*This midterm exam has **5 questions** on **6 pages**, for a total of 50 marks.*

Duration: **50 minutes**

Last name: ______________________ First name: ______________________

Student-No: ______________________________ Course Section: **202, 204**

Signature: ____

Question:	1	2	3	4	5	Total
Points:	10	10	10	10	10	50
Score:						

Please read the following carefully before starting to write.

- This is a closed-book examination. **None of the following are allowed**: books, notes, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- **You must show your work to receive full credit on a problem (except question 3 and question 5).**
- Please do your work on the exam provided. Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- You may not leave during the first 30 minutes or final 15 minutes of the exam.

$\boxed{\text{10 marks}}$ 1. Find the inverse matrix A^{-1} of $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \\ 1 & 3 & 0 \end{pmatrix}$.

Solution:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -3 & -2 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2/3 & -1/3 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1/3 & 1/3 & 0 \\ 0 & 1 & 0 & -1/3 & -1/3 & 1 \\ 0 & 0 & 1 & 2/3 & -1/3 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & -1/3 & -1/3 & 1 \\ 0 & 0 & 1 & 2/3 & -1/3 & 0 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1 & 1 & -2 \\ -1/3 & -1/3 & 1 \\ 2/3 & -1/3 & 0 \end{pmatrix}.$$

$\boxed{\text{10 marks}}$ 2. Let $A = \begin{pmatrix} 1 & 2 & -1 \\ c & 0 & 2 \\ -1 & c & -1 \end{pmatrix}$.

(a) [7 points] Find $\det(A)$.

(b) [3 points] Find all real number c such that the matrix A is invertible.

Solution:

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & -1 \\ c & 0 & 2 \\ -1 & c & -1 \end{pmatrix} &= \det \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2c & 2+c \\ 0 & c+2 & -2 \end{pmatrix} \\ &= 4c - (2+c)^2 \\ &= -c^2 - 4 \\ &\leq -4. \end{aligned}$$

So the determinant is never 0, hence the matrix is invertible for any real number c.

10 marks

3. Find the 2×2 matrices which describe the following linear maps.
You only need to write answers, no justification necessary.

a) [3 points] projection onto the y-axis.

Solution: $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

b) [3 points] reflection about the line $y = x$.

Solution: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

c) [4 points] rotation by 90 degrees *clockwise* followed by projection onto the y-axis.

Solution: $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$.

$\boxed{\text{10 marks}}$ 4. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T(\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}) = \begin{pmatrix} x \\ -y \\ w-z \end{pmatrix}.$$

(a) [7 points] Find the standard matrix A for T.

Solution:

$$T(\mathbf{e}_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, T(\mathbf{e}_2) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, T(\mathbf{e}_3) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, T(\mathbf{e}_4) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The standard matrix of T is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

(b) [3 points] Is T onto? Explain how you get your answer.

Solution:

T is onto. A has a pivot in each row. *or* $\dim Col(A) = 3$ = dimension of the codomain. *or* $Col(A) = \mathbb{R}^3$.

10 marks 5. For each of the following state whether the statement is TRUE or FALSE (**no justification is necessary**).

1. There exists a linear map $T : \mathbb{R}^4 \to \mathbb{R}^3$ which is onto but *not* one-to-one.

   Solution: True

2. If A, B, C are square matrices and $AB = C$ and A is invertible then $B = CA^{-1}$.

   Solution: False

3. Suppose that $A = (\mathbf{u}, \mathbf{v}, \mathbf{w})$ is a 3×3 matrix where $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are its columns. If $\det(A) = 3$ then $\det(2\mathbf{w}, \mathbf{u}, \mathbf{w} - \mathbf{v}) = 6$.

   Solution: False

4. Suppose that the standard matrix A of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ has linearly independent column vectors. If $T(x) = T(y)$ then $x = y$.

   Solution: True

5. If A is a 2×2 matrix which satisfies $A^2 = 2A$, then either A is the 2×2 zero matrix or $A = 2I_2$.

   Solution: False

   For example: $A = \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$.