Set 5, Due: March 25, 2024

1. Let $L \subset \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be an immersed submanifold. Ddefine the cone over L by

$$C(L) = \{rx : r \geq 0, x \in L\}$$

and call L the link of $C(L)$. Find the relation between the mean curvatures of $C(L)$ in $\mathbb{R}^{n+1}$ (away from the origin), L in $\mathbb{R}^{n+1}$, L in $\mathbb{S}^n$.

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3. Consider the upper half-plane $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$.

(1) Let g be the Riemannian metric given by

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = \frac{1}{y}.$$

Compute $\lim_{\epsilon \to 0} L_g(c_\epsilon)$ where c_ϵ is the vertical segment $x = 0, \epsilon \leq y \leq 1$. Is g complete? Explain your answer.

(2) Let g be the Riemannian metric (called Lobatchevski metric, Poincaré metric) defined by

$$g_{11} = g_{22} = \frac{1}{y^2}, \quad g_{12} = 0.$$

Show g is complete.

4. Let M and N be Riemannian manifolds and let $f : M \to N$ be a diffeomorphism. Assume that N is complete and that there exists a constant $c > 0$ such that

$$\left| df_p(v) \right| \leq c|v|$$

for all $p \in M$ and $v \in T_pM$. Show that M is complete.