1. TRUE or FALSE:

   (a) If A is an $n \times n$ matrix with nonzero determinant and $AB = AC$ then $B = C$.

   (b) A square matrix with zero diagonal entries is never invertible.

   (c) A linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ is one-to-one if and only if its standard matrix has nonzero determinant.

   (d) Every spanning subset of $\mathbb{R}^4$ contains a basis for $\mathbb{R}^4$.

   (e) A linearly independent subset of $\mathbb{R}^n$ has at most n elements.

   (f) Every subspace of $\mathbb{R}^3$ contains infinitely many vectors.

   (g) The system of linear equations $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is in the column space of A.

2. Indicate if each of the following is a linear subspace:

   (a) The set of all vectors parallel to a fixed vector $\mathbf{v}$.

   (b) All the vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ with $x_1 + x_3 = 1$.

   (c) The intersection of two subspaces of $\mathbb{R}^n$.

   (d) The set of vectors in $\mathbb{R}^3$ with two equal components.

3. Determine if each of the following matrices is invertible? If not, explain why. If so, compute its inverse.

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 0 & -4 \\ 3 & -2 & -2 & 8 \\ 1 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

4. Compute the determinant of the following matrices:

$$\begin{pmatrix} 0 & 2 & 3 \\ -1 & -1 & 4 \\ 2 & -2 & 2 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 5 & 5 \\ 0 & 2 & 6 & 12 \\ 1 & 4 & 7 & 12 \\ 2 & 8 & 14 & 15 \end{pmatrix}$$

5. Suppose $A = (\mathbf{a}_1 \cdots \mathbf{a}_n)$ is an $n \times n$ matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$. For each of the following statements, indicate whether they are true or false and justify your answer.

   (a) If $\det(A) = 0$, then the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is linearly dependent.

   (b) If $\det(A) = 0$, then $\mathbf{a}_n$ is a linear combination of $\{\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}\}$.

   (c) If $\{\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}\}$ is linearly independent and $\det(A) = 0$, then $\mathbf{a}_n$ is a linear combination of $\{\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}\}$.

   (d) If the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a solution, then the determinant of the matrix $B = (\mathbf{a}_1 \cdots \mathbf{a}_{n-1}\, \mathbf{b})$, obtained from replacing the last column of A by $\mathbf{b}$, is zero.

   (e) The exists a vector $\mathbf{c} \in \mathbb{R}^n$ which is not in the span of $\{\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}\}$.

   (f) If $\{\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}\}$ is linearly independent then there exists a vector $\mathbf{c}$ such that the determinant of the matrix $C = (\mathbf{a}_1 \cdots \mathbf{a}_{n-1}\, \mathbf{c})$, obtained from replacing the last column of A by $\mathbf{c}$, has nonzero determinant.

6. Given a set of vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ in $\mathbb{R}^n$, if A is the matrix $A = (\mathbf{a}_1 \ \ldots \ \mathbf{a}_n)$, we write

$$\det(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \det(A)$$

   (a) Suppose that $\det(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}) = 2$ for a set of vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ in $\mathbb{R}^4$. Find:

$$\det(\mathbf{w} + 2\mathbf{v}, \mathbf{v}, \mathbf{z}, 3\mathbf{u})$$