18.100B Problem Set 9

Due Thursday April 21 by 2:30pm. When solving homework problems, you may cite the theorems proved in class. However, you may not cite theorems from Apostol that were not discussed / proved in class unless noted in the problem description.

Part A

1 Problem 1 contained an error, so has been removed.

2 (a, 5 points). (Apostol 7.14). Let $f, \alpha \colon [a,b] \to \mathbb{R}$. Suppose that f is continuous and α has bounded variation. Let $V(a) = 0$ and for $x \in (a,b]$, let $V(x)$ be the variation of α. Prove that

$$\Big| \int_a^b f d\alpha \Big| \le \int_a^b |f| dV. \tag{1}$$

(note: you can't immediately assume that $|f| \in \mathcal{R}(V)$. But since f is continuous and α has bounded variation, this shouldn't be hard to prove)

(b, 5 points). When does equality hold in (1)? Prove that your answer is correct.

3 (5 points) (Apostol 7.15) Let $\alpha \colon [a,b]$ be a function of bounded variation on $[a,b]$, and let $\alpha_1, \alpha_2, \ldots$ be a sequence of functions, each of bounded variation on $[a,b]$. Suppose that the total variation of $\alpha - \alpha_n$ on $[a,b]$ tends to 0 as $n \to \infty$, and assume that $\alpha_n(a) = \alpha(a)$ for each $n = 1, 2, \ldots$. Let $f \colon [a,b] \to \mathbb{R}$ be continuous. Prove that

$$\lim_{n\to\infty} \int_a^b f(x) d\alpha_n(x) = \int_a^b f(x) d\alpha.$$

(since f is continuous and α and each α_n have bounded variation, you can assume that the integrals in the above equation exist).

Part B

4 (5 points) Let $f_1, f_2, f_3, \ldots$ be a sequence of functions from $[a,b] \to \mathbb{R}$. Suppose that for each $x \in [a,b]$, $f_1(x) \le f_2(x) \le f_3(x) \le \ldots$, and $|f_j(x)| \le 1$ for each $j = 1, 2, \ldots$. Define $f(x) = \sup\{f_j(x) : j = 1, 2, \ldots\}$ (convince yourself that f is well defined). If the Riemann integral $\int_a^b f_j(x)dx$ exists for each j, is it true that $\int_a^b f(x)dx$ exists? Prove that the answer is yes, or provide a counter-example.

5 (a, 7 points) (Apostol 7.35) Let f be Riemann integrable on $[a,b]$. Suppose $0 \le f(x) \le M$ for all $x \in [a,b]$ (here $M > 0$), and $\int_a^b f(x)dx = I > 0$. Let $h = \frac{I}{2(M+b-a)}$. Prove that the set $\{x \in [a,b] \colon f(x) > h\}$ contains a finite number of disjoint open intervals, the sum of whose lengths is at least h.

Hint: Choose a partition P of $[a,b]$ so that every Riemann sum $\sum_{k=1}^n f(t_k)\Delta x_k$ is $\ge I/2$ (i.e. regardless of the choice of $t_k \in [x_{k-1}, x_k]$; you should explain why you can do this. Define

$$A = \{k \colon f(x) > h \text{ for all } x \in [x_{k-1}, x_k]\}.$$

You want to show that the sum of the lengths of the intervals $[x_{k-1}, x_k]$ with $k \in A$ is at least h.

(b, 3 points) Using part a, prove the following theorem: Let f be bounded and Riemann integrable on $[a, b]$. If $\int_a^b f dx > 0$, then there is an open interval I so that $f(x) > 0$ for all $x \in I$.