Theorem of Hadamard:

M^n: complete, simply connected
$K(p,\sigma) \le 0$. $\forall p, \sigma$

Then M is diffeo. to $\mathbb{R}^n$

(in fact, $\exp_p : T_pM \to M$ is diffeo.)

Lemma: M complete, $K \le 0$
Then $\forall p \in M$, the conjugate locus
$C(p) = \phi$. $\exp_p$ is local diffeom.

Pf. $J \not\equiv 0$ a Jacobi field along
geodesic $\gamma : [0, \infty) \to M$
$\gamma(0) = p$, $J(0) = 0$

$$\langle J, J\rangle'' = 2\langle J', J'\rangle + 2\langle J'', J\rangle$$
$$= 2\langle J', J'\rangle - 2\langle R(\gamma', J)\gamma', J\rangle$$
$$\ge 0 \quad \because K \le 0.$$

$\therefore \langle J, J\rangle' \nearrow \quad \because J'(0) \ne 0,\ J(0) = 0$

For $0 < t \ll 1$,

$\langle J, J\rangle(t) > \langle J, J\rangle(0) = 0$

$\therefore \gamma$ has no conjugate pts to $\gamma(0)$.

If $\langle J, J\rangle(t_1) = 0$ 1st. then $\langle J, J\rangle'(t_1) = 0 \Rightarrow \langle J, J\rangle'(t) = 0 \ \forall t < t_1$ ↯

$\langle J, J\rangle'$ strictly increasing at 0