18.100B Problem Set 5

Due Thursday March 10 by 2:30pm. The problem set has been divided into two parts (A,B) to facilitate grading. Please hand in two separate packets with your name, the homework number (i.e. Homework 5), and the part (A, B) clearly labeled at the top of each packet. Don't forget to staple your homework (but don't staple the two packets together!).

New: When solving homework problems, you may cite the theorems proved in class. However, you may not cite theorems from Apostol that were not discussed/proved in class. Of course, you can include a proof of these theorems in your homework, but generally there will be a shorter and more direct proof, and you are encouraged to look for that.

Part A

1. (5 pts). Let

$$f(x) = \begin{cases} e^x, & x \leq 0, \\ \cos(x), & x > 0. \end{cases}$$

Prove that $f(x)$ is uniformly continuous on $\mathbb{R}$.

2. (5 pts). Let $C([0,1])$ be the set of all continuous functions $f : [0,1] \to \mathbb{R}$. Define a metric on $C([0,1])$ by

$$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

Prove that $d(f,g)$ is indeed a metric, and that $(C([0,1]), d)$ is a complete metric space.

3. (5 pts). Let $\mathbb{C} = \{x + iy : \ x, y \in \mathbb{R}\}$ be the complex numbers. We can turn $\mathbb{C}$ into a metric space using the metric $d((x_1 + iy_1), (x_2 + iy_2)) = \big((x_2 - x_1)^2 + (y_2 - y_1)^2\big)^{1/2}$. Let $f \colon \mathbb{C} \to \mathbb{C}$ be a continuous function. Suppose that $f(0) = -1$ and $f(1) = 1$ (here $0 = 0 + 0i$, $1 = 1 + 0i$, and $-1 = -1 + 0i$). Is it true that there exists a point $c \in \mathbb{C}$ so that $f(c) = 0$? Either prove that the answer is yes or provide a counter-example.

4. (5 pts). Let (M,d) be a metric space, where d is the discrete metric. What are the compact subsets of M? (prove that your answer is correct)

Part B

5. (5 pts). Apostol 4.13: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, and suppose $f(x) = 0$ whenever x is rational. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

6. (5 pts). Define $f \colon [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & \text{if } x \notin \mathbb{Q}, \\ 1/n, & \text{if } x = m/n \text{ in lowest terms, with } n > 0. \end{cases}$$

Prove that f is continuous only at the irrational points of $[0,1]$.

7. (5 pts). Recall the definition of a convex set in $\mathbb{R}^n$ from HW 4. Prove that every convex set in $\mathbb{R}^n$ is connected.