Name: Answer Key **Student Number:** ____________

Math 221: Matrix Algebra
Midterm 1 - May 23, 2012

Instructions: There are five questions and 100 points on this exam. You will need only a pen or pencil and eraser; nothing else is permitted. Unless otherwise indicated, write your final answers clearly in complete sentences; *failure to do so will cost points.* Point values indicated for each question are estimates and subject to change.

(1) (12 points) Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ which is *not* one-to-one, and let A denote the standard matrix of T. Indicate whether the following are true or false *by writing the complete word* True *or* False (you will lose points for simply writing T or F).

(a) A is a square matrix. True

(b) The columns of A are linearly dependent. True

(c) A has a pivot in every column. False

(d) T is not onto. True

(2) (16 points) For each of the following mappings, write *linear* if the mapping is a linear transformation, and otherwise write *not linear*. You do *not* need to justify your answers.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ x^2 \\ 0 \end{pmatrix}$$ not linear

(b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ x+1 \\ 0 \end{pmatrix}$$ not linear

(c) $T : \mathbb{R} \to \mathbb{R}^3$ defined by

$$T(x) = \begin{pmatrix} x \\ -2x \\ 0 \end{pmatrix}$$ linear

(d) $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = \mathbf{0}$. linear

(3) (24 points) Suppose that a linear system has a coefficient matrix A whose reduced echelon form $REF(A)$ is

$$REF(A) = \begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Express the solution set for the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ in parametric vector form.

x_2 and x_4 are free variable

$x_1 = x_2 + x_4 \quad x_3 = x_4 \quad x_5 = 0$

The solution set is

$$\left\{ x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} : x_3, x_4 \in R \right\}$$

(b) Suppose that $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}$, and $\mathbf{a_5}$ are the columns of A, so that

$$A = \begin{pmatrix} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} & \mathbf{a_4} & \mathbf{a_5} \end{pmatrix}$$

Express the zero vector $\mathbf{0} \in \mathbb{R}^4$ as a linear combination of the columns of A in which not all the coefficients are zero.

We can use part (a). Let $x_2 = 1 \quad x_4 = 0$

This gives us $x_1 = 1 \quad x_3 = 0$

So we get

$$\vec{0} = 1\vec{a}_1 + 1\vec{a}_2 + 0\vec{a}_3 + 0\vec{a}_4 + 0\vec{a}_5$$

(c) Do the columns of A span $\mathbb{R}^4$? *Justify your answer.*

The colums of A do not span R^4.

As A does not have a pivot position in every row, the system

$A\vec{x} = \vec{b}$ is not consistent for all $\vec{b} \in R^4$.

(4) (24 points) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that

$$T(\mathbf{e}_1 - 2\mathbf{e}_2) = 3\mathbf{e}_1 - \mathbf{e}_3 \qquad T(-\mathbf{e}_1 + \mathbf{e}_2) = -2\mathbf{e}_2$$

Let A be the standard matrix of T.

(a) How many rows and columns does A have?

A has 3 rows and 2 columns.

(b) Find the matrix A. $T\left(\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$ $T\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{array}\right) \xrightarrow[\text{reduce}]{\text{row}} \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & -1 & -1 \\ 0 & 1 & -2 & -1 \end{array}\right)$$

Therefore $T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = T\left(-1\begin{pmatrix} 1 \\ -2 \end{pmatrix} - 2\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -3+0 \\ 0+4 \\ 1+0 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$

and $T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = T\left(-\begin{pmatrix} 1 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -3+0 \\ 0+2 \\ 1+0 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$

Thus $A = \begin{pmatrix} -3 & -3 \\ 4 & 2 \\ 1 & 1 \end{pmatrix}$ is the standard matrix of T.

(c) Is T one-to-one? Is T onto? You do *not* need to justify your answers.

T is one-to-one as the columns of A are linearly independent

T is not onto as A does not have a pivot in every row.

(5) (24 points) In each part, determine whether the given set of vectors is linearly independent. *State a reason for your conclusion.*

(a) $\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ A set with one non-zero vector is always linearly independent.

(b) $\left\{\begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} \begin{pmatrix} -2 \\ -4 \\ 10 \end{pmatrix}\right\}$ A set of two vectors which are scalar multiples of each other is always linearly dependent.

(c) $\left\{\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ 6 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}\right\}$ No vector in this set can be expressed as a linear combination of the other vectors. Therefore this set is linearly independent.