18.100B Problem Set 2

Due Thursday February 18 by 2:30pm. Be sure to review the course collaboration policies.

NEW: The problem set has been divided into three parts (A,B,C) to facilitate grading. Please hand in three separate packets, with your name, the homework number (i.e. Homework 2), and the part (A,B, or C) clearly labled at the top of each packet. Don't forget to staple your homework (but don't staple the three packets together!).

Note: in this problem set (and throughout the course), the notation $A \subset B$ means the same thing as $A \subseteq B$, i.e. A is contained in B; this includes the possibility that $A = B$. If we want to say that A is contained in B and $A \neq B$, we will write $A \subsetneq B$.

Part A: Functions, sets and cardinality

1. Let A, B, and C be sets, and let $f : A \to B$ and $g : B \to C$ be functions. Prove that if $g \circ f : A \to C$ is a bijection, then f is an injection and g is a surjection.

2. An algebraic number is a (complex) number that is a root of a polynomial equation in one variable with integer coefficients. Prove that the set of algebraic numbers is countable. You may assume that a polynomial in one variable of degree D has at most D (complex) roots. (We haven't discussed the complex numbers in this course, but that isn't important for this problem.)

Part B: Point-set topology in $\mathbb{R}^n$

3. Set $S \subset \mathbb{R}^n$ be a set. Define

$$\mathrm{int}(S) = \{x \in S : x \text{ is an interior point of } S\}.$$

Prove that $\mathrm{int}(\mathrm{int}(S)) = \mathrm{int}(S)$.

Remark: In particular, this shows that $\mathrm{int}(S)$ is open.

4. Let $S \subset \mathbb{R}^n$, $T \subset \mathbb{R}^n$ be sets. Suppose that S is open, and $S \subset T$. Prove that $S \subset \mathrm{int}(T)$.

Remark: Thus $\mathrm{int}(T)$ is the largest (in the sense of set inclusion) open set contained in T.

5. Prove that a set $S \in \mathbb{R}^n$ is open if and only if it is a union of open balls.

Part C: Intervals in $\mathbb{R}$

6. Let $I = (a, b) \subset \mathbb{R}$ be a finite interval (i.e. both a and b are real numbers). We define the measure of I to be $\mathrm{mes}(I) = b - a$. If I is not a finite interval, then we define $\mathrm{mes}(I) = \infty$.

If $\mathcal{I} = \{I_1, I_2, \ldots\}$ is a countable set of disjoint intervals, we define

$$\mathrm{mes}\left(\bigcup_{I \in \mathcal{I}} I\right) = \sum_{I \in \mathcal{I}} \mathrm{mes}(I). \tag{1}$$

In lecture, we proved that any open set $S \subset \mathbb{R}$ can be uniquely written as a countable union of disjoint open intervals (or see Apostol §3.4). Thus for any open set $S \subset \mathbb{R}$, $\mathrm{mes}(S)$ is well-defined.

(A) Let $\mathcal{I}$ be a countable (finite or infinite) set of intervals—note that the intervals in $\mathcal{I}$ need not be disjoint, so Equation (1) need not hold for $\bigcup_{I\in\mathcal{I}} I$. Prove that there exists a subset $\mathcal{I}' \subset \mathcal{I}$ so that the intervals in $\mathcal{I}'$ are disjoint, and

$$\operatorname{mes}\Big(\bigcup_{I\in\mathcal{I}} I\Big) \leq 3\operatorname{mes}\Big(\bigcup_{I\in\mathcal{I}'} I\Big). \tag{2}$$

Note: you aren't allowed to make the intervals $I \in \mathcal{I}$ smaller. Instead, you are supposed to find a subset of the set of intervals.

(B) Can the constant 3 in Equation (2) be replaced by a smaller number? If so, prove it. If not, write down an example (or family of examples) showing why not.