Submanifolds

$f : M^n \longrightarrow \bar{M}^{n+k}$ immersion.

$$T_p\bar{M} = T_pM \oplus T_pM^{\perp}$$

Define
$$\nabla_X Y = (\bar{\nabla}_{\bar{X}} \bar{Y})^T$$

$\forall\, X, Y \in T_pM$, $\bar{X}, \bar{Y}$ are local extension of X, Y to $\bar{M}$.

Prop. ∇ is the Rie. conn. of the induced metric

Pf ∇ is a map from $TM \times TM$ to TM

$$\nabla_{f_1X_1+f_2X_2} Y = (\bar{\nabla}_{f_1\bar{X}_1+f_2\bar{X}_2} \bar{Y})^T = (f_1 \bar{\nabla}_{\bar{X}_1}\bar{Y} + f_2 \bar{\nabla}_{\bar{X}_2}\bar{Y})^T$$
$$= f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y.$$

$$\nabla_X f_1Y_1 + f_2Y_2 = (\bar{\nabla}_{\bar{X}} f_1\bar{Y}_1 + f_2\bar{Y}_2)^T$$
$$= f_1 \nabla_X Y_1 + f_2 \nabla_X Y_2$$

$$\nabla_X Y - \nabla_Y X = (\bar{\nabla}_{\bar{X}}\bar{Y})^T - (\bar{\nabla}_{\bar{Y}}\bar{X})^T = [\bar{X}, \bar{Y}]^T$$
$$= [X, Y]$$

$$X\langle Y, Z\rangle = \bar{X}\langle \bar{Y}, \bar{Z}\rangle$$
$$= \langle \bar{\nabla}_{\bar{X}}\bar{Y}, \underset{\text{tn}}{\bar{Z}}\rangle + \langle \bar{\nabla}_{\bar{X}}\bar{Z}, \underset{\text{tn}}{\bar{Y}}\rangle$$
$$= \langle \nabla_X Y, Z\rangle + \langle \nabla_X Z, Y\rangle$$

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