(99)

**Cor** (If) $c$ is (piecewise $C^\infty$) geodsic on $[0,a]$ ~~then~~ $\Longleftrightarrow$

$E'(0) = 0 \quad \forall\ V$ proper

Pf "$\Rightarrow$" $c$ geod. $c$ regular, $\frac{D}{dt}\frac{\partial c}{\partial t} = 0$.

if $V(0) = V(a) = 0 \quad \Rightarrow \quad E'(0) = 0$

"$\Leftarrow$" piecewise $C^\infty$, $g(t_i) = 0$

$\tilde{g} > 0$ for $t \neq t_i$

Consider $V(t) = g(t)\frac{D}{dt}\frac{dc}{dt}$

$t_0 \quad t_1 \quad t_2 \quad \ldots$

~~Cbx~~ $\exists f$ with v.f. $V$

$\therefore\ 0 = \frac{1}{2}E'(0) = -\int_0^a g\left\langle \frac{D}{dt}\frac{dc}{dt}, \frac{D}{dt}\frac{dc}{dt}\right\rangle$

$\therefore$ $c$ is geod on each $(t_i, t_{i+1})$

Consider

$\bar{V}$, $\bar{V}(0) = \bar{V}(a) = 0$, $\bar{V}(t_i) = \frac{dc}{dt}(t_i^+) - \frac{dc}{dt}(t_i^-)$

$\because$ $c$ is geodsic on $(t_i, t_{i+1})$

$0 = \frac{1}{2}\frac{dE}{dt}(0) = -\sum_i \left\langle \frac{dc}{dt}(t_i^+) - \frac{dc}{dt}(t_i^-), \frac{dc}{dt}(t_i^+) - \frac{dc}{dt}(t_i^-)\right\rangle$

$\Rightarrow \frac{dc}{dt}(t_i^+) = \frac{dc}{dt}(t_i^-)$

$\therefore$ $c$ is $C^1$, $\because$ $c$ geod. on $(0,a)$ $\frac{D}{dt}\frac{dc}{dt}(t_i) = 0$

uniqueness of sol'n of ODE $\Rightarrow c \in C^\infty \Rightarrow c$ geod.

Warning: even $c(0) = c(a)$
$c$ may not be $C^\infty$ at $c(0) = c(a)$