18.100B Problem Set 6

Due Thursday March 17 by 2:30pm. The problem set has been divided into three parts (A,B,C) to facilitate grading. Please hand in three separate packets with your name, the homework number (i.e. Homework 5), and the part (A, B,C) clearly labeled at the top of each packet. Don't forget to staple your homework (but don't staple the three packets together!).

New: When solving homework problems, you may cite the theorems proved in class. However, you may not cite theorems from Apostol that were not discussed/proved in class. Of course, you can include a proof of these theorems in your homework, but generally there will be a shorter and more direct proof, and you are encouraged to look for that.

Part A

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and suppose that f is differentiable at every point $x \in \mathbb{R}$ (with finite derivative). Suppose that f' is a continuous function, and that $f'(x) > 0$ for $x < -1$ and $f'(x) < 0$ for $x > 1$.

We will describe an algorithm intended to find a local extremum of f. The algorithm is as follows: Let $x_1 \in \mathbb{R}$ (this is the starting point) and let $t_1 > 0$ be a real number (this is the "step size"). At stage i, define

$$x_{i+1} = \begin{cases} x_i + t_i, & \text{if } f'(x_i) > 0, \\ x_i - t_i, & \text{if } f'(x_i) < 0, \\ x_i, & \text{if } f'(x_i) = 0. \end{cases}$$

Define

$$t_{i+1} = \begin{cases} t_i, & \text{if } f'(x_i)f'(x_{i-1}) \geq 0, \\ t_i/2, & \text{if } f'(x_i)f'(x_{i-1}) < 0. \end{cases}$$

(A) (5 points). Prove that the sequence $\{x_i\}$ is Cauchy, no matter what the initial point x_1 and the initial step size $t_i > 0$ is.

(B) (5 points). Let $x \in \mathbb{R}$ be the point to which the sequence $\{x_i\}$ converges. Prove that $f'(x) = 0$.

2. (5 pts) Let $f : [a, b] \to \mathbb{R}$. Suppose f is continuous on $[a, b]$ and differentiable on (a, b) (we assume $b > a$, so (a, b) is non-empty). Suppose that there exists a number $t \geq 0$ so that at each point $x \in (a, b)$, we have $|f'(x)| < t$. Prove that $|f(b) - f(a)| < t(b - a)$.

Part B

3. (5 points) (Apostol 5.18). Let $f : [a, b] \to \mathbb{R}$. Suppose that f is continuous on $[a, b]$ and differentiable at every point of (a, b), and the derivative is always finite. Suppose that $f(a) = f(b) = 0$. Prove that for every $t \in \mathbb{R}$, there is a point $c \in (a, b)$ with $f'(c) = tf(c)$.

4. (5 points) Let $f[-1, 1] \to \mathbb{R}$ be a continuous function. Define the function $g(x) : [-1, 1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} xf(x), & x \in \mathbb{Q}, \\ -xf(x), & x \notin \mathbb{Q}. \end{cases}$$

$g(x)$ is differentiable at 0 if and only if some condition on f holds. Find this condition on f (and prove that your answer is correct).

Part C

5. Let $C^1([0,1])$ be the set of all functions $f : [0,1] \to \mathbb{R}$ that satisfy

1. f is continuous at every point of $[0,1]$.
2. f is differentiable at every point of $(0,1)$ (with finite derivative).
3. The one-sided derivatives $f'_+(0)$ and $f'_-(1)$ exist (and are finite).
4. The function $f'(x)$ is continuous on $(0,1)$.
5. $\lim_{x\to 0+} f'(x) = f'_+(0)$ and $\lim_{x\to 1-} f'(x) = f'_-(1)$.

If $f, g \in C^1([0,1])$, define

$$d(f,g) = \sup_{x\in(0,1)} |f(x) - g(x)| + \sup_{x\in(0,1)} |f'(x) - g'(x)|.$$

(A) (2 points). Observe that in the above definition, the sup is over all $x \in (0,1)$ rather than $[0,1]$. For $|f(x) - g(x)|$, does this matter? What about for $|f'(x) - g'(x)|$?

(B) (5 points) Prove that $(C^1([0,1]), d)$ is a complete metric space. Hints: Problem 2 from this HW and Problem 2 from HW5 may be helpful.