15 marks 1. Find the general solution of the following system of equations. Write your answer in parametric vector form.

$$\begin{cases} x_1 + 2x_2 + 2x_4 &= 1 \\ -x_1 + x_2 + 6x_3 + 4x_4 &= 2 \\ 2x_1 + x_2 - x_3 + x_4 &= 0 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 1 \\ -1 & 1 & 6 & 4 & 2 \\ 2 & 1 & -1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 1 \\ 0 & 3 & 6 & 6 & 3 \\ 0 & -3 & -1 & -3 & -2 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 5 & 3 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & -4 & -2 & -1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 5 & 3 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & -4 & -2 & -1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 3/5 & 1/5 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2/5 & -1/5 \\ 0 & 1 & 0 & 4/5 & 3/5 \\ 0 & 0 & 1 & 3/5 & 1/5 \end{array}\right)$$

$$\begin{cases} x_1 = -\frac{1}{5} - \frac{2}{5}x_4 \\ x_2 = \frac{3}{5} - \frac{4}{5}x_4 \\ x_3 = \frac{1}{5} - \frac{3}{5}x_4 \end{cases}$$

pramametrized vector form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} \\ \frac{3}{5} \\ \frac{1}{5} \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -\frac{2}{5} \\ -\frac{4}{5} \\ -\frac{3}{5} \\ 1 \end{pmatrix}$$

10 marks 2. Are the following 3 vectors linearly independent? Explain why or why not.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}.$$

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 1 \\ 0 & -5 & -2 \\ 0 & -2 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 2/5 \\ 0 & 1 & 3/2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} \boxed{1} & 3 & 1 \\ 0 & \boxed{1} & 2/5 \\ 0 & 0 & \boxed{11/10} \end{pmatrix}$$

pivots.

$\therefore$ every column of A is a pivot column

$\therefore$ v_1, v_2, v_3 are linearly independent.

10 marks 3. Consider the following matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & -1 & 1 \\ 2 & 4 & c \\ 2 & 2 & 6 \end{pmatrix}.$$

For what values of c is null(A) equal to $\{0\}$ where $\mathbf{0}$ is the zero vector?

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & -1 & 1 \\ 2 & 4 & c \\ 2 & 2 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 4 & c-2 \\ 0 & 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 4 & c-2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & c-10 \\ 0 & 0 & 0 \end{pmatrix}$$

① If $c \neq 10$, $A \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

For $Ax = 0$

$\therefore$ $x_1 = x_2 = x_3 = 0$. $\mathrm{Nul}(A) = \{0\}$

② If $c = 10$, $A \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

then $\begin{cases} x_1 = -x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \end{cases}$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$, for any x_3 in $\mathbb{R}$, so $\mathrm{Nul}(A) \neq \{0\}$.

Answer: any $c \neq 10$

10 marks 4. Let $\mathbf{v}_1 = \begin{pmatrix}1\\2\\1\end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix}2\\0\\-1\end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix}0\\4\\3\end{pmatrix}$ and $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Find a basis for V (make sure to show your work).

$$A = \begin{pmatrix}1 & 2 & 0\\2 & 0 & 4\\1 & -1 & 3\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 2 & 0\\0 & -4 & 4\\0 & -3 & 3\end{pmatrix} \longrightarrow \begin{pmatrix}\boxed{1} & 2 & 0\\0 & \boxed{1} & -1\\0 & 0 & 0\end{pmatrix}$$

pivots

So v_1, v_2 are pivot columns of A

$V = \text{span}\{v_1, v_2, v_3\} = \text{Col}(A)$

$\{v_1, v_2\}$ forms a basis of V.

10 marks 5. For each of the following state whether the statement is TRUE or FALSE (no justification is necessary).

(1) If vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ in $\mathbb{R}^3$ are linearly independent then so must be the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

T

(2) Let $v_1, ..., v_k$ be vectors in $\mathbb{R}^n$. If $k > n$ then $v_1, ..., v_k$ are linearly dependent.

T

(3) For a consistent linear system $Ax = b$ where A is a matrix with m rows and n columns and b is a nonzero vector in $\mathbb{R}^m$, the solution set is a subspace of $\mathbb{R}^n$.

F

(4) If A is a matrix with m rows and n columns, then $\text{rank}(A) \leq m$ and $\text{rank}(A) \leq n$.

T

(5) Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis of a subspace W of $\mathbb{R}^3$. Then the $\mathcal{B}$-coordinate vector $[\mathbf{v}]_{\mathcal{B}}$ of a vector $\mathbf{v}$ in W is a vector in $\mathbb{R}^3$.

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