Week 7

Matrix multiplican

Motivated by composition of linear transformations.

Def $A_{m\times k}$, $B_{k\times n}$

$$AB = (Av_1 \ \cdots \ Av_n),$$ where $B = (v_1 \ \cdots \ v_n)$

Write
$$\begin{array}{cc} & \\ AB = \begin{pmatrix} c_{11} & \cdots & \cdots & c_{1n} \\ \vdots & & c_{ij} & \vdots \\ c_{m1} & \cdots & \cdots & c_{mn} \end{pmatrix} & i^{th} \text{ row} \\ j^{th} \text{ column} & \end{array}$$

$$i^{th} \rightarrow \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \boxed{a_{i1} \ \cdots \ a_{ik}} \\ a_{m1} & \cdots & a_{mk} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & \boxed{\begin{matrix} b_{1j} \\ \vdots \\ b_{kj} \end{matrix}} & \cdots & b_{1n} \\ \vdots & & & & \vdots \\ b_{k1} & & & & b_{kn} \end{pmatrix}$$

j^{th}

$$c_{ij} = a_{i1}b_{1j} + \cdots + a_{ik}b_{kj}.$$

Example ...

When A is a square matrix, i.e. # row = # column

$A^2, A^3, \ldots$ can all be defined, i.e. power of A.

Fact 1 $T : \mathbb{R}^n \to \mathbb{R}^m$ linear, $S : \mathbb{R}^m \to \mathbb{R}^k$ linear

then $S \circ T : \mathbb{R}^n \to \mathbb{R}^k$ is linear.

check $S \circ T(c_1 u + c_2 v) = S(c_1 T(u) + c_2 T(v))$

$= c_1 S \circ T(u) + c_2 S \circ T(v)$.

Fact 2 T has standard matrix A, $(T(x) = Ax)$

S . . . . B. $(S(x) = Bx)$

then $S \circ T$ . . . . BA.

check Let C be standard matrix of $S \circ T$.

i^{th} column of C : ~~$C(e_i)$~~ Ce_i

$\therefore S \circ T(e_i) = Ce_i$ ← i^{th} column of C

$$\begin{aligned} &\,\| \\ S(&Ae_i) = B(Ae_i) \\ &= (BA)e_i \end{aligned}$$ ← i^{th} column of BA

$\therefore C = BA$.

Ex.

① $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\neq \quad \neq \quad\quad //$

zero matrix

$\exists\, A \neq 0,\ B \neq 0$, s.t. $AB = 0$.

② $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

$\exists\, A, B$. $AB \neq BA$

③ $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}$

$\therefore \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}$

$\therefore CA = CB \not\Longrightarrow A = B.$

Recall:

$T : \mathbb{R}^n \to \mathbb{R}^m$ linear transf. $\Longleftrightarrow T(x) = Ax$

composition of T's $\Longrightarrow$

* **Fact 3**

$A(B_1 + B_2) = AB_1 + AB_2$

~~$cA(\ldots) = c$~~ $\quad c(AB) = (cA)B = A(cB)$

$c \in \mathbb{R}$

$c(A+B) = cA + cB$

$A(BC) = (AB)C$

Fact 4. A : $m \times n$,

$AI_n = A = I_m A$

Def.

$A+B = \begin{pmatrix} \cdots & a_{ij}+b_{ij} & \cdots \end{pmatrix}$

$cA = \begin{pmatrix} \cdots & ca_{ij} & \cdots \end{pmatrix}$

We have for matrices:

$$+, \quad \cdot, \quad \div ?$$

For real numbers, $a \neq 0$, $\Rightarrow b \div a = a^{-1}b = ba^{-1}$

$$a \cdot a^{-1} = 1 = a^{-1} \cdot a$$

$\therefore$ a^{-1} is the inverse of a.

Given $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, can we find B

s.t. $AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$?

$$AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\therefore$ We cannot hope to get an inverse for every non-zero matrix.

<u>Size of A</u>: $A_{m\times n}$, if $\exists\, B_{pq}$, s.t. AB & BA are defined ($= I$) $\leftarrow$ square.

$AB \Rightarrow p = n$, $BA \Rightarrow q = m$ $\quad \therefore \boxed{m = n}$

$\therefore$ We consider <u>square matrices</u>.

<u>Def</u>. $A: n \times n$. A is <u>invertible</u> if $\exists\, B$, $n \times n$, s.t. $AB = I_n = BA$.

and B is called the <u>inverse matrix</u> of A.

write $B = A^{-1}$.

<u>Fact</u> ① If $A_{n\times n}$ invertible, then A^{-1} invertible. $(A^{-1})^{-1} = A$

② If A, B invertible $n\times n$, then AB invertible & $(AB)^{-1} = B^{-1}A^{-1}$.

Why? ① A invertible $\Rightarrow$ $A A^{-1} = I_n$, $A^{-1}A = I_n$

$\therefore A = (A^{-1})^{-1}$

② $(B^{-1}A^{-1})\,AB = B^{-1}(A^{-1}A)\,B = B^{-1}I_n B = B^{-1}B = I_n$

$(AB)(B^{-1}A^{-1}) = I_n$

How do we know A is invertible, and if yes how to find A^{-1}?

<u>2x2 case</u> $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

The <u>determinant</u> of A, $\det(A) = ad - bc$.

<u>Result</u>: ① If $\det A \neq 0$, then $A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

② If $\det A = 0$ then A non invertible.

Why? ① $A \cdot \left(\frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right) = I_2$

$(\quad\quad)\, A = I_2$.

② If $ad = bc$. then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d \\ -c \end{pmatrix} = \begin{pmatrix} ad - bc \\ \cancel{bc - ad} \\ 0 \end{pmatrix} = \begin{pmatrix} ad - bc \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

If $\begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} d \\ -c \end{pmatrix}$ then $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

If $\begin{pmatrix} -b \\ a \end{pmatrix} \neq 0$ then $A\begin{pmatrix} -b \\ a \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$n\times n$ case (include 2×2).

Thm. $A : n\times n$, If $(A \mid I_n) \xrightarrow{\text{rref}} (I_n \mid B)$
then A invertible and $B = A^{-1}$.
Otherwise, A not invertible.

Reason: A invertible $\Leftrightarrow$ $Ax=0$ has only trivial sol'n.

Why? "$\Rightarrow$" $\quad Ax=0 \Rightarrow AA^{-1}x = A^{-1}0 = 0$
$\Rightarrow I_n x = 0 \Rightarrow x = 0$

"$\Leftarrow$" $\quad Ax=0$ only trivial sol'n

$$\Rightarrow A \xrightarrow{\text{rref}} \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$\therefore (A \mid I_n) \xrightarrow[A]{\text{rref}} (I \mid B)$$

$B = (x_1 \cdots x_n)$

$\therefore$ n systems
$$Ax_1 = e_1$$
$$\vdots$$
$$Ax_n = e_n$$

$\therefore AB = I_n$

reverse row op.

$$(B \mid I_n) \longrightarrow (I_n \mid A)$$

$\therefore BA = I_n$.

Ex. $\det A \neq 0$. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \xrightarrow[(\text{or } c\neq 0)]{a\neq 0} \left(\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d-\frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & \frac{-\frac{c}{a}}{d-\frac{bc}{a}} & \frac{1}{d-\frac{bc}{a}} \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & \frac{-c}{\det A} & \frac{a}{\det A} \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} - \frac{\frac{b}{a}(-c)}{\det A} & -\frac{\frac{b}{a}a}{\det A} \\ 0 & 1 & \frac{-c}{\det A} & \frac{a}{\det A} \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & \frac{\cancel{\frac{1}{a}(A+bc)}\,d}{\det A} & \frac{-b}{\det A} \\ 0 & 1 & \frac{-c}{\det A} & \frac{a}{\det A} \end{array}\right)$$