18.100B Problem Set 3

Due Thursday February 25 by 2:30pm. The problem set has been divided into two parts (A,B) to facilitate grading. Please hand in two separate packets with your name, the homework number (i.e. Homework 3), and the part (A or B) clearly labeled at the top of each packet. Don't forget to staple your homework (but don't staple the two packets together!).

Note: in this problem set (and throughout the course), the notation $A \subset B$ means the same thing as $A \subseteq B$, i.e. A is contained in B; this includes the possibility that $A = B$. If we want to say that A is contained in B and $A \neq B$, we will write $A \subsetneq B$.

Part A: Clean up from lecture

In this section, we'll prove two results that were stated in lecture but weren't proved, plus a useful result I didn't get to.

1. (5 pts) The multi-dimensional nested interval theorem:

Let $\mathcal{S} = \{[a^{(1)}, b^{(1)}],\ [a^{(2)}, b^{(2)}],\ [a^{(3)}, b^{(3)}], \ldots\}$ be a (countable) collection of closed intervals in $\mathbb{R}^n$. Suppose that

$$[a^{(1)}, b^{(1)}] \supset [a^{(2)}, b^{(2)}] \supset [a^{(3)}, b^{(3)}] \supset \ldots.$$

Prove that $\bigcap_{i=1}^{\infty} [a^{(i)}, b^{(i)}]$ is not empty. Hint: you can assume the one-dimensional version of the result (proved in lecture), and use it a black box.

2. (5 pts) Let $S \subset \mathbb{R}^n$ be a compact set. Prove that S is closed.

Remark: This was the final missing piece of the Heine-Borel theorem: a set $S \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

3. (5 pts) Let $S \subset \mathbb{R}^n$. Define

$$\overline{S} = \{x \in \mathbb{R}^n \colon x \text{ is an adherent point of } S\}.$$

Let

$$\mathcal{S} = \{T \subset \mathbb{R}^n \colon T \text{ is closed, and } S \subset T\}.$$

Prove that $\overline{S} = \bigcap_{T \in \mathcal{S}} T$.

Remark: Since we know that an arbitrary intersection of closed sets is closed, the above problem shows that $\overline{S}$ is closed; it is the smallest (in the sense of set inclusion) closed set that contains S. $\overline{S}$ is called the *closure* of S.

Part B

4. (10 pts) Let $S \subset \mathbb{R}^n$ be a set. Define

$$\text{bdry}(S) = \{x \in \mathbb{R}^n \colon \text{for each } r > 0,\ B(x, r) \cap S \neq \emptyset \text{ and } B(x, r) \cap (\mathbb{R}^n \backslash S) \neq \emptyset\}.$$

In words, "bdry(S)" is the set of points in $\mathbb{R}^n$ so that every ball centered at that point meets at least one point in S, and also meets at least one point not in S.

A. Prove that $\mathrm{bdry}(S) = \overline{S} \setminus \mathrm{int}(S)$. Here $\mathrm{int}(S)$ was defined in HW2.

B. Let $B(0,1) \subset \mathbb{R}^n$ be the unit ball. What is $\mathrm{bdry}(B(0,1))$? (it's not enough to state your answer; you need to prove it is correct.)

C. Part B suggests that the boundary of a set is in some sense "smaller" than the set. But this need not be the case. Give an example of a non-empty set $S \subset \mathbb{R}^n$ where $\mathrm{bdry}(S) = S$. Give an example of a countable set $S \subset \mathbb{R}^n$ where $\mathrm{bdry}(S)$ is uncountable. Prove that your examples have the claimed properties.

Remark: Some sources use the notation ∂S in place of $\mathrm{bdry}(S)$.

5. (5 pts). By the Heine-Borel theorem, we know that $(0,1) \subset \mathbb{R}$ is not compact. Find a cover $\mathcal{F}$ of $(0,1)$ that has no finite sub-cover (and prove that $\mathcal{F}$ has the stated property).

6. (5 pts) Apostol, Problem 3.22: Prove that a collection of disjoint open sets in $\mathbb{R}^n$ must be countable. Give an example of a collection of disjoint closed sets in $\mathbb{R}^n$ that is uncountable (and prove that the set is uncountable. We've already proved that $\mathbb{R}$ is uncountable, so you can use this fact.)