V not smooth at t_i

Prop. (2nd Variation of E)

$\gamma : [0,a] \to M$ geodesic

f : proper variation of γ

or γ : broken geodesic

Then $$\frac{1}{2}E''(0) = -\int_0^a \left\langle V(t), \frac{D^2V}{dt^2} + R\left(\frac{d\gamma}{dt}, V\right)\frac{d\gamma}{dt} \right\rangle dt$$

$$- \sum_{i=1}^{k} \left\langle V(t_i), \frac{DV}{dt}(t_i^+) - \frac{DV}{dt}(t_i^-) \right\rangle$$

where

V is the variational field of f.

Pf Recall:

$$\frac{1}{2}E' = -\int_0^a \left\langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t} \right\rangle dt + \sum_{i=0}^{k} \left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle \Bigg|_{t_i}^{t_{i+1}}$$

$$\therefore \frac{1}{2}E'' = -\int_0^a \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t} \right\rangle dt - \int_0^a \left\langle \frac{\partial f}{\partial s}, \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t} \right\rangle dt$$

$$+ \sum_{i=0}^{k} \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle \Bigg|_{t_i}^{t_{i+1}} + \sum_{i=0}^{k} \left\langle \frac{\partial f}{\partial s}, \frac{D}{ds}\frac{\partial f}{\partial t} \right\rangle \Bigg|_{t_i}^{t_{i+1}}$$

$\frac{D}{ds}\frac{\partial f}{\partial s} = 0$ at $t=0, a$, $s=0$ & f: smooth

At $s=0$, 1) $\cancel{\frac{\partial f}{\partial t} = 0}$ because f is proper

2) $\frac{D}{dt}\frac{\partial f}{\partial t} = 0$ because γ is geodesic

3) $\frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t} = \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial t} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}$