \frac{\left(\tau^4 f\right)_k (x-h_p) - \left(\tau^4 f\right)_k}{h}\right| dx &= \int_{B_1} \left|\psi^k\right| \left|\int_0^1 D\left(\tau^4 f\right)_k (x - th_p) dt\right| dx \\
&\le \int_0^1 \int_{B_1} \left|\psi^k\right| \left|D\left(\tau^4 f\right)_k (x-th_p)\right| dx dt \\
&\le \int_0^1 \|\psi\|_{L^2(B_1)} \left\|D^2\left(\tau^4 f\right)\right\|_{L^2(B_1)} dt \\
&= \|\psi\|_{L^2(B_1)} \left\|D^2\left(\tau^4 f\right)\right\|_{L^2(B_1)}
\end{aligned}$$

which can be treated as in (4.76)

$$\int_{B_1} \left|\psi^k \left(\tau^4 f\right)_k^{-h_p}\right| dx \le C\left(D^2\tau, \|f\|_{W^{1,\infty}}, \|\psi\|_{L^2(B_1)}\right)\left(\frac{1}{\varepsilon} + \varepsilon \int_{B_1} \tau^4 \left|D^2 f\right|^2 dx\right).$$

Finally, treating the last term in (4.75) similarly as for (4.77), we can bound (4.75) in lower order terms of f.

Combining and using the appropriately chosen τ, we choose ε appropriately in the above equation and in (4.76), to get

$$\frac{\Lambda}{2}\int_{B_{1/2}} \left|D^2 f\right|^2 dx \le C\left(\|f\|_{W^{1,\infty}(B_1)}, \|\psi\|_{L^2(B_1)}, \|\zeta\|_{L^2(B_1)}, \|\beta\|_{L^\infty(B_1)}\right),$$

therefore complete the proof. □

Our next result is key in achieving $C^{2,\alpha}$ regularity of u.

PROPOSITION 4.6.2. For a fixed h_p with $|h| < \frac{1}{100}$ suppose that $f \in W^{2,2}(B_1)$ satisfies the uniformly elliptic double divergence equation (4.70) weakly on $B_{3/4}(0)$. Suppose that $\gamma^{jl}, \psi^k, \zeta \in L^{2q}$ with $q = \frac{n}{2-2\alpha}, \alpha \in (0,1)$. Then, there is an $\varepsilon_0(n, \Lambda, \alpha) > 0$, such that if (4.73) holds as in Theorem 4.11 then we have $Df \in C^\alpha(B_{1/4})$ and the estimates:

(4.78)

$$||Df||_{C^\alpha(B_{1/4})} \le C(\Lambda, \alpha, ||f||_{W^{2,2}(B_{1/2})}, \left\|\gamma^{jl}\right\|_{L^{2q}(B_1),} \left\|\psi^k\right\|_{L^{2q}(B_1)}, \|\zeta\|_{L^{2q}(B_1)}).$$

PROOF. Pick an arbitrary point $y \in B_{1/4}$. Then $B_r(y) \subset B_{3/4}$ for any fixed $r < 1/2$.

We write $v = f - w$, where w satisfies the following constant coefficient partial differential equation on $B_r(y) \subset B_{3/4}$:

$$\begin{gathered}
\int_{B_r(y)} a_0^{ij,kl} w_{ik} \eta_{jl} dx = 0, \quad \forall \eta \in C_0^\infty(B_r(y)) \\
w = f, \quad Dw = Df \quad \text{ on } \partial B_r(y).
\end{gathered}$$

Here $a_0^{ij,kl}$ is the symbol occurring in our assumption (4.73). This solution exists by Lemma 4.10 and is smooth on the interior of $B_r(y)$ [**?**, Theorem 6.33].

We may extend v to a function (still named v) on $B_{3/4}$ by defining $v = 0$ on $B_{3/4}\backslash B_r(y)$. As the original $v \in H_0^2(B_r(y))$ is the limit of $C_c^\infty(B_r(y))$ functions $\eta^{(m)}$ it follows that the extended v must also remain in $H_0^2(B_{3/4})$.

Now because v is the $W^{2,2}(B_r(y))$ limit of functions $\eta^{(m)} \in C_c^\infty(B_r(y)) \subset C_c^\infty(B_{3/4})$ we may also write

$$\begin{aligned}
\int_{B_r(y)} a_0^{ij,kl} v_{ik} v_{jl} dx &= \lim_{m\to\infty} \int_{B_r(y)} a_0^{ij,kl} v_{ik} (\eta^{(m)})_{jl} dx \\
&= \lim_{m\to\infty} \int_{B_r(y)} a_0^{ij,kl} f_{ik} (\eta^{(m)})_{jl} dx \\
&= \lim_{m\to\infty} \int_{B_{3/4}} a_0^{ij,kl} f_{ik} (\eta^{(m)})_{jl} dx \\
&= \int_{B_{3/4}} a_0^{ij,kl} f_{ik} v_{jl} dx. \qquad (4.79)
\end{aligned}$$

Now taking limits of (4.70) for $\eta^{(m)} \to v$ we conclude that

$$\int_{B_{3/4}} \left(\beta^{ij,kl} f_{ik} v_{jl} + \gamma^{jl} v_{jl} + \psi^k v_k^{-h_p} + \zeta v^{-h_p} \right) dx = 0. \qquad (4.80)$$

Now we subtract (4.80) from (4.79)

$$\begin{aligned}
&\int_{B_r(y)} a_0^{ij,kl} v_{ik} v_{jl} dx = \int_{B_{3/4}} a_0^{ij,kl} f_{ik} v_{jl} dx - \int_{B_{3/4}} \left(\beta^{ij,kl} f_{ik} v_{jl} + \gamma^{jl} v_{jl} + \psi^k v_k^{-h_p} + \zeta v^{-h_p} \right) dx \\
&\quad = \int_{B_{3/4}} \left(a_0^{ij,kl} - \beta^{ij,kl} \right) f_{ik} v_{jl} dx - \int_{B_{3/4}} \gamma^{jl} v_{jl} dx - \int_{B_{3/4}} \left(\psi^k v_k^{-h_p} + \zeta v^{-h_p} \right) dx. \qquad (4.81)
\end{aligned}$$

First we note that our condition (4.73), for an ε_0 yet to determined, gives us

$$\int_{B_{3/4}} \left| (a_0^{ij,kl} - \beta^{ij,kl}) f_{ik} v_{jl} \right| dx \le \varepsilon_0 \left\| D^2 f \right\|_{L^2(B_r(y))} \left\| D^2 v \right\|_{L^2(B_r(y))}, \qquad (4.82)$$

making use of the fact that v is supported in $B_r(y)$. Next, by Hölder's inequality

$$\int_{B_{3/4}} |\gamma^{jl} v_{jl}| dx \le C(n) \left\| \gamma \right\|_{L^2(B_r(y))} \left\| D^2 v \right\|_{L^2(B_r(y))} \le C(n) \left\| \gamma \right\|_{L^{2q}(B_r(y))} r^{\frac{n-2+2\alpha}{2}} \left\| D^2 v \right\|_{L^2(B_r(y))} \qquad (4.83)$$

where $q = \frac{n}{2(1-\alpha)}$.

For the third term

$$
\begin{aligned}
\int_{B_{3/4}} \left|\psi^k \frac{v_k(x-h_p)-v_k(x)}{h}\right| dx &= \lim_{m\to\infty} \int_{B_{3/4}} \left|\psi^k \frac{\left(\eta^{(m)}\right)_k(x-h_p)-\left(\eta^{(m)}\right)_k(x)}{h}\right| dx \\
&= \lim_{m\to\infty} \int_{B_{3/4}} \left|\psi^k \int_0^1 \left(-D_{pk}\eta^{(m)}(x-th_p)\right) dt\right| dx \\
&\le \lim_{m\to\infty} \int_{B_{3/4}} \left|\psi^k\right| \int_0^1 \left|D_{pk}\eta^{(m)}(x-th_p)dt\right| dx \\
&\le \lim_{m\to\infty} \int_0^1 \int_{B_{3/4}} \left|\psi^k\right| \left|D_{pk}\eta^{(m)}(x-th_p)\right| dxdt \qquad \text{(Fubini-Tonelli's Theorem)} \\
&\le \int_0^1 \int_{B_{3/4}} \left|\psi^k\right| \left|D^2 v(x-th_p)\right| dxdt \qquad (\eta^{m)} \to v \text{ in } W^{2,2}) \\
&= \int_0^1 \int_{B_{r+h}(y)} \left|\psi^k\right| \left|D^2 v(x-th_p)\right| dxdt \qquad (\operatorname{supp} v \subset B_r(y)) \\
&\le \|\psi\|_{L^2(B_{r+h}(y))} \left\|D^2 v\right\|_{L^2(B_r(y))} \qquad \text{(Cauchy-Schwarz inequality)} \\
&\le C(n)\,\|\psi\|_{L^{2q}(B_{r+h}(y))}\, r^{\frac{n-2+2\alpha}{2}} \left\|D^2 v\right\|_{L^2(B_r(y))}. \qquad \text{(Hölder's inequality)}
\end{aligned} \tag{4.84}
$$

A similar computation yields

$$
\begin{aligned}
\int_{B_{3/4}} \left|\zeta(x) v^{-h_p}(x) dx\right| &\le \|\zeta\|_{L^2(B_{r+h}(y))} \cdot \|Dv\|_{L^2(B_r(y))} \\
&\le C(n) \|\zeta\|_{L^{2q}(B_{r+h}(y))} r^{\frac{n-2+2\alpha}{2}} \cdot C_p |B_r(y)|^{\frac{1}{n}} \left\|D^2 v\right\|_{L^2(B_r(y))}
\end{aligned} \tag{4.85}
$$

where C_p is from the Poincaré inequality [**17**, (7.44)].

Now since $a_0^{ij,kl}$ has an ellipticity constant Λ, plugging the bounds (4.82), (4.83), (4.84), (4.85) into (4.81), we have (collecting dimensional constants into a new $C(n)$)

$$
\begin{aligned}
\Lambda \left\|D^2 v\right\|^2_{L^2(B_r(y))} \le\ & \varepsilon_0 \left\|D^2 f\right\|_{L^2(B_r(y))} \left\|D^2 v\right\|_{L^2(B_r(y))} + C(n) \|\gamma\|_{L^{2q}} r^{\frac{n-2+2\alpha}{2}} \left\|D^2 v\right\|_{L^2(B_r(y))} \\
& + C(n) \|\psi\|_{L^{2q}} r^{\frac{n-2+2\alpha}{2}} \left\|D^2 v\right\|_{L^2(B_r(y))} + C(n) \|\zeta\|_{L^{2q}} r^{\frac{n-2+2\alpha}{2}} \left\|D^2 v\right\|_{L^2(B_r(y))}.
\end{aligned}
$$

Dividing by $\left\|D^2 v\right\|_{L^2(B_r(y))}$ and collecting

$$
\Lambda \left\|D^2 v\right\|_{L^2(B_r(y))} \le \varepsilon_0 \left\|D^2 f\right\|_{L^2(B_r(y))} + C(n) \left(\|\gamma\|_{L^{2q}} + \|\psi\|_{L^{2q}} + \|\zeta\|_{L^{2q}}\right) r^{\frac{n-2+2\alpha}{2}}.
$$

That is

$$
\Lambda^2 \left\|D^2 v\right\|^2_{L^2(B_r(y))} \le 2\varepsilon_0^2 \left\|D^2 f\right\|^2_{L^2(B_r(y))} + K r^{n-2+2\alpha}
$$

for (again modifying $C(n)$)

$$
K = C(n) \left(\|\gamma\|^2_{L^{2q}} + \|\psi\|^2_{L^{2q}} + \|\zeta\|^2_{L^{2q}}\right).
$$

Recalling $f = v + w$ and Corollary 4.6.1

$$
\int_{B_\rho(y)} \left|D^2 f\right|^2 \le 4C_1 \left(\frac{\rho}{r}\right)^n \left\|D^2 f\right\|^2_{L^2(B_r(y))} + (2 + 8C_1) \left\|D^2 v\right\|^2_{L^2(B_r(y))}
$$

for C_1 depending on the ellipticity of $a_0^{ij,kl}$ we see

(4.86)
$$\int_{B_\rho(y)} \left|D^2 f\right|^2 \leq 4C_1 \left(\frac{\rho}{r}\right)^n \left\|D^2 f\right\|^2_{L^2(B_r(y))} + \frac{2\,(2+8C_1)}{\Lambda^2}\left(\varepsilon_0^2 \left\|D^2 f\right\|^2_{L^2(B_r(y))} + Kr^{n-2+2\alpha}\right).$$

Now, we would like to apply Lemma 4.9. To this end, let

$$\begin{aligned}
\phi(\rho) &= \int_{B_\rho} \left|D^2 f\right|^2 \\
A &= 4C_1 \\
\varepsilon &= \frac{2\,(2+8C_1)}{\Lambda^2}\varepsilon_0^2 \\
B &= \frac{2\,(2+8C_1)}{\Lambda^2} K \\
\alpha &= n \\
\beta &= n-2+2\alpha \\
\gamma &= n-1 \\
R &= \frac{1}{2}.
\end{aligned}$$

To be clear, in order to avoid notational double-dipping, the notations appearing on the left hand side of expressions above refer to constants as they are named in Lemma 4.9, while the right hand side refers to constants as they appear previously in this proof so far. We observe that (4.86) can be written using notation on the left side of the above table as

$$\phi(\rho) \leq A\left[\left(\frac{\rho}{r}\right)^\alpha + \varepsilon\right]\phi(r) + Br^\beta \tag{4.87}$$

for all $0 < \rho \leq r < \frac{1}{2}$. There exists a constant $\varepsilon^*(A, \alpha, \beta, \gamma)$ so that (4.87) allows us to conclude that there is a constant $C > 0$ such that

$$\phi(\rho) \leq C\left[\left(\frac{\rho}{r}\right)^{n-1}\phi(r) + Br^{n-2+2\alpha}\right]$$

whenever

$$\frac{2\,(2+8C_1)}{\Lambda^2}\varepsilon_0^2 \leq \varepsilon^*(A, \alpha, \beta, \gamma). \tag{4.88}$$

We pick one such ε_0. Thus

$$\begin{aligned}
\phi(r) &\leq C\left[2^{n-1} r^{n-1}\phi(\frac{1}{2}) + Br^{n-2+2\alpha}\right] \\
&\leq C' r^{n-2+2\alpha}
\end{aligned}$$

where C' depends on $\int_{B_{1/2}} \left|D^2 f\right|^2, \Lambda, n, \alpha$, and $\frac{2(2+8C_1)}{\Lambda^2}K$.

We now have that

$$\int_{B_r} \left|D^2 f\right|^2 \leq C' r^{n-2+2\alpha}.$$

Noting that we chose an arbitrary point in $B_{1/4}(0)$ we may apply Morrey's Lemma [**?**, Lemma 3, page 8] to Df to get the desired conclusion. □

Proof of Theorem 4.11. Applying Proposition 4.6.1 we see that $u \in W^{3,2}$, with estimates controlled by $\|u\|_{W^{2,\infty}}$. The difference quotient $f = u^{h_p}$ satisfies (4.70) where now $f \in W^{2,2}$ with estimates. Using the supremum norms of DF, a^k, b and that $u \in W^{2,\infty}$, the conditions on $\gamma^{jl}, \psi^k, \zeta$ in Proposition 4.6.2 are fulfilled, namely, they are in L^{2q}. In light of Proposition 4.6.2 we conclude $u_p^h \in C^{1,\alpha}$ with the estimate (4.78) where we note that now

$$\begin{aligned}
\|f\|_{W^{1,\infty}} &= \left\| \frac{u(x) - u(x-h_p)}{h} \right\|_{W^{1,\infty}} \\
&= \text{ess sup} \left(\left| \frac{u(x) - u(x-h_p)}{h} \right| + \left| \frac{Du(x) - Du(x-h_p)}{h} \right| \right) \\
&\leq \text{Lip}(u) + \text{Lip}(Du) \\
&\leq \text{ess sup} \left(|u| + |Du| + |D^2 u| \right) \\
&= \|u\|_{W^{2,\infty}}
\end{aligned}$$

Letting $h \to 0$ in (4.78) yields the estimate that holds on $B_{1/4}$. Now take any interior point x_0 and consider the equation

$$\partial_{y_l} \partial_{y_j} \tilde{F}^{jl}(y, Dv, D^2 v) = \partial_{y_k} \tilde{a}^k(y, Dv, D^2 v) - \tilde{b}(y, Dv, D^2 v) \tag{4.89}$$

with

$$\begin{aligned}
\tilde{F}^{jl}(y, Dv, D^2 v) &= F^{jl}(x_0 + ry, rDv(x_0 + ry), D^2 v(x_0 + ry)) \\
\tilde{a}^k(y, Dv, D^2 v) &= ra^k(x_0 + ry, rDv(x_0 + ry), D^2 v(x_0 + ry) \\
\tilde{b}(x, Dv, D^2 v) &= r^2 b(x_0 + ry, rDv(x_0 + ry), D^2 v(x_0 + ry).
\end{aligned}$$

Suppose that

$$B_r(x_0) \subset B_1.$$

Define

$$v(y) = \frac{u(x_0 + ry)}{r^2}.$$

One can check that v satisfies (4.89) on B_1 whenever u satisfies (4.61).

Noting that

$$\frac{\partial \tilde{F}^{jl}}{\partial v_{ik}}(y, Dv, D^2 v) = \frac{\partial F^{jl}}{\partial u_{ik}}(x_0 + ry, rDv(x_0 + ry), D^2 v(x_0 + ry))$$

we see equation (4.89) and the solution v will satisfy the closeness condition (4.73) as well. This rescaling argument allows us to claim an estimate holds at any interior point in B_1. $\square$

4.6.4. Proof of Theorem 4.12. We start by boosting regularity from $C^{2,\alpha}$ to $C^{3,\alpha}$.

PROPOSITION 4.6.3. Suppose that $u \in C^{2,\alpha}(B_1)$ satisfies the Λ-uniform equation (4.61) on B_1, and let $0 < \delta < \alpha$. Then $D^3 u \in C^{\alpha - \delta/2}(B_{1/5})$ and satisfies the following estimate:

$$||D^3 u||_{C^{\alpha-\delta/2}(B_{1/5})} \leq C(||u||_{W^{2,\infty}(B_1)}, \Lambda, \alpha, \delta). \tag{4.90}$$

PROOF. We assume that u enjoys uniform $C^{2,\alpha}$ estimates on $B_{9/10}$. As before we take a difference quotient of the solution u to (4.61) to get (4.70) with $f = u^{h_p}$, for some $h < 1/100$. Since $D^2u \in C^{\alpha}(\bar{B}_{9/10})$, the measurable coefficients are now integrals of Hölder continuous functions, when defined for any $x \in B_{3/4}$ as follows:

$$\beta^{ij,kl}(x) = \int_0^1 \frac{\partial F^{jl}}{\partial u_{ik}}(\xi_0 + t\vec{V})dt \in C^{\alpha}(B_{3/4})$$
$$\gamma^{jl}(x) = \int_0^1 \left(\frac{\partial F^{jl}}{\partial u_k}(\xi_0 + t\vec{V})u_k^{h_p} + \frac{\partial F^{jl}}{\partial x_p}(\xi_0 + t\vec{V})\right) dt \in C^{\alpha}(B_{3/4}).$$

Note also that $\psi^k(x) \in C^{\alpha}(B_{3/4})$. In particular

$$\left|\beta^{ij,kl}(x) - \beta^{ij,kl}(y)\right| \leq C_3 \left|x-y\right|^{\alpha} \ .$$

Again, fixing $y \in B_{1/4}$ for a fixed $r < \frac{1}{2}$ we let w solve the boundary value problem

$$\int_{B_r(y)} \beta^{ij,kl}(0) w_{ij}\eta_{kl}\, dx = 0, \quad \forall \eta \in C_0^{\infty}(B_r(y))$$
$$w = f,\ Dw = Df \quad \text{on } \partial B_r(y)$$

and repeat verbatim the steps leading to (4.81), with $a_0^{ij,kl}$ being replaced by $\beta^{ij,kl}(0)$, again taking $v = f - w \in H_0^2(B_r(y))$. Thus by (4.70)

$$\int_{B_r(y)} \beta^{ij,kl}(0)v_{ij}v_{kl}dx = \int_{B_r(y)} \left(\beta^{ij,kl}(0) - \beta^{ij,kl}(x)\right) f_{ik}v_{jl}dx - \int_{B_r(y)} \left(\gamma^{jl}v_{jl} + \psi^k v_k^{-h_p} + \zeta v^{-h_p}\right) dx.$$

Now this time, we define

$$\Upsilon(r) = \sup\left\{\left|\beta^{ij,kl}(x) - \beta^{ij,kl}(x')\right| \mid x, x' \in B_r(y)\right\} \tag{4.91}$$

which enjoys an estimate from the Hölder estimate on D^2u :

$$\Upsilon(r) \leq C_4 r^{\alpha}. \tag{4.92}$$

Since $v \in H_0^2(B_r(y))$, we have, via integration by parts, that

$$\int_{B_1} \gamma^{jl}(y)v_{jl}(x)\, dx = 0$$
$$\int_{B_1} \psi^k(y)v_k^{-h_p}(x)\, dx = 0$$

and

$$\int_{B_1} \zeta(y)v^{-h_p}(x)dx = \zeta(y)\frac{1}{h}\left(\int_{B_1} v(x-h_p)dx - \int_{B_1} v(x)dx\right) = 0$$

so we may write

$$\int_{B_1} \left(\gamma^{jl}v_{jl} + \psi^k v_k^{-h_p} + \zeta v^{-h_p}\right) dx$$
$$= \int_{B_1} \left(\left[\gamma^{jl}(x) - \gamma^{jl}(y)\right] v_{jl} + \left[\psi^k(x) - \psi^k(y)\right] v_k^{-h_p} + \left[\zeta(x) - \zeta(y)\right] v^{-h_p}\right) dx.$$

Now

$$\int_{B_1} \left|[\gamma^{jl}(x) - \gamma^{jl}(y)]v_{jl}\right| dx \leq \|\gamma(x) - \gamma(y)\|_{L^2(B_r(y))} \left\|D^2 v\right\|_{L^2(B_r(y))} \leq C_5 \left(r^{2\alpha}r^n\right)^{\frac{1}{2}} \left\|D^2 v\right\|_{L^2(B_r)}$$

and similarly,

$$\int_{B_1} \left|[\psi^k(x) - \psi^k(y)]v_k^{-h_p}\right| dx \leq C_6 \left(r^{2\alpha}r^n\right)^{\frac{1}{2}} \left\|D^2 v\right\|_{L^2(B_r(y))}$$

$$\begin{aligned}\int_{B_{1/2}} \left|[\zeta(x) - \zeta(y)]v^{-h_p}(x)\right| dx &\leq C_7 \left(r^{2\alpha}r^n\right)^{\frac{1}{2}} \|Dv\|_{L^2(B_r(y))} \\ &\leq C_7 \left(r^{2\alpha}r^n\right)^{\frac{1}{2}} C_p|B_r|^{\frac{1}{n}} \left\|D^2 v\right\|_{L^2(B_r(y))} \\ &\leq C_p'C_7 \left(r^{2\alpha}r^n\right)^{\frac{1}{2}} \left\|D^2 v\right\|_{L^2(B_r(y))}\end{aligned}$$

where C_p is from the Poincaré inequality [**17**, (7.44)], $C_p' = C_p|B_1|$, and

$$\begin{aligned}|\gamma(x) - \gamma(y)| &\leq C_5 r^\alpha \\ |\psi(x) - \psi(y)| &\leq C_6 r^\alpha \\ |\zeta(x) - \zeta(y)| &\leq C_7 r^\alpha.\end{aligned}$$

(Recall the components of these functions are smooth as functions of D^2u so these will be Hölder continuous now as D^2u is Hölder continuous.) Note that, for Λ the ellipticity constant for β we have

$$\Lambda \left\|D^2 v\right\|^2_{L^2(B_r(y))} \leq \Upsilon(r) \left\|D^2 f\right\|_{L^2(B_r(y))} \left\|D^2 v\right\|_{L^2(B_r(y))} + (C_5+C_6+C_p'C_7) \left(r^{2\alpha}r^n\right)^{\frac{1}{2}} \left\|D^2 v\right\|_{L^2(B_r(y))}.$$

That is

$$\left\|D^2 v\right\|_{L^2(B_r(y))} \leq \frac{1}{\Lambda}\left\{\Upsilon(r) \left\|D^2 f\right\|_{L^2(B_r(y))} + (C_5 + C_6 + C_p'C_7) \left(r^{2\alpha}r^n\right)^{\frac{1}{2}}\right\}$$

or

$$\left\|D^2 v\right\|^2_{L^2(B_r(y))} \leq \frac{2}{\Lambda^2}\left\{\Upsilon^2(r) \left\|D^2 f\right\|^2_{L^2(B_r(y))} + (C_5 + C_6 + C_p'C_7)^2 r^{2\alpha}r^n\right\}.$$

Using Corollary 4.6.1, for any $0 < \rho \leq r$ we get

$$\begin{aligned}\int_{B_\rho(y)} \left|D^2 f - (D^2 f)_\rho\right|^2 &\leq 4C_2 \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(y)} \left|D^2 f - (D^2 f)_r\right|^2 + (8 + 16C_2) \int_{B_r(y)} \left|D^2 v\right|^2 \\ &\leq 4C_2 \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(y)} \left|D^2 f - (D^2 f)_r\right|^2 \\ &\quad + \frac{2}{\Lambda^2}\left\{\Upsilon^2(r) \left\|D^2 f\right\|^2_{L^2(B_r(y))} + (8 + 16C_2)(C_5 + C_6 + C_p'C_7)^2 r^{2\alpha}r^n\right\}.\end{aligned} \tag{4.93}$$

Next, to get decay on the $\left\|D^2 f\right\|^2_{L^2}$ factor, we will find an $r_0 < 1/2$ to be determined, such that for $r < r_0$ we have

$$\int_{B_\rho(y)} \left|D^2 f\right|^2 \leq C_9 \rho^{n-\delta}$$

where $\delta = 1 - \tilde{\alpha} < \alpha$. In order to do this, first observe (4.86). We may replace ε_0 by $\Upsilon^2(r)$ by virtue of (4.91). We let $\tilde{\alpha} = 1 - \delta$, which will result in a different value $\tilde{q}$ in the derivation leading up to (4.86). By repeating the derivation of (4.86) replacing only ε_0 by $\Upsilon^2(r)$, α by $\tilde{\alpha} = 1 - \delta$, and K by a $\tilde{K}$ determined by the different norms arising from now the exponent $\tilde{q} = n/(2 - 2\tilde{\alpha})$, we get

$$\begin{aligned}\int_{B_\rho(y)} \left|D^2 f\right|^2 &\le 4C_1 \left(\frac{\rho}{r}\right)^n \left\|D^2 f\right\|^2_{L^2(B_r(y))} + \frac{2\,(2+8C_1)}{\Lambda^2}\left(\Upsilon^2(r)\left\|D^2 f\right\|^2_{L^2(B_r(y))} + \tilde{K} r^{n-2+2\tilde{\alpha}}\right)\\ &= \left(4C_1\left(\frac{\rho}{r}\right)^n + \frac{2\,(2+8C_1)}{\Lambda^2}\Upsilon^2(r)\right)\left\|D^2 f\right\|^2_{L^2(B_r(y))} + \frac{2\,(2+8C_1)}{\Lambda^2}\tilde{K} r^{n-2+2\tilde{\alpha}}.\end{aligned}$$

As before, denote

$$\begin{aligned}\phi(\rho) &= \int_{B_\rho(y)} \left|D^2 f\right|^2\\ A &= 4C_1\\ \varepsilon &= \frac{2\,(2+8C_1)}{\Lambda^2}\Upsilon^2(r_0)\\ B &= \frac{2\,(2+8C_1)}{\Lambda^2}\tilde{K}\\ \alpha &= n\\ \beta &= n - 2\delta\\ \gamma &= n - \delta.\end{aligned}$$

Now by (4.92) and Lemma 4.9, there exists r_0 small enough such that

$$\frac{2\,(2+8C_1)}{\Lambda^2}\Upsilon^2(r_0) \le \varepsilon^*\left(A, \alpha, \beta, \gamma\right),$$

for the ε^* provided by Lemma 4.9, and we have for $\rho < r_0$

$$\phi(\rho) \le C_8\left\{\left(\frac{\rho}{r}\right)^{n-\delta}\phi(r) + B r^{n-2\delta}\right\}.$$

Hence

$$\begin{aligned}\phi(\rho) &\le C_8 \frac{1}{r_0^{n-\delta}}\rho^{n-\delta}\left\|D^2 f\right\|_{L^2(B_{r_0})} + B\rho^{n-2\delta}\\ &\le C_9 \rho^{n-\delta}.\end{aligned}$$

Turning back to (4.93), we now have, for $r < r_0$

$$\begin{aligned}\int_{B_\rho(y)} \left|D^2 f - (D^2 f)_\rho\right|^2 &\le 4C_2\left(\frac{\rho}{r}\right)^{n+2}\int_{B_r(y)}\left|D^2 f - (D^2 f)_r\right|^2\\ &\quad + \frac{2}{\Lambda^2}\left\{\Upsilon^2(r) C_9 r^{n-\delta} + (C_5 + C_6 + C_p' C_7)^2 r^{2\alpha} r^n\right\}\\ &\le 4C_2\left(\frac{\rho}{r}\right)^{n+2}\int_{B_r(y)}\left|D^2 f - (D^2 f)_r\right|^2 + \frac{2}{\Lambda^2} C_4 C_9 r^{2\alpha+n-\delta}\\ &\quad + \frac{2}{\Lambda^2}(C_5 + C_6 + C_p' C_7)^2 r^{2\alpha} r^n.\end{aligned}$$

Now we can apply Lemma 4.9 yet again, this time with

$$\begin{aligned}\phi(\rho) &= \int_{B_\rho(y)} \left|D^2 f - (D^2 f)_\rho\right|^2\\ A &= 4C_2\\ \alpha &= n+2\\ B &= \frac{2}{\Lambda}\left[C_4C_9 + C(C_5 + C_6 + C_p' C_7)^2\right]\\ \beta &= n + 2\alpha - \delta\\ \gamma &= n + 2\alpha.\end{aligned}$$

We then conclude that

$$\int_{B_r(y)} \left|D^2 f - (D^2 f)_r\right|^2 \leq C_{10} r^{n+2\alpha-\delta}$$

for $r < r_0$ (and will be necessarily true for $r \in [r_0, \frac{1}{2}]$ as well, perhaps modifying C_{10}). Noting that this applies for any $y \in B_{1/4}$ we apply [**19**, Theorem 3.1] to $D^2 f$ to conclude that $D^2 f \in C^{(2\alpha-\delta)/2}(B_{1/5})$. Noting $f = u^{h_p}$ we may take a limit and conclude that u must enjoy uniform $C^{3,\alpha}$ estimates on $B_{1/5}$. □

We now apply the regularity bootstrapping procedure as in [**4**] to obtain smoothness.

Proof of Theorem 4.12. We may scale the estimate provided in Proposition 4.6.3 to get $u \in C^{3,\alpha}(B_r)$ for any $r < 1$. Letting $f = u^{h_{p_1}}$ we may apply the dominated convergence theorem while passing the limit as $h \to 0$ to the equation (4.70) and conclude that, for $v = u_{p_1}$

$$\int_{B_1} \left(\beta^{ij,kl} v_{ik}\eta_{jl} + \gamma^{jl}\eta_{jl} - \psi^k \eta_{kp_1} - \zeta\eta_{p_1}\right) dx = 0$$

where

$$\beta^{ij,kl}(x) = \frac{\partial F^{jl}}{\partial u_{ik}}(x, Du, D^2u) \in C^{1,\alpha}(B_r)$$

$$\gamma^{jl}(x) = \frac{\partial F^{jl}}{\partial u_k}(x, Du, D^2u)) f_k(x) + \frac{\partial F^{jl}}{\partial x_{p_1}}(x, Du, D^2u) \in C^{1,\alpha}(B_r).$$

Noting that the functions ψ^k, ζ are $C^{1,\alpha}$ when u in $C^{3,\alpha}$, we can integrate by parts in the last two terms to get

$$\int_{B_1} \left(\beta^{ij,kl} v_{ik}\eta_{jl} + \gamma^{jl}\eta_{jl} + \partial_{x_{p_1}}\psi^k\eta_k + \partial_{x_{p_1}}\zeta\eta\right) dx = 0.$$

Following the difference quotient procedure leading to (4.70), this time in the direction p_2

$$\int_{B_1} \left([\beta^{ij,kl} v_{ik} + \gamma^{jl}]^{h_p}\eta_{jl} + \partial_{x_{p_1}}\psi^k\eta_k^{-h_{p_2}} + \partial_{x_{p_1}}\zeta\eta^{-h_{p_2}}\right) dx = 0.$$

Expanding

$$\int_{B_1} \left[\left(\left(\beta^{ij,kl}\right)^{h_{p_2}} v_{ik} + \left(\gamma^{jl}\right)^{h_{p_2}} + \left(\beta^{ij,kl}\right) v_{ik}^{h_{p_2}}\right)\eta_{jl} + \partial_{x_{p_1}}\psi^k\eta_k^{-h_{p_2}} + \partial_{x_{p_1}}\zeta\eta^{-h_{p_2}}\right] dx = 0.$$

Observe that each of the terms $\left(\beta^{ij,kl}\right)^{h_{p_2}} v_{ik}, \left(\gamma^{jl}\right)^{h_{p_2}}, \partial_{x_{p_1}}\psi^k, \partial_{x_{p_1}}\zeta$ are C^α with uniform estimates on B_r.

So letting

$$\begin{aligned}\tilde{\gamma}^{jl} &= \left(\beta^{ij,kl}\right)^{h_{p_2}} v_{ik} + \left(\gamma^{jl}\right)^{h_{p_2}}\\ \tilde{\psi}^k &= \partial_{x_{p_1}} \psi^k\\ \tilde{\zeta} &= \partial_{x_{p_1}} \zeta\end{aligned}$$

we see that $\tilde{v} = v^{h_{p_2}}$ satisfies

$$\int_{B_1} \left(\beta^{ij,kl}\tilde{v}_{ik}\eta_{jl} + \tilde{\gamma}^{jl}\eta_{jl} + \tilde{\psi}^k \eta_k^{-h_{p_2}} + \tilde{\zeta}\eta^{-h_{p_2}}\right) dx = 0 \tag{4.94}$$

which is of identical form as equation (4.70). By our Λ-uniform assumption on (4.61), the above equation is uniformly elliptic, as β has not changed. Now we apply verbatim the proof of Proposition 4.6.3, noting that all coefficients in sight are Hölder continuous, we get $D^2\tilde{v} \in C^{\alpha'}$. Since $\tilde{v}$ is the difference quotient of a derivative of u, we may take $h \to 0$ and conclude that $u_{p_1p_2} \in C^{2,\alpha'}(B_r)$ with estimates for any $\alpha' < \alpha$, for $r < 1$, thus $u \in C^{4,\alpha'}(B_r)$.

Note that when bootstrapping from $C^{m-1,\alpha}$ to $C^{m,\alpha'}$ via (4.94) for

$$\tilde{v} = u^{h_{p_{m-2}}}_{p_1p_2\cdots p_{m-3}}$$

we may take the limit of (4.94) to get

$$\int_{B_1} \left(\beta^{ij,kl}\tilde{v}_{p_{m-2}ik}\eta_{jl} + \tilde{\gamma}^{jl}\eta_{jl} - \tilde{\psi}^k \eta_{kp_{m-2}} - \tilde{\zeta}\eta_{p_{m-2}}\right) dx = 0$$

but now $\tilde{\psi}^k, \tilde{\zeta} \in C^{m-3,\alpha}$ so we may integrate by parts and take another difference quotient in another direction p_{m-1} to obtain another expression very similar to (4.94), again with Hölder regularity holding for all the coefficients and one higher order of derivative arising in $\tilde{v}$. Repeating the proof of Proposition 4.6.3, we conclude $u_{p_1p_2\cdots p_{m-1}} \in C^{2,\alpha}(B_r)$. In this way we can obtain estimates of any order. ☐

Proof of Theorem 4.4. Observing that condition (4.60) is equivalent to condition (4.73), the result follows immediately from Theorems 4.11 and 4.12. ☐

4.7. Derivation of the Euler-Lagrange equations on a Riemannian ball

We start by deriving the equation for a manifold that is volume stationary among gradient graphs.

DEFINITION 4.14. Let Γ be the set of gradient graphs of functions $u \in C^{1,1}(B_1)$ with $Du(0) = 0$ and $\|Du\|_{L^\infty} \leq 1$, where $B_1 \subset \mathbb{R}^n$, and

$$\Gamma(u) = \{(x, Du(x)) : x \in B_1\} \subset B_2^{2n}.$$

Let h be a Riemannian metric on the euclidean ball B_2^{2n} in $\mathbb{R}^{2n}$ with $h(0) = \delta_0$. We say that $\Gamma(u)$ is *volume stationary in* (B_1, h) *among gradient graphs in* Γ, if

$$\frac{d}{dt}\mathrm{Vol}_h(\Gamma(u + t\eta))\bigg|_{t=0} = 0, \quad \forall \eta \in C_c^\infty(B_1)$$

where Vol_h is volume measured in h.

The volume functional Vol_h acting on Γ is given by

$$\mathrm{Vol}_h(\Gamma(u)) = \int_{B_1} \sqrt{\det(g_{ij}(x))}\, dx$$

where, in the standard euclidean basis $\{e_1, \ldots, e_n, e_{1+n}, \ldots, e_{2n}\}$ of $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, the induced metric g from h on $\Gamma(u) \subset \mathbb{R}^n \times \mathbb{R}^n$ is

$$\begin{aligned} g_{ij} &= h(e_i + \sum_k u_{ki} e_{k+n}, e_j + \sum_l u_{lj} e_{l+n}) \\ &= h_{ij} + \sum_k u_{ki} h_{k+n,j} + \sum_l u_{lj} h_{l+n,i} + \sum_{k,l} u_{ki} u_{lj} h_{l+n,k+n} \end{aligned} \tag{4.95}$$

with $1 \leq i, j \leq n$. We may write

$$\begin{aligned} h_{ij}(x, Du(x)) &= \delta_{ij} + \mathcal{A}_{ij}(x, Du(x)) \\ h_{l+n,k+n}(x, Du(x)) &= \delta_{kl} + \mathcal{B}_{kl}(x, Du(x)) \\ h_{k+n,j}(x, Du(x)) &= \mathcal{C}_{kj}(x, Du(x)). \end{aligned}$$

Note that $\mathcal{C}$ need not be symmetric, while $\mathcal{A}$ and $\mathcal{B}$ are symmetric. In block diagonal form of matrices we have

$$h = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} \mathcal{A} & \mathcal{C} \\ \mathcal{C}^T & \mathcal{B} \end{pmatrix}. \tag{4.96}$$

Now we have

$$g_{ij} = \delta_{ij} + u_{ik}\delta^{kl}u_{lj} + \mathcal{A}_{ij} + u_{im}u_{pj}\delta^{mk}\delta^{pl}\mathcal{B}_{kl} + u_{ki}\delta^{kl}\mathcal{C}_{lj} + u_{kj}\delta^{kl}\mathcal{C}_{li}. \tag{4.97}$$

Therefore, as a matrix-valued function defined on (x, Du), the induced metric g is quadratic in D^2u. In particular,

$$\left|\frac{\partial g_{ij}}{\partial u_{kl}}\right| \leq C(n, \|h\|_{C^0}) \sup \left|D^2 u\right| + C(n) \sup_{i,j} |h_{i+n,j}|, \tag{4.98}$$

where (and in sequel) we set

$$\|h\|_{C^0} = \sup_{B_2} \{|h_{pq}|, 1 \leq p, q \leq 2n\}. \tag{4.99}$$

Now, we compute the first variation of Vol_h. Take a variation generated by $\eta \in C_c^\infty(B_1)$ for the path

$$\gamma[t](x) = u(x) + t\eta(x), \tag{4.100}$$

which varies the manifold $\Gamma(u)$ along the y-direction in B_2^{2n}. Denote the induced metric from h on $\Gamma(u + t\eta)$ by $g(t)$. Straightforwardly,

$$\begin{aligned} g_{ij}(t) = \delta_{ij} &+ (u_{ik} + t\eta_{ik})\, \delta^{kl}\, (u_{lj} + t\eta_{lj}) \\ &+ \mathcal{A}_{ij}(x, Du(x) + tD\eta(x)) \\ &+ (u_{im} + t\eta_{im})\, (u_{pj} + t\eta_{pj})\, \delta^{mk}\delta^{pl}\mathcal{B}_{kl}(x, Du(x) + tD\eta(x)) \\ &+ (u_{ki} + t\eta_{ki})\, \delta^{kl}\mathcal{C}_{lj}(x, Du(x) + tD\eta(x)) \\ &+ (u_{kj} + t\eta_{kj})\, \delta^{kl}\mathcal{C}_{li}(x, Du(x) + tD\eta(x)). \end{aligned}$$

Next, we compute the derivative at $t=0$

$$\begin{aligned}\left.\frac{d}{dt}g_{ij}(t)\right|_{t=0} &= \left(u_{ik}\delta^{kl}\eta_{lj}+\eta_{ik}\delta^{kl}u_{lj}\right)+\left(u_{im}\eta_{pj}+\eta_{im}u_{pj}\right)\delta^{mk}\delta^{pl}\mathcal{B}_{kl}(x,Du(x))\\ &\quad+\eta_{ki}\delta^{kl}\mathcal{C}_{lj}(x,Du(x))+\eta_{kj}\delta^{kl}\mathcal{C}_{li}(x,Du(x))\\ &\quad+\left\{\begin{array}{c} D_y\mathcal{A}_{ij}(x,Du(x))\\ +u_{ki}\delta^{kl}D_y\mathcal{C}_{lj}(x,Du(x))+u_{kj}\delta^{kl}D_y\mathcal{C}_{li}(x,Du(x))\\ +u_{im}u_{pj}\delta^{mk}\delta^{pl}D_y\mathcal{B}_{kl}(x,Du(x))\end{array}\right\}\cdot D\eta.\end{aligned}$$

Then

$$\begin{aligned}\left.\frac{d}{dt}\mathrm{Vol}_h(\gamma[t])\right|_{t=0} &= \left.\int_{B_1}\frac{1}{2}\sqrt{g[t]}g^{ij}[t]\ \frac{d}{dt}g_{ij}[t]dx\right|_{t=0}\\ &=\frac{1}{2}\int_{B_1}\sqrt{g}g^{ij}\left(u_{ik}\delta^{kl}\eta_{lj}+\eta_{ik}\delta^{kl}u_{lj}+\left(u_{im}\eta_{pj}+\eta_{im}u_{pj}\right)\delta^{mk}\delta^{pl}\mathcal{B}_{kl}(x,Du(x))\right)dx\\ &\quad+\frac{1}{2}\int_{B_1}\sqrt{g}g^{ij}\left(\eta_{ki}\delta^{kl}\mathcal{C}_{lj}(x,Du(x))+\eta_{kj}\delta^{kl}\mathcal{C}_{li}(x,Du(x))\right)dx\\ &\quad+\frac{1}{2}\int_{B_1}\sqrt{g}g^{ij}\left\{\begin{array}{c} D_y\mathcal{A}_{ij}(x,Du(x))\\ +u_{ki}\delta^{kl}D_y\mathcal{C}_{lj}(x,Du(x))+u_{kj}\delta^{kl}D_y\mathcal{C}_{li}(x,Du(x))\\ +u_{im}u_{pj}\delta^{mk}\delta^{pl}D_y\mathcal{B}_{kl}(x,Du(x))\end{array}\right\}\cdot D\eta\ dx.\end{aligned}$$

Dropping dependencies for easier presentation, and making use of symmetries

$$\begin{aligned}\left.\frac{d}{dt}\mathrm{Vol}_h(\gamma[t])\right|_{t=0} &= \int_{B_1}\sqrt{g}g^{ij}\left(u_{ik}\delta^{kl}+u_{im}\delta^{mq}\delta^{lk}\mathcal{B}_{qk}\right)\eta_{lj}dx+\int_{B_1}\sqrt{g}g^{ij}\eta_{kj}\delta^{kl}\mathcal{C}_{li}dx\\ &\quad+\frac{1}{2}\int_{B_1}\sqrt{g}g^{ij}\ \left\{D_y\mathcal{A}_{ij}+2u_{ik}\delta^{kl}D_y\mathcal{C}_{lj}+u_{im}u_{pj}\delta^{mk}\delta^{pl}D_y\mathcal{B}_{kl}\right\}\cdot D\eta\ dx.\end{aligned}$$

Then we arrive at the Euler-Lagrange equation of Vol_h for variations in Γ:

LEMMA 4.15. *For $1\le i,j,k,l\le n$, let*

$$\begin{aligned}a^{ij,kl}(x,Du,D^2u) &= \sqrt{g}g^{ij}\delta^{kl}+\sqrt{g}g^{ij}\mathcal{B}_{lk} &&(4.101)\\ b^{jk}(x,Du,D^2u) &= \sqrt{g}g^{ij}\mathcal{C}_{ki}\\ c^{k}(x,Du,D^2u) &= \frac{1}{2}\sqrt{g}g^{ij}\left(D_{y^k}\mathcal{A}_{ij}+2u_{ik}D_{y^k}\mathcal{C}_{kj}+u_{ik}u_{lj}D_{y^k}\mathcal{B}_{kl}\right)\end{aligned}$$

$$F^{jl}(x,Du,D^2u)=a^{ij,kl}u_{ik}+b^{jl} \tag{4.102}$$

Then the Euler-Lagrange equation of Vol_h under variations in Γ is

$$\int F^{jl}\eta_{jl}+c^k\eta_k\,dx=0,\quad \textit{for all } \eta\in C_c^\infty(B_1). \tag{4.103}$$

LEMMA 4.16. *For any $s>0$ there exists $\varepsilon_1(s,n)<1$ depending only on s and n such that if*

$$\begin{aligned}h(0)&=I_{2n}\\ \left\|D^2u\right\|_{L^\infty(B_1)}&\le\varepsilon_1\\ \left\|Dh\right\|_{L^\infty(B_1)}&\le\varepsilon_1\end{aligned}$$

all hold we have

$$\left\|a^{ij,kl}(x,Du(x),D^2u(x))-\delta^{ij}\delta^{kl}\right\|_{L^\infty(B_1)}<s.$$

(Here and below the norms $\|\cdot\|$ *are defined as in* (4.99)*.)*

PROOF. From (4.101)

$$a^{ij,kl} = \delta^{ij}\delta^{kl} + \left(\sqrt{g}g^{ij} - \delta^{ij}\right)\delta^{kl} + \sqrt{g}g^{ij}\mathcal{B}_{lk}$$

It will be convenient to define the following function

$$\omega(z) = \sup_{M \in S^{n\times n}, \|M\| \leq z} \left\| \sqrt{\det(I+M)}\,(1+M)^{ij} - \delta^{ij} \right\|$$

which is clearly continuous for small values of z and vanishes at $z=0$. This allows us to write

$$\left\|a^{ij,kl} - \delta^{ij}\delta^{kl}\right\| \leq \omega\left(\|g - \delta_{ij}\|\right) + \left(1 + \omega\left(\|g-\delta_{ij}\|\right)\right)|\mathcal{B}_{lk}|$$

Noting from (4.96)

$$\mathcal{A}(0) = 0$$
$$\mathcal{B}(0) = 0$$
$$\mathcal{C}(0) = 0$$

and

$$\sup_{B_2^{2n}} \{|\mathcal{A}|, |\mathcal{B}|, |\mathcal{C}|\} \leq 2\varepsilon_1$$

we may inspect (4.97) and see that

$$\|g_{ij} - \delta_{ij}\| \leq C(n)\left(\varepsilon_1 + 3\varepsilon_1^2 + \varepsilon_1^3\right)$$

Then

$$\left\|a^{ij,kl} - \delta^{ij}\delta^{kl}\right\| \leq \omega\left(C(n)\varepsilon_1\right) + \left(1 + \omega\left(C(n)\varepsilon_1\right)\right)\varepsilon_1$$

Because ω is continuous near 0 we choose an ε_1 such that

$$\left\|a^{ij,kl} - \delta^{ij}\delta^{kl}\right\| < s.$$

□

THEOREM 4.17. *Suppose that* $u(x)$ *is a* $C^{1,1}$ *function on* B_1 *such that* $Du = 0, D^2u(0) = 0$ *and*

$$\left\|D^2 u\right\|_{L^\infty(B_1)} \leq \varepsilon_1(\varepsilon_0, n)$$

for ε_1 *determined by Lemma 4.16 and* $\varepsilon_0(\frac{1}{2}, n)$ *determined by (4.60). If* $\Gamma(u) = \{(x, Du)\}$ *is volume stationary among gradient graphs over the* x*-plane in for a Riemannian metric* h *on the euclidean ball* B_2^{2n} *in* $\mathbb{R}^{2n}$*, then* u *is smooth in a neighborhood of* 0.

PROOF. We start by perfoming a rescaling. Consider the map

$$S : B_{2R}^{2n} \to B_2^{2n}$$

given by

$$S(x,y) = \left(\frac{x}{R}, \frac{y}{R}\right).$$

This gives us a metric $\tilde{h}$ on B_{2R}^{2n} via

$$\tilde{h} = S^* h$$

which satisfies

$$\left\|D\tilde{h}\right\| = \frac{1}{R}\|Dh\|.$$

In particular, by choosing R large, we can scale so that

$$\left\|D\tilde{h}\right\| \leq \varepsilon_1(\varepsilon_0).$$

Notice that by letting

$$\tilde{u} = R^2 u\left(\frac{x}{R}\right) \text{ on } B_R$$

the gradient graph $\tilde{u}$ is precisely the pullback of the gradient graph of u via the scaling S :

$$\begin{aligned} S\left(x, D\tilde{u}(x)\right) &= \frac{1}{R} \cdot \left(x, RDu\left(\frac{x}{R}\right)\right) \\ &= \left(\frac{x}{R}, Du\left(\frac{x}{R}\right)\right). \end{aligned}$$

Note also that

$$D^2\tilde{u}(x) = D^2 u\left(\frac{x}{R}\right) \tag{4.104}$$

will satisfy the same bounds. Now restricting $\tilde{h}$ to B_2^{2n} and $\tilde{u}$ to B_1 we can apply Lemma 4.16, observe (4.102) and conclude that the Euler-Lagrange equation (4.103) satisfies the condition in Theorem 4.4. Thus $\tilde{u}$ is smooth inside B_1. Rescaling, we see that u is smooth insde $B_{1/R}$. □

4.8. HSL submanifolds in a symplectic manifold

The following is our main regularity result:

THEOREM 4.18. *Let (M, ω, h) be a symplectic manifold. Suppose that L is a C^1 Hamiltonian stationary Lagrangian submanifold (possibly open but without boundary) embedded in M. Then L is smooth.*

PROOF. Fix an arbitrary point p in $L \subset M$. By Proposition **??**, we can choose Darboux coordinates around $\Upsilon_{p,v}$ at p, choosing v so that $d\Upsilon_{p,v}|_0\left(\mathbb{R}^n\right) = T_pL$. Now the submanifold

$$L_0 = \Upsilon_{p,v}^{-1}(L \cap \Upsilon_{p,v}\left(B_\varepsilon^{2n}\right)) \subset B_\varepsilon^{2n} \subset \mathbb{C}^n$$

is Lagrangian and Hamiltonian stationary in $\left(B_\varepsilon^{2n}, \Upsilon_{p,v}^* h, \omega_0\right)$. As a Lagrangian submanifold tangent to $\mathbb{R}^n$ at 0, L_0 must be represented in a neighborhood of 0 as the gradient graph of function u satisfying $Du(0) = 0$ and $D^2u(0) = 0$. Because L_0 is C^1, the Hessian D^2u is continuous: We can choose $0 < \varepsilon_2 < \varepsilon$ if necessary such that

$$\left\|D^2 u\right\|_{C^0(B_{\varepsilon_2/2})} < \varepsilon_1$$

and so that the projection of $L_0 \cap B_\varepsilon^{2n}$ to $\mathbb{R}^n$ contains $B_{\varepsilon_2/2}$, for the ε_1 provided by Theorem 4.17. Next we make use of the dilation map in Proposition **??** (5), choosing $t < \frac{1}{2}\varepsilon_2$, small enough so that

$$\|\partial h_{p,v}^t\|_{C^0} \leq C_1 t < \varepsilon_1.$$

We now have the following: A rescaled submanifold $\tilde{L}_0$, still Lagrangian, and Hamiltonian stationary with respect to the metric $h_{p,v}^t$, which satisfies

$$\left\|Dh_{p,v}^t\right\| < \varepsilon_1$$

so that the projection of $\tilde{L}_0 \cap B_{2\varepsilon/t}^{2n}$ to $\mathbb{R}^n$ contains B_1. Noting that the scaling does not change the Hessian D^2u (recall (4.104)), we see that we are in the setting of Theorem 4.17. Since ω_0 is the standard symplectic form, the condition of being Lagrangian Hamiltonian stationary is equivalent to being critical for gradient graphs. Theorem 4.17 now gives us that u is smooth in a neighborhood of 0, so L is smooth in a neighborhood of p. As p was arbitrary, L is smooth everywhere. $\square$

4.9. Compactness of space of HSL submanifolds

The discussion will be based on works of Chen-Warren [**9**] when the ambient space is $\mathbb{C}^n$, Chen-Ma [**10**] for HSL surfaces in a Kähler surface, and for ambient space is a symplectic manifold [**11**].

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