PROBLEM 1. [5 Pts] Find the general solution of the system of equations. Write your answer in parametric form.

$$\begin{cases} x_1 - 5x_2 - 9x_3 + 8x_3 = -7 \\ \qquad x_2 + 3x_3 - 4x_4 = 2 \\ \qquad 2x_2 + 6x_3 - 8x_4 = 4 \end{cases}$$

Augmented matrix

$$\begin{bmatrix} 1 & -5 & -9 & 8 & -7 \\ 0 & 1 & 3 & -4 & 2 \\ 0 & 2 & 6 & -8 & 4 \end{bmatrix} \longrightarrow$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 6 & -12 & 3 \\ 0 & 1 & 3 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \iff \begin{matrix} x_1 + 6x_3 - 12x_4 = 3 \\ x_2 + 3x_3 - 4x_4 = 2 \end{matrix}$$

Free: x_3, x_4 Basic: x_1, x_2

Solution:
$$\begin{cases} x_1 = 3 - 6x_3 + 12x_4 \\ x_2 = 2 - 3x_3 + 4x_4 \\ x_3 = \text{free} \\ x_4 = \text{free} \end{cases}$$

4pts

Parametric form

$$\begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \cdot \begin{bmatrix} -6 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 12 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

1 pt

PROBLEM 2. [9 Pts] For each matrix below, determine whether its columns span $\mathbb{R}^3$. Give reasons for your answers.

a. $\begin{bmatrix} -3 & 11 \\ 2 & -4 \\ -4 & 3 \end{bmatrix}$ **b.** $\begin{bmatrix} 2 & 7 & 0 \\ -4 & -6 & 5 \\ -2 & 9 & 10 \end{bmatrix}$ **c.** $\begin{bmatrix} 1 & 0 & -5 & 8 \\ 0 & 2 & -3 & 5 \\ -2 & 6 & 1 & 4 \end{bmatrix}$

a. Two columns do not span $\mathbb{R}^3$

b. Row reduce:

$$\begin{bmatrix} 2 & 7 & 0 \\ -4 & -6 & 5 \\ -2 & 9 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 7 & 0 \\ 0 & 8 & 5 \\ 0 & 16 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} \underline{2} & 7 & 0 \\ 0 & \underline{8} & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Since no pivot in last row $\Rightarrow$ do <u>not</u> span

c. $$\begin{bmatrix} 1 & 0 & -5 & 8 \\ 0 & 2 & -3 & 5 \\ -2 & 6 & 1 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -5 & 8 \\ 0 & 2 & -3 & 5 \\ 0 & 6 & -9 & 20 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} \underline{1} & 0 & -5 & 8 \\ 0 & \underline{2} & -3 & 5 \\ 0 & 0 & 0 & \underline{5} \end{bmatrix}$$

Since pivot in each row $\Rightarrow$ <u>do</u> span.

PROBLEM 3. [12] Determine which of the following sets of vectors are linearly independent. Give reasons for your answers.

a. $\begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ 15 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ b. $\begin{bmatrix} -3 \\ 5 \\ -2 \end{bmatrix}, \begin{bmatrix} -9 \\ 15 \\ -6 \end{bmatrix}$

c. $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -8 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$ d. $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

a. <u>No</u> Too many vectors

b. <u>No</u> Second vector = 3 · first vector

c. Row-reduce:

$$\begin{bmatrix} 1 & -2 & -1 \\ -1 & 4 & 3 \\ 3 & -8 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 2 \\ 0 & -2 & 5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$

<u>Yes</u>, pivot in every column

d. <u>No</u> because of the zero vector.

PROBLEM 4. A company manufactures two products. For each unit of product #1, the company spends \$4 on materials, \$8 on labor, \$1 on packaging, and \$5 on overhead expenses. For each unit of product #2, the company spends \$6 on materials, \$10 on labor, \$2 on packaging, and \$5 on overhead. The company wants to know how much of each product to make in order to use exactly all of its budgeted resources of \$600 for materials, \$1100 for labor, \$175 for packaging, and \$600 for overhead.

a.[4] Set up (but do not solve) a *vector equation* that describes this problem. Include a statement about what the variables in the equation represent.

b.[2]Write an equivalent *matrix equation* for this problem. (Do not solve it.)

	Product #1	Product #2	Budget
Material	4	6	600
Labor	8	10	1100
Packaging	1	2	175
Overhead	5	5	600

a. x_1 = # of units of Product #1

x_2 = ——— " ——— #2

$$x_1 \cdot \begin{bmatrix} 4 \\ 8 \\ 1 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 6 \\ 10 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 600 \\ 1100 \\ 175 \\ 600 \end{bmatrix}$$

b.

$$\begin{bmatrix} 4 & 6 \\ 8 & 10 \\ 1 & 2 \\ 5 & 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 600 \\ 1100 \\ 175 \\ 600 \end{bmatrix}$$

PROBLEM 5. The following 3 equations define 3 lines in the plane with coordinates x_1, x_2 (the exact location of the third line depends on the constant h):

$$2(x_1+4) = 3x_2, \qquad \frac{x_1+1}{2} = x_2, \qquad 3x_1 + 4x_2 = h.$$

a.[3] Set up (but do not solve) a *matrix equation* for finding a point (x_1, x_2) that lies on all three lines.

b.[3] Find a value of h such that the three lines have a common point.

c.[2] Find this common point.

System of eqns:

$$\begin{cases} 2x_1 - 3x_2 = -8 \\ x_1 - 2x_2 = -1 \\ 3x_1 + 4x_2 = h \end{cases}$$

(a)

$$\begin{bmatrix} 2 & -3 \\ 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -8 \\ -1 \\ h \end{bmatrix}$$

(b) Row reduce

$$\left[\begin{array}{cc|c} 2 & -3 & -8 \\ 1 & -2 & -1 \\ 3 & 4 & h \end{array}\right] \to \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -6 \\ 0 & 10 & h+3 \end{bmatrix}$$

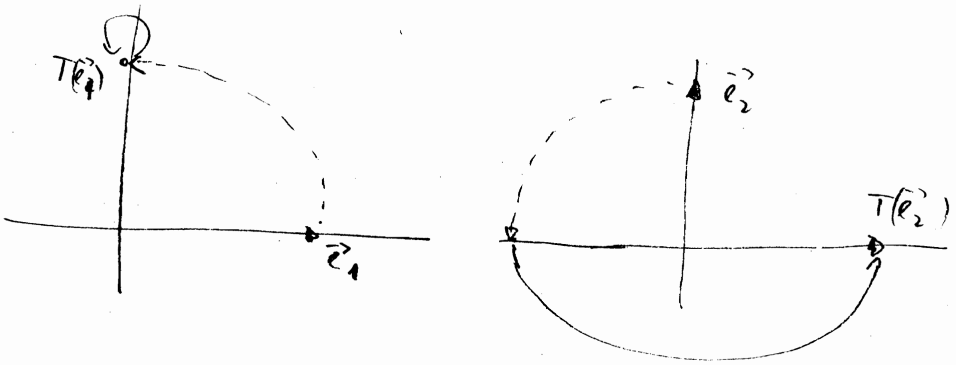
$$\to \begin{bmatrix} 1 & 0 & -13 \\ 0 & 1 & -6 \\ 0 & 0 & h+63 \end{bmatrix} \qquad h = -63$$

(c)

$$\begin{cases} x_1 = -13 \\ x_2 = -6 \end{cases}$$

PROBLEM 6. [5] Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates points counterclockwise $90°$ and then reflects the result in the vertical x_2-axis.

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix}$$



$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

PROBLEM 7. [10] Mark each statement either True or False. You do **not** have to justify your answer.

a. In some cases, it is possible for five vectors in $\mathbb{R}^5$ to be linearly independent.

b. If a matrix A is $m \times n$ and if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$, then the columns of A must be linearly independent in $\mathbb{R}^m$.

c. If A is a 5×5 matrix such that $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$, then the columns of A span $\mathbb{R}^5$.

d. If a system of linear equations has two different solutions, then it has infinitely many solutions.

e. If $\mathbf{v}_1$ and $\mathbf{v}_2$ span a plane in $\mathbb{R}^3$ and if $\mathbf{v}_3$ is in that plane, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set.

a. True

b. False

c. True

d. True

e. True