Application of the Rauch Comparison Theorem

Theorem 0.1. *Let K be the sectional curvature of (M, g) with $0 < L \leq K \leq H$ where L, H are constants. Let d be the distance between two consecutive conjugate points on a geodesic γ. Then $\pi/\sqrt{H} \leq d \leq \pi/\sqrt{L}$.*

Proof. Let $S^n(H)$ be the tound sphere with sectional curvature $= H$. Let J be a Jacobi field along a geodesic $\gamma : [0, l] \to M$, $J(0) = 0, J \perp \gamma'$; take a geodesic $\tilde{\gamma} : [0, l] \to S^n(H)$ with $|\tilde{\gamma}'| = |\gamma'|$ and $\tilde{J}$ a Jacobi field along $\tilde{J}$ with $\tilde{J}(0) = 0, \tilde{J} \perp \tilde{\gamma}'$ and $|\tilde{J}'(0)| = |J'(0)|$. Note that $\tilde{\gamma}$ has no congujate points on $(0, \pi/\sqrt{H})$ (recall we wrote down Jacobi fields which equal 0 at $t = 0$ for space forms). Rauch comparison implies $|J(t)| \geq |\tilde{J}(t)| > 0, t \in (0, \pi/\sqrt{H})$, so J has no conjugate points there, in turn $d \geq \pi/\sqrt{H}$. Next, consider $S^n(L)$. If $d > \pi/\sqrt{L}$, then $0 < |J(t) \leq |\tilde{J}(t)|, t \in (0, d)$ but this is not tru for $S^n(L)$. □

Remark. Compare to "model" spaces, namely, the space forms.

Proposition 0.2. *Given $M^n, \tilde{M}^n$ with $K_p(\sigma) \leq K_{\tilde{p}}(\tilde{\sigma})$ for all $p \in M, \tilde{p} \in \tilde{M}, \sigma, \tilde{\sigma}$, let $i : T_pM \to T_{\tilde{p}}\tilde{M}$ be a linear isometry. Suppose that $exp_p|_{B_r(0)}$ is diffeomprphic and $exp_{\tilde{p}}|_{\tilde{B}_r(0)}$ is non-singular; and $c : [0, a] \to exp_p(B_r(0)), \tilde{c} : [0, a] \to exp_{\tilde{p}}(\tilde{B}_r(0))$ with $\tilde{c} = exp_{\tilde{p}} \circ i \circ exp_p^{-1} \circ c$. Then $l(c) \geq l(\tilde{c})$.*

Proof. Take a curve $\bar{c}(s) = exp_p^{-1}c(s)$ in T_pM. Denote $f(t, s) = \gamma_s(t) = exp_p t\bar{c}(s)$ the radial geodesic for any fixed $s \in [0, a]$ for $0 \leq t \leq 1$, and $J_s(t) := \frac{\partial f}{\partial s}$ is a Jacobi field along $\gamma_s, J_s(0) = 0, J_s(1) = \frac{\partial f}{\partial s}(1, s) = c'(s)$ (draw a picture to see what's going on).

$$\frac{DJ_s}{dt}(0) = \frac{D}{dt}\left[(d\, exp_p)_{t\bar{c}(s)} t\bar{c}'(s)\right]_{t=0} = \bar{c}'(s).$$

In $\tilde{M}$, set $\tilde{f}(t, s) = exp_{\tilde{p}} ti(\bar{c}(s)) = \tilde{\gamma}_s(t)$. Let $\tilde{J}_s(t) = \frac{\partial \tilde{f}}{\partial s}(t, s)$ be a Jacobi field with $\tilde{J}_s(0) = 0, \tilde{J}_s(1) = \tilde{c}'(s), \frac{D\tilde{J}_s}{dt}(0) = i(\bar{c}'(s))$. We have

$$\langle \tilde{J}_s'(0), \tilde{\gamma}_s'(0)\rangle = \langle i\bar{c}'(s), i\gamma_s'(0)\rangle = \langle \bar{c}'(s), \gamma_s'(0)\rangle = \langle J_s'(0), \gamma_s'(0)\rangle.$$

By Rauch comparison,

$$|\tilde{c}'(s)| = |\tilde{J}_s(1)| \leq |J_s(1)| = |c'(s)|$$

so

$$l(\tilde{c}) = \int_0^a |\tilde{c}'| \leq \int_0^a |c'| = l(c).$$

□

We state, without proof, Bishop's volume comparison theorem:

Theorem 0.3. *Let (M, g) be a complete manifold (more detail later, for now, say metrically complete). If $Ric_M \geq H$ where H is a constant, then $Vol_M(B_r(0)) \leq Vol(B_r^H(0))$ where $B_r^H(0)$ is the geodesic ball in the simply connected complete manifold of constant sectional curvature H. Moreover, $Vpl_M(B_r(0))/Vol(B_r^H(0))$ is non-increasing in r and tends to 1 as $r \to 0$.*