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On the complete cd-index of a Bruhat interval

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Abstract We study the non-negativity conjecture of the complete **cd**-index of a Bruhat interval as defined by Billera and Brenti. For each **cd**-monomial M we construct a set of paths, such that if a "flip condition" is satisfied, then the number of these paths is the coefficient of the monomial M in the complete **cd**-index. When the monomial contains at most one **d**, then the condition follows from Dyer's proof of Cellini's conjecture. Hence the coefficients of these monomials are non-negative. We also relate the flip condition to shelling of Bruhat intervals.

Keywords Complete **cd**-index · Coxeter groups · Bruhat order · Shelling

1 Introduction

Let (W, S) be a Coxeter system and $u < v$ two elements in W related in the Bruhat order. Billera and Brenti in [1] define a polynomial $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ in the non-commuting variables $\mathbf{c}, \mathbf{d}$, called the complete **cd**-index of the interval $[u, v]$. They conjecture that this polynomial has non-negative coefficients. In this article we study the non-negativity conjecture by constructing for each interval $[u, v]$ in the Bruhat order and each **cd**-monomial M a set of paths $T_M(u, v)$, such that if a condition, called the *flip condition* is satisfied, then the number of paths in $T_M(u, v)$ is equal to the coefficient of M in the complete **cd**-index $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$. We conjecture the flip condition to be true for all intervals and all monomials, which then would imply the non-negativity conjecture.

Using the notation explained in the next section, we briefly describe the flip condition in its different forms and give evidence for it to hold. To construct the set of

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paths $T_M(u,v)$ we need to fix a reflection order $\mathcal{O}$. Let $\overline{T}_M(u,v)$ be the set of paths constructed using the reverse order $\overline{\mathcal{O}}$. By induction on the length of $[u,v]$, both sets have the same number of paths, equal to the coefficient of the monomial M in $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$. Let $F: T_M(u,v) \to \overline{T}_M(u,v)$ be a bijection, called a flip. The (strong) flip condition states that if M starts with $\mathbf{c}$, then one can choose F in such a way that if $F(x)=y$, then the first reflection in x is less than or equal to the first reflection in y. (This condition is then used to define $T_{M'}(w,v)$ for longer intervals $[w,v]$, where $w<u<v$.)

As a special case, consider $M=\mathbf{c}^n$ for some $n<l[u,v]$, where $l[u,v]$ is the rank of the interval $[u,v]$. Then $T_M(u,v)$ is the set of ascending paths of length n from u to v (ascending with respect to the reflection order $\mathcal{O}$), and $\overline{T}_M(u,v)$ is the set of descending paths of length n. A result of Dyer [4] states that for any $x\in T_M(u,v)$ and $y\in \overline{T}_M(u,v)$, the first reflection in x always precedes the first reflection in y. Hence the flip condition for this M is true for any choice of F. As we will see below, this result suffices to prove that $|T_M(u,v)|$ is the coefficient of M in $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ in case the monomial M contains at most one $\mathbf{d}$. It follows that the complete **cd**-index $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ has non-negative coefficients if the rank $l[u,v]$ is at most 6: the top degree terms are non-negative because the Bruhat order is Gorenstein* and the lower order terms contain at most one $\mathbf{d}$.

The flip condition can be described in an equivalent form as follows. The polynomial $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ is computed by summing the ascent–descent sequences of all paths from u to v. Let us fix a reflection t and sum the ascent–descent sequences of all those paths of length n from u to v that have their first reflection $\le t$. This sum can be expressed in the form

$$f_n(\mathbf{c},\mathbf{d}) + A g_{n-1}(\mathbf{c},\mathbf{d})$$

for some homogeneous **cd**-polynomials f_n, g_{n-1} of degree $n, n-1$, respectively. The (strong) flip condition is equivalent to g_{n-1} having non-negative coefficients.

The second form of the flip condition can be related to the shelling of the Bruhat interval. When C is a regular CW-complex that is topologically an $(n-1)$-ball or an $(n-1)$-sphere, then the **cd**-index of C can be expressed in the form:

$$f_n(\mathbf{c},\mathbf{d}) + A g_{n-1}(\mathbf{c},\mathbf{d}),$$

for some homogeneous polynomials f_n and g_{n-1} with non-negative coefficients [6]. The Bruhat order on the interval $[u,v]$ is shellable with respect to the lexicographic ordering of maximal chains (using the reflection order $\mathcal{O}$) [3]. This implies that paths of maximal length from u to v with first reflection $\le t$ are the paths in the poset of a regular CW-complex C that is topologically a ball or a sphere. This means that the f_n and g_{n-1} in the two formulas above coincide, and in particular that the flip condition holds for paths of maximal length.

We consider the two positive results described above as evidence for the conjecture that the flip condition holds in general.

The approach to computing the **cd**-index by counting paths in $T_M(u,v)$ is motivated by the theory of sheaves on posets [6]. One can define a sheaf on an appropriate poset constructed using length n paths from u to v in the Bruhat graph. Then the flip condition states that one can carry out the same operations on this sheaf as in the case of the constant sheaf on a Gorenstein* poset described in [6]. The result of these operations is a vector space whose dimension is the coefficient of M in the **cd**-index. However, since the sheaf for the Bruhat graph is constructed from the paths in the graph, the operations reduce to counting paths with a given ascent–descent sequence. Therefore, we only work with paths in the Bruhat graph and do not mention sheaves again.

In the next section we recall the definition of the complete **cd**-index in terms of a reflection order. We then construct the sets $T_M(u, v)$ and give the condition for these sets to count the coefficient of M in the complete **cd**-index.

2 The complete cd-index

We fix a Coxeter system (W, S) (see [2, 5]) and a reflection order $\mathcal{O}$ (see [3]). The latter is a total order on the set of reflections of (W, S), satisfying a condition on dihedral subgroups. The reverse of the order $\mathcal{O}$ is also a reflection order. We denote it by $\overline{\mathcal{O}}$.

Let $l(x)$ be the length function on W. We write $u \prec v$ if $l(u) < l(v)$ and $u^{-1}v$ is a reflection. The relation $\prec$ generates the Bruhat order on W. The Bruhat graph has vertex set W and an edge from u to v if $u \prec v$.

Let $u < v$ in the Bruhat order. A path of length n from u to v in the Bruhat graph is a sequence

$$x = (u = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_n \prec x_{n+1} = v).$$

(Note a slightly unusual convention for the length. For example, the path $(u \prec v)$ has length 0.) We let $B_n(u, v)$ be the set of all paths of length n from u to v, and $B(u, v) = \bigcup_n B_n(u, v)$. We label an edge $x_i \prec x_{i+1}$ with the reflection $t_i = x_i^{-1} x_{i+1}$. The ascent–descent sequence of the path is

$$w(x) = \beta_1 \beta_2 \cdots \beta_n,$$

where

$$\beta_i = \begin{cases} A & \text{if } t_{i-1} < t_i, \\ D & \text{if } t_{i-1} > t_i. \end{cases}$$

The reflections t_i here are related by the reflection order $\mathcal{O}$.

Example 2.1 Figure 1 shows the Bruhat graph of the interval $[2134, 4321]$ in the Coxeter system where the group is the symmetric group S_4 generated by transpositions (12), (23), (34). The full Bruhat graph of this system can be found in [2]. The reflections here are the transpositions in S_4 and they are ordered as follows:

$$(12) < (13) < (14) < (23) < (24) < (34).$$

We number the reflections so that (12) has number 1, (13) has number 2, and so on. The edges in the Bruhat graph are then labeled with the numbers of the corresponding reflections. For example, the path $2134 \prec 2143 \prec 4123 \prec 4132 \prec 4312 \prec 4321$ has

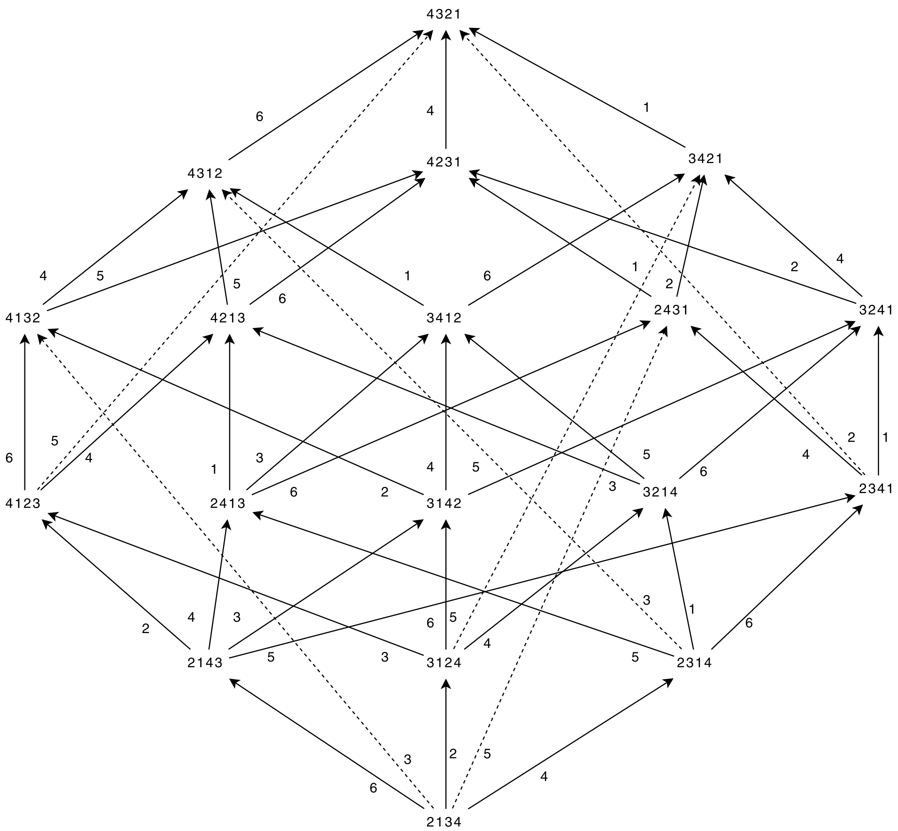

Fig. 1 The Bruhat graph of the interval [2134, 4321] in S_4

labels 62646, hence its ascent–descent sequence is $DADA$. The path $2134 \prec 3124 \prec 3421 \prec 4321$ has labels 251 and ascent–descent sequence AD.

Let $\mathbb{Z}\langle A, D\rangle$ be the polynomial ring in non-commuting variables A and D. Summing the ascent–descent sequences of all paths from u to v gives a polynomial in A and D:

$$\widetilde{\phi}_{u,v}(A, D) = \sum_{x \in B(u,v)} w(x).$$

The complete **cd**-index is obtained from this polynomial by a change of variable. Let $\mathbf{c} = A + D$ and $\mathbf{d} = AD + DA$. This gives an inclusion of rings

$$\mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle \subset \mathbb{Z}\langle A, D\rangle.$$

Billera and Brenti [1] prove that the polynomial $\widetilde{\phi}_{u,v}(A, D)$ lies in this subring, hence can be expressed in terms of $\mathbf{c}$ and $\mathbf{d}$:

$$\widetilde{\phi}_{u,v}(A, D) = \widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d}).$$

The polynomial $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ is the complete **cd**-index of the interval $[u,v]$. It does not depend on the chosen reflection order $\mathcal{O}$.

The rings $\mathbb{Z}\langle\mathbf{c},\mathbf{d}\rangle$ and $\mathbb{Z}\langle A,D\rangle$ are graded so that $A,D,\mathbf{c}$ have degree 1 and $\mathbf{d}$ has degree 2. If $l(u,v)=n+1$, then $B(u,v)$ can contain paths of length $n,n-2,n-4,\dots$. It follows that the polynomial $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ has nonzero terms of the same degree.

We define the involution $f\mapsto\overline{f}$ in the ring $\mathbb{Z}\langle A,D\rangle$ by $\overline{f(A,D)}=f(D,A)$. Elements of $\mathbb{Z}\langle\mathbf{c},\mathbf{d}\rangle$ are invariant by this involution. Since the polynomial $\widetilde{\phi}_{u,v}(A,D)$ is invariant by the involution, it follows that for every AD-monomial w, the number of paths $x\in B(u,v)$ with $w(x)=w$ is equal to the number of paths $y\in B(u,v)$ with $w(y)=\overline{w}$.

We will consider below homogeneous polynomials $p(A,D)\in\mathbb{Z}\langle A,D\rangle$ that can be expressed in the form $f_n(\mathbf{c},\mathbf{d})+g_{n-1}(\mathbf{c},\mathbf{d})D$ for some homogeneous **cd**-polynomials f_n and g_{n-1}. If such an expression exists, then it is unique. We can recover g_{n-1} by computing $p(A,D)-\overline{p(A,D)}=g_{n-1}(\mathbf{c},\mathbf{d})(D-A)$, and then subtracting $g_{n-1}(\mathbf{c},\mathbf{d})D$, we recover f_n. More generally, every homogeneous $p(A,D)\in\mathbb{Z}\langle A,D\rangle$ of degree n can be expressed in a unique way as

$$p(A,D)=f_n(\mathbf{c},\mathbf{d})+f_{n-1}(\mathbf{c},\mathbf{d})D+f_{n-2}(\mathbf{c},\mathbf{d})D^2+\cdots+f_0D^n$$

for some homogeneous **cd**-polynomials f_i.

If $M(\mathbf{c},\mathbf{d})$ is a **cd**-monomial, consider the AD-monomial $M(A,DA)$. This correspondence gives a bijection between **cd**-monomials and AD-monomials in which every D is followed by an A. Below we will often use the letter M to denote either the **cd**-monomial $M(\mathbf{c},\mathbf{d})$ or the AD-monomial $M(A,DA)$, with the distinction being clear from the context. For example, we define $T_{M(\mathbf{c},\mathbf{d})}(u,v)=T_{M(A,DA)}(u,v)$.

3 Coefficients of the complete cd-index

Let $u<v$ in the Bruhat order and let $M(\mathbf{c},\mathbf{d})$ be a **cd**-monomial of degree n. We wish to express the coefficient of M in $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ as a number of certain paths in $B_n(u,v)$. We start by defining a number $s_M(x)$ for every path $x\in B_n(u,v)$, giving the contribution of x to the coefficient of M. The numbers $s_M(x)$ are in the set $\{-1,0,1\}$. We then study the case when $s_M(x)$ is non-negative for every x and call it the flip condition. If the flip condition is satisfied, the number of paths x with $s_M(x)=1$ is the coefficient of M in $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$.

Recall that for any AD-monomial w, the number of paths $x\in B(u,v)$ with $w(x)=w$ is equal to the number of paths $y\in B(u,v)$ with $w(y)=\overline{w}$.

Definition 3.1 A flip $F=F_{u,v}$ is an involution

$$F_{u,v}:B(u,v)\to B(u,v),$$

such that $w(F(x))=\overline{w(x)}$.

We fix a flip $F_{u,v}$ for every $u < v$.

Let $x = (u \prec x_1 \prec x_2 \prec \cdots \prec x_n \prec v) \in B_n(u,v)$, and let $1 \le m \le n$. We apply the flip $F_{x_m,v}$ to the tail of x to get $y = (u \prec x_1 \prec \cdots \prec x_m \prec y_{m+1} \prec \cdots \prec y_n \prec v)$. If $w(x) = \beta_1 \cdots \beta_m \cdots \beta_n$, then $w(y) = \beta_1 \cdots \beta_{m-1}\alpha_m \overline{\beta}_{m+1} \cdots \overline{\beta}_n$, where α_m could be either A or D. Define

$$s_{m,A}(x) = \begin{cases} 1 & \text{if } \beta_m = A \\ 0 & \text{otherwise,} \end{cases}$$

$$s_{m,D}(x) = \begin{cases} 1 & \text{if } \beta_m = D, \alpha_m = A \\ -1 & \text{if } \beta_m = A, \alpha_m = D \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.2 Let $M(A, DA)$ be the AD-monomial $\gamma_1 \cdots \gamma_n$ and define

$$s_M(x) = s_{M(A,DA)}(x) = \prod_{m=1}^{n} s_{m,\gamma_m}(x).$$

Note that the definition of $s_M(x)$ depends on the flip F that was fixed before.

Let $x \in B_n(u,v)$, and let $y = F_{u,v}(x)$. Then the ascent–descent sequence of y when computed using the reverse reflection order $\overline{\mathcal{O}}$ is the same as the ascent–descent sequence of x computed using the order $\mathcal{O}$. Let us denote by $\overline{s}_M(y)$ the number $s_M(y)$ computed as above, but using the order $\overline{\mathcal{O}}$.

Definition 3.3 We say that the flip F is compatible with the reflection order $\mathcal{O}$ if

$$s_M(x) = \overline{s}_M(y)$$

for any $u < v$, M and $x \in B(u,v)$, $y = F_{u,v}(x)$.

Theorem 3.4 *Assume that F is compatible with the reflection order $\mathcal{O}$. For any* **cd**-*monomial $M(\mathbf{c},\mathbf{d})$ of degree n, the coefficient of M in $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ is equal to*

$$\sum_{x \in B_n(u,v)} s_M(x).$$

Proof Write $N(A,D) = M(A,DA)$ and for $0 \le m \le n$ let $N = N_m N_{n-m}$, where N_m, N_{n-m} are AD-monomials of degree $m, n-m$, respectively. Define

$$P_m = \sum_{x \in B_n(u,v)} w(u \prec x_1 \prec \cdots \prec x_{m+1}) \cdot s_{N_{n-m}}(x_m \prec x_{m+1} \prec \cdots \prec x_n \prec v).$$

Note that P_n is the degree n part of $\widetilde{\phi}_{u,v}$ and hence can be expressed as a homogeneous **cd**-polynomial of degree n. The statement of the theorem is that P_0 is the coefficient of M in P_n.

Lemma 3.5 *For $0 \le m \le n$ there exist homogeneous* **cd**-*polynomials f_m and g_{m-1} of degree m and $m-1$, respectively, such that*

$$P_m = f_m(\mathbf{c}, \mathbf{d}) + g_{m-1}(\mathbf{c}, \mathbf{d})D.$$

Moreover, P_{m-1} can be computed from P_m as follows.

(1) *If N_m ends with A, then $P_{m-1} = f_{m-1}(\mathbf{c}, \mathbf{d}) + g_{m-2}(\mathbf{c}, \mathbf{d})D$, where $f_m = f_{m-1}\mathbf{c} + g_{m-2}\mathbf{d}$.*
(2) *If N_m ends with D, then $P_{m-1} = g_{m-1}(\mathbf{c}, \mathbf{d})$.*

Proof We use induction on m. When $m = n$, then P_m is a homogeneous **cd** polynomial of degree n. Assume that $P_m = f_m(\mathbf{c}, \mathbf{d}) + g_{m-1}(\mathbf{c}, \mathbf{d})D$ and let us prove the "moreover" statement.

If N_m ends with A, let $x \in B_n(u, v)$ with $w(x) = \beta_1 \cdots \beta_m \cdots \beta_n$. Then

$$s_{AN_{n-m}}(x_{m-1} \prec x_m \prec \cdots \prec x_n \prec v) = \begin{cases} s_{N_{n-m}}(x_m \prec \cdots \prec x_n \prec v) & \text{if } \beta_m = A \\ 0 & \text{otherwise.} \end{cases}$$

Thus, to compute P_{m-1} from P_m, we consider only those monomials that end with A and then delete this last A. When contracting $P_m = f_m(\mathbf{c}, \mathbf{d}) + g_{m-1}(\mathbf{c}, \mathbf{d})D$ with A from the right, we get $f_{m-1}(\mathbf{c}, \mathbf{d}) + g_{m-2}(\mathbf{c}, \mathbf{d})D$, where $f_m = f_{m-1}\mathbf{c} + g_{m-2}\mathbf{d}$.

Now suppose N_m ends with D. By induction on m, the polynomials f_m and g_{m-1} depend only on $u < v$ and monomial M, not on the reflection order or the flip F. Let us denote by $\overline{w}$ and $\overline{s}_{N_{n-m}}$ the quantities computed using the same flip F, but with the reverse reflection order $\overline{\mathcal{O}}$. This does not change the polynomial P_m. Then $\overline{w}(x)$ is obtained from $w(x)$ by switching A and D. If

$$(x_m \prec y_{m+1} \prec \cdots \prec y_n \prec v) = F_{x_m,v}(x_m \prec x_{m+1} \prec \cdots \prec x_n \prec v),$$

then

$$\overline{s}_{N_{n-m}}(x_m \prec y_{m+1} \prec \cdots \prec y_n \prec v) = s_{N_{n-m}}(x_m \prec x_{m+1} \prec \cdots \prec x_n \prec v)$$

by the compatibility condition on F.

Let $F_m : B_n(u, v) \to B_n(u, v)$ be the involution that flips the tail of a path:

$$F_m(u \prec x_1 \prec \cdots \prec x_n \prec v) = (u \prec x_1 \prec \cdots \prec x_m \prec y_{m+1} \prec \cdots \prec y_n \prec v).$$

Since this is a bijection, we may compute P_m with respect to $\overline{\mathcal{O}}$ by summing over $F_m(x)$. We call this new polynomial Q_m. By the previous discussion $Q_m = P_m$.

$$\begin{aligned} Q_m &= \sum_{x \in B_n(u,v)} \overline{w}(u \prec x_1 \prec \cdots \prec y_{m+1}) \cdot \overline{s}_{N_{n-m}}(x_m \prec y_{m+1} \prec \cdots \prec y_n \prec v) \\ &= \sum_{x \in B_n(u,v)} \overline{w}(u \prec x_1 \prec \cdots \prec y_{m+1}) \cdot s_{N_{n-m}}(x_m \prec x_{m+1} \prec \cdots \prec x_n \prec v). \end{aligned}$$

Let us now compute $P_m - \overline{Q}_m = g_{m-1}(\mathbf{c}, \mathbf{d}) \cdot (D - A)$:

$$\sum_{x \in B_n(u,v)} \big(w(u \prec x_1 \prec \cdots \prec x_{m+1}) - w(u \prec x_1 \prec \cdots \prec y_{m+1})\big) \\ \cdot s_{N_{n-m}}(x_m \prec \cdots \prec x_n \prec v).$$

For $x \in B_n(u, v)$, let

$$w(u \prec x_1 \prec \cdots \prec x_m \prec x_{m+1}) = \beta_1 \cdots \beta_{m-1}\beta_m,$$
$$w(u \prec x_1 \prec \cdots \prec x_m \prec y_{m+1}) = \beta_1 \cdots \beta_{m-1}\alpha_m.$$

Then x contributes to the sum if and only if $\beta_m \neq \alpha_m$. The contribution is

$$\pm w(u \prec x_1 \prec \cdots \prec x_m)(D - A)s_{N_{n-m}}(x_m \prec \cdots \prec x_n \prec v),$$

where the sign is positive if $\beta_m = D, \alpha_m = A$ and negative otherwise. Notice that, with the same sign,

$$\pm s_{N_{n-m}}(x_m \prec \cdots \prec x_n \prec v) = s_{DN_{n-m}}(x_{m-1} \prec x_m \prec \cdots \prec x_n \prec v).$$

This means that $P_{m-1} = g_{m-1}$. □

Now suppose m is such that $M(\mathbf{c}, \mathbf{d}) = M_m(\mathbf{c}, \mathbf{d}) \cdot M_{n-m}(\mathbf{c}, \mathbf{d})$ where M_m, M_{n-m} are **cd**-monomials of degree $m, n - m$, respectively. Then the inductive computation of P_{m-1} from P_m in the lemma can be restated as follows. Let $f_m = f_{m-1}\mathbf{c} + g_{m-2}\mathbf{d}$.

(1) If M_m ends with **c**, then

$$P_{m-1} = f_{m-1} + g_{m-2} \cdot D.$$

(2) If M_m ends with **d**, then

$$P_{m-2} = g_{m-2}.$$

If we only consider the degree m term f_m of P_m, then in the first case f_{m-1} is obtained from f_m by contracting with **c** from the right. In the second case f_{m-2} is obtained from f_m by contracting with **d** from the right. It follows that P_0 is the number that is obtained from P_n by contracting with the monomial M. In other words, P_0 is the coefficient of M in P_n. □

4 Non-negativity of the complete cd index

There are two problems with computing the coefficients of the complete **cd**-index as described in the previous section. The first is that the formula involves negative signs. The second problem is that it is not clear how to define a flip F that is compatible with the reflection order.

In this section we define the "flip condition" requiring that all terms $s_M(x)$ that go into the computation of the coefficient of M in the complete **cd**-index are non-negative. In this case we define a set of paths $T_M(u,v) \subset B(u,v)$, such that $|T_M(u,v)|$ is the coefficient of M in $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$. It also turns out that the flip condition gives an optimal way of defining the flip $F_{u,v}$. The flip condition for the interval $[u,v]$ only involves the flips $F_{w,v}$ where $u < w < v$, hence this gives an inductive procedure for defining F, checking the flip condition and constructing the set $T_M(u,v)$.

Let $u < v$ in the Bruhat order, and let $M(\mathbf{c},\mathbf{d})$ be a **cd**-monomial of degree n. Let $M(A, DA)$ be the AD-monomial $\gamma_1 \cdots \gamma_n$.

Definition 4.1 Let

$$T_M(u,v) = T_{\gamma_1\cdots\gamma_n}(u,v) = \bigl\{x \in B_n(u,v) | s_{m,\gamma_m}(x) = 1 \text{ for all } 1 \le m \le n\bigr\}.$$

Remark 4.2 Using the definition of s_{m,γ_m}, a path x lies in $T_M(u,v)$ if and only if

(1) $w(x) = \gamma_1 \cdots \gamma_n$.
(2) For any m such that $\gamma_m = D$, let

$$(x_m \prec y_{m+1} \prec \cdots \prec y_n \prec v) = F_{x_m,v}(x_m \prec \cdots \prec x_n \prec v).$$

Then $w(x_{m-1} \prec x_m \prec y_{m+1}) = A$.

The paths $x \in T_M(u,v)$ all satisfy $s_M(x) = 1$. The following condition implies that these are the only paths $x \in B_n(u,v)$ with $s_M(x) \neq 0$.

Definition 4.3 The *flip condition* holds for the interval $[u,v]$ and monomial M if for every $x \in B_n(u,v)$ the following is satisfied. If $s_{m,\gamma_m}(x) = -1$ for some m, then there exists $l > m$ such that $s_{l,\gamma_l}(x) = 0$.

This condition can be re-written using the definition of s_{m,γ_m} by saying that the flip condition is violated for some $x \in B_n(u,v)$ if there exists m such that

(1) $(x_m \prec \cdots \prec x_n \prec v) \in T_{\gamma_{m+1}\cdots\gamma_n}(x_m, v)$. (Equivalently, $s_{l,\gamma_l}(x) = 1$ for $l > m$.)
(2) $\gamma_m = D$ and if

$$(x_m \prec y_{m+1} \prec \cdots \prec y_n \prec v) = F_{x_m,v}(x_m \prec \cdots \prec x_n \prec v),$$

then $w(x_{m-1} \prec x_m \prec x_{m+1}) = A$ and $w(x_{m-1} \prec x_m \prec y_{m+1}) = D$. (Equivalently, $s_{m,\gamma_m}(x) = -1$.)

From Theorem 3.4 we now get:

Corollary 4.4 *Assume that F is compatible with the reflection order $\mathcal{O}$. If the flip condition holds for the interval $[u,v]$ and monomial M, then $|T_M(u,v)|$ is the coefficient of M in $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$.*

Example 4.5 Consider the Bruhat interval $[u,v] = [2134, 4321]$ shown in Fig. 1. We compute the sets $T_M(u,v)$ for different monomials.

When $M = \mathbf{c}^n$, $n = 4$ or $n = 2$, then the condition (2) in Remark 4.2 is trivially true, hence $T_M(u,v)$ consists of all ascending paths of length n from u to v. Thus, $T_{\mathbf{c}^2}(u,v) = \{346, 235\}$ and $T_{\mathbf{c}^4}(u,v) = \{23456\}$.

When $M = \mathbf{d}$, we consider paths with ascent–descent sequence $M(A, DA) = DA$. There are three such paths: 436, 514, 625. We need to check that when we flip such a path $(u \prec x_1 \prec x_2 \prec v)$ to $(u \prec x_1 \prec y_2 \prec v)$, then the resulting path must have ascent–descent sequence AD. For the three paths, the interval $[x_1, v]$ contains exactly one ascending and one descending path of length 1, hence the flip $F_{x_1,v}$ is uniquely defined. The flips of the three paths are the paths 462, 521, 652. Only the first one of these has the correct ascent–descent sequence. Hence $T_{\mathbf{d}}(u, v) = \{436\}$.

For a slightly longer computation, let us find the set $T_{\mathbf{d}^2}(u, v)$. For this we need to find all paths $(u \prec x_1 \prec x_2 \prec x_3 \prec x_4 \prec v)$ with ascent–descent sequence $DADA$ and check the ascent–descent sequences after applying the flips $F_{x_3,v}$ and $F_{x_1,v}$. There are six paths with ascent–descent sequence $DADA$,

$$62646, 64614, 63416, 63524, 41516, 41624.$$

In all six cases, the flip $F_{x_3,v}$ is again mapping the unique ascending chain of length 1 to the unique descending chain of length 1. Applying the flip $F_{x_3,v}$ we get paths 62654, 64621, 63461, 63541, 41561, 41641. Among these, only the third and the fifth have the required ascent–descent sequence $DAAD$. This reduces the candidate paths to two: 63416 and 41516. To define the flip $F_{x_1,v}$, we first need to construct the sets $T_{ADA}(x_1, v)$ and $\overline{T}_{ADA}(x_1, v)$. For the first path 63416 we find that $T_{ADA}(2143, v) = \{3416\}$ and $\overline{T}_{ADA}(2143, v) = \{4361\}$. (Both computations involve finding all paths $x = (2143 \prec x_2 \prec x_3 \prec x_4 \prec v)$ with $w(x) = ADA$ and checking the flip $F_{x_3,v}$ for them.) Thus, applying the flip $F_{x_3,v}$ to the path 63416 gives 64361. This path does not have the required ascent–descent sequence $ADAD$. For the second path 41516, we similarly find $T_{ADA}(2314, v) = \{1516\}$ and $\overline{T}_{ADA}(2314, v) = \{5361\}$. The result of applying the flip to 41516 is 45361 with the required ascent–descent sequence $ADAD$. Thus, $T_{\mathbf{d}^2}(u, v) = \{41516\}$.

It follows from the discussion in the introduction that the interval $[u, v]$ in this example satisfies the flip condition for any monomial (either the paths have maximal length or the monomial contains at most one $\mathbf{d}$). This implies that the complete $\mathbf{cd}$-index $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ has non-negative coefficients. By the computation above, $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d}) = 2\mathbf{c}^2 + \mathbf{d} + \mathbf{c}^4 + x\mathbf{c}^2\mathbf{d} + y\mathbf{cdc} + z\mathbf{dc}^2 + \mathbf{d}^2$ for some $x, y, z \geq 0$.

Let us now turn to the definition of the flip $F_{u,v}$. Note that the flip condition on the interval $[u, v]$ and the definition of $T_M(u, v)$ involve flips $F_{w,v}$ for proper subintervals $[w, v] \subsetneq [u, v]$ only. Hence, to construct $F_{u,v}$, we may assume that $T_M(u, v)$ is constructed for any M. Similarly, let $\overline{T}_M(u, v)$ denote the set $T_M(u, v)$ constructed using the reverse reflection order $\overline{\mathcal{O}}$. Assuming the flip condition on $[u, v]$, the sets $T_M(u, v)$ and $\overline{T}_M(u, v)$ have the same number of elements. We claim that the set $B(u, v)$ contains the disjoint union

$$\bigsqcup_M \bigl(T_M(u, v) \sqcup \overline{T}_M(u, v)\bigr)$$

over all $\mathbf{cd}$-monomials M. This follows from the fact that the AD-monomials $M(A, DA)$ and $\overline{M(A, DA)}$ are all distinct (e.g. $M(A, DA) \neq \overline{M'(A, DA)}$ because the first ends with A, the second with D). From this we find that any bijection

$$\bigsqcup_M T_M(u,v) \to \bigsqcup_M \overline{T}_M(u,v)$$

can be extended to a flip $F_{u,v} : B(u,v) \to B(u,v)$.

Definition 4.6 Assume that the flip condition holds for the interval $[u,v]$ and all **cd**-monomials M. Define $f_{u,v} : T_M(u,v) \to \overline{T}_M(u,v)$ as the bijection that preserves the lexicographic ordering of paths. (The path with the smallest first reflection maps to a path with the smallest first reflection, using the order $\mathcal{O}$ on both sides.) Let the flip $F_{u,v}$ be an extension of the bijection

$$\bigsqcup_M T_M(u,v) \xrightarrow{f_{u,v}} \bigsqcup_M \overline{T}_M(u,v)$$

to an involution on $B(u,v)$.

The flip $F_{u,v}$ is used to check the flip condition on larger intervals $[z,v]$ for $z < u$, and to construct the sets $T_{M'}(z,v)$. In both cases the flip $F_{u,v}$ is only applied to paths that lie in $T_M(u,v)$ (or to paths in $\overline{T}_M(u,v)$ when constructing $\overline{T}_{M'}(z,v)$), hence we only need the flip defined on these sets.

We claim that the flip $F_{u,v}$ constructed above automatically satisfies the compatibility condition in Definition 3.3. Indeed $s_M(x) = \overline{s}_M(F_{u,v}(x)) = 1$ for any $x \in T_M(u,v)$ and $s_M(x) = \overline{s}_M(F_{u,v}(x)) = 0$ for any $x \notin T_M(u,v)$.

The flip $F_{u,v}$ is optimal in the following sense. Consider a path $(z \prec u \prec x_1 \prec \cdots \prec x_n \prec v)$ and the AD-monomial $\gamma_0\gamma_1 \cdots \gamma_n$. We claim that if the path z violates the flip condition for $m = 0$ and for the flip $F_{u,v}$ defined above, then it violates the flip condition for any $\tilde{F}_{u,v}$. Equivalently, if the flip condition holds for some $\tilde{F}_{u,v}$ then it holds for the $F_{u,v}$ defined above. Indeed, the path violates the flip condition for $m = 0$ when $x = (u \prec x_1 \prec \cdots \prec x_n \prec v) \in T_M(u,v)$ and after applying $F_{u,v}$ we get the path $(u \prec y_1 \prec \cdots \prec y_n \prec v)$, such that $w(z \prec u \prec x_1) = A$ and $w(z \prec u \prec y_1) = D$. This implies that

$$u^{-1}y_1 < z^{-1}u < u^{-1}x_1.$$

Here $u^{-1}x_1$ is the first reflection in $(u \prec x_1 \prec \cdots \prec v) \in T_M(u,v)$ and $u^{-1}y_1$ is the first reflection in $(u \prec y_1 \prec \cdots \prec v) \in \overline{T}_M(u,v)$. Since $F_{u,v}$ preserves ordering by first reflection, it follows that $T_M(u,v)$ has fewer paths with first reflection less than $z^{-1}u$ than does $\overline{T}_M(u,v)$. Thus, no matter how the flip $\tilde{F}_{u,v}$ is chosen, some $x \in T_M(u,v)$ with first reflection greater than $z^{-1}u$ maps to $y \in \overline{T}_M(u,v)$ with first reflection smaller than $z^{-1}u$. Hence the flip condition is violated for any $\tilde{F}_{u,v}$. (Note that in the argument above $\gamma_0 = D$, hence for $\gamma_0 \cdots \gamma_n$ to come from a **cd**-monomial, we need $\gamma_1 = A$ and thus $M = \mathbf{c}M'$. Then the path $(z \prec u \prec x_1 \prec \cdots \prec x_n \prec v)$ violates the flip condition for the interval $[z,v]$ and monomial $\mathbf{d}M'$.)

Definition 4.7 We say that the *strong flip condition* holds for the interval $[u, v]$ and monomial M if for any $x \in T_M(u, v)$ and $y = F_{u,v}(x)$,

$$u^{-1}x_1 \le u^{-1}y_1$$

whenever M starts with $\mathbf{c}$, $M = \mathbf{c}M'$.

The strong flip condition is stronger than the flip condition because the flip condition allows $u^{-1}y_1 < u^{-1}x_1$ as long as there is no $z^{-1}u$ between them for some $z \prec u$.

When $M = \mathbf{c}^n$, then the strong flip condition was proved by Dyer [4]. Since to check the flip condition, we only need to check the flip for each occurrence of $\mathbf{d}$ in M, it follows from this that the flip condition holds for any interval and any monomial M that contains at most one $\mathbf{d}$. Thus, the coefficients of such monomials are nonnegative in any $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$.

5 Shelling of the Bruhat interval

In this section we give an equivalent formulation of the flip condition that is related to shelling of Bruhat intervals.

Let t be a reflection. Denote

$$B_n(u, v)_{\le t} = \{(u \prec x_1 \prec \cdots \prec x_n \prec v) \in B_n(u, v) | u^{-1}x_1 \le t\}.$$

Also let

$$T_M(u, v)_{\le t} = T_M(u, v) \cap B_n(u, v)_{\le t}, \qquad \overline{T}_M(u, v)_{\le t} = \overline{T}_M(u, v) \cap B_n(u, v)_{\le t}.$$

Theorem 5.1 *The AD-polynomial*

$$\widetilde{\phi}^{\le t}_{u,v} = \sum_{x \in B_n(u,v)_{\le t}} w(x)$$

can be expressed in the form $f_n(\mathbf{c}, \mathbf{d}) + Ag_{n-1}(\mathbf{c}, \mathbf{d})$ for some homogeneous **cd**-*polynomials f_n, g_{n-1} of degree $n, n-1$, respectively. Assuming that the flip condition holds for the interval $[u, v]$ and monomial M, then $|\overline{T}_M(u, v)_{\le t}|$ is the coefficient of M in f_n and $|T_M(u, v)_{\le t}|$ is the coefficient of M in $f_n + \mathbf{c}g_{n-1}$.*

Before we prove this theorem, let us derive an equivalent form of the flip condition from it. Suppose the flip condition holds for the interval $[u, v]$ and monomial $M = \mathbf{c}M'$, but is violated for some $(z \prec u \prec x_1 \prec \cdots \prec x_n \prec v)$ and monomial $\mathbf{d}M'$. Let $t = z^{-1}u$. Then, as in the previous section, we must have

$$|T_M(u, v)_{\le t}| < |\overline{T}_M(u, v)_{\le t}|.$$

By the theorem, the difference between the two numbers is the coefficient of M in $\mathbf{c}g_{n-1}$. Clearly this argument can also be reversed to get an equivalent condition. For simplicity we will state it without specifying the intervals and monomials.

Corollary 5.2 *The flip condition holds for all intervals and all monomials if and only if in the expression* $\widetilde{\phi}_{u,v}^{\leq t} = f_n(\mathbf{c},\mathbf{d}) + A g_{n-1}(\mathbf{c},\mathbf{d})$ *the polynomial* $g_{n-1}(\mathbf{c},\mathbf{d})$ *has non-negative coefficients for all intervals* $[u,v]$ *and all reflections* $t = z^{-1}u$, *where* $z \prec u$. □

The strong flip condition defined at the end of previous section is equivalent to g_{n-1} having non-negative coefficients for any interval $[u,v]$ and any reflection t.

By the theorem, the flip condition also implies that the polynomial $f_n(\mathbf{c},\mathbf{d})$ has non-negative coefficients.

Proof To prove the first statement of the theorem, it suffice to show that the AD-polynomial

$$\sum_{u^{-1}x_1=t} w(x),$$

where the sum runs over all $x \in B_n(u,v)$ having t as its first reflection, has the stated form. This sum can be written as

$$\begin{aligned} &A \sum_{y\in B_{n-1}(x_1,v)_{\leq t}} w(y) + D \sum_{y\in B_{n-1}(x_1,v)_{>t}} w(y) \\ &\quad = A \sum_{y\in B_{n-1}(x_1,v)_{\leq t}} w(y) + D\biggl(\sum_{y\in B_{n-1}(x_1,v)} w(y) - \sum_{y\in B_{n-1}(x_1,v)_{\leq t}} w(y)\biggr) \\ &\quad = (A-D) \sum_{y\in B_{n-1}(x_1,v)_{\leq t}} w(y) + D \sum_{y\in B_{n-1}(x_1,v)} w(y). \end{aligned}$$

Here the subscript $> t$ has similar meaning to $\leq t$. Using induction, we can write this as

$$\begin{aligned} &(A-D)\bigl(f_{n-1}(\mathbf{c},\mathbf{d}) + A g_{n-2}(\mathbf{c},\mathbf{d})\bigr) + D h_{n-1}(\mathbf{c},\mathbf{d}) \\ &\quad = (2A - A - D) f_{n-1} + \bigl(A^2 + AD - AD - DA\bigr) g_{n-2} + (A + D - A) h_{n-1} \\ &\quad = (-\mathbf{c} f_{n-1} - \mathbf{d} g_{n-2} + \mathbf{c} h_{n-1}) + A(2 f_{n-1} + \mathbf{c} g_{n-2} - h_{n-1}), \end{aligned}$$

for some **cd**-polynomials $f_{n-1}, g_{n-2}, h_{n-1}$.

The proof of the second statement is very similar to the proof of Theorem 3.4, so we only sketch it.

Let $N(A,D) = M(A,DA)$ and for $0 \leq m \leq n$ write $N = N_m N_{n-m}$, where N_m, N_{n-m} are AD-monomials of degree $m, n-m$, respectively. Define

$$P_m^{\leq t} = \sum_{x\in B_n(u,v)_{\leq t}} w(u \prec x_1 \prec \cdots \prec x_{m+1}) \cdot s_{N_{n-m}}(x_m \prec x_{m+1} \prec \cdots \prec x_n \prec v),$$

$$Q_m^{\leq t} = \sum_{x\in B_n(u,v)_{\leq t}} \overline{w}(u \prec x_1 \prec \cdots \prec x_{m+1}) \cdot \overline{s}_{N_{n-m}}(x_m \prec x_{m+1} \prec \cdots \prec x_n \prec v).$$

Note that

$$P_n^{\leq t} = \widetilde{\phi}_{u,v}^{\leq t} = f_n(\mathbf{c}, \mathbf{d}) + A g_{n-1}(\mathbf{c}, \mathbf{d}),$$

$$Q_n^{\leq t} = \overline{P}_n^{\leq t} = f_n(\mathbf{c}, \mathbf{d}) + D g_{n-1}(\mathbf{c}, \mathbf{d}).$$

On the other hand,

$$P_0^{\leq t} = \left|T_M(u, v)_{\leq t}\right|, \qquad Q_0^{\leq t} = \left|\overline{T}_M(u, v)_{\leq t}\right|.$$

Lemma 5.3 *There exist* **cd***-polynomials* $f_m, g_{m-1}, h_{m-1}, l_{m-2}$ *of degree* m, $m-1, m-1, m-2$, *respectively, such that*

$$P_m^{\leq t} = (f_m + A g_{m-1}) + (h_{m-1} + A l_{m-2}) D.$$

For $m > 0$,

$$Q_m^{\leq t} = (f_m + D g_{m-1}) + (h_{m-1} + D l_{m-2}) D.$$

Moreover, $P_{m-1}^{\leq t}$, $Q_{m-1}^{\leq t}$ *can be computed from* $P_m^{\leq t}$, $Q_m^{\leq t}$ *as follows.*

(1) *If* N_m *ends with* A *and* $m \geq 1$, *then*

$$P_{m-1}^{\leq t} = (f_{m-1} + A g_{m-2}) + (h_{m-2} + A l_{m-3}) D,$$

where $f_m = f_{m-1}\mathbf{c} + h_{m-2}\mathbf{d}$ *and* $g_{m-1} = g_{m-2}\mathbf{c} + l_{m-3}\mathbf{d}$.

(2) *If* N_m *ends with* A *and* $m = 1$, *let* $P_1^{\leq t} = \alpha\mathbf{c} + A\beta + \gamma D$ *for* $\alpha, \beta, \gamma \in \mathbb{Z}$. *Then* $P_0^{\leq t} = \alpha + \beta$ *and* $Q_0^{\leq t} = \alpha$.

(3) *If* N_m *ends with* D, *then*

$$P_{m-1}^{\leq t} = h_{m-1} + A l_{m-2}.$$

Proof If N_m ends with A, we contract $P_m^{\leq t}$ and $Q_m^{\leq t}$ with A from the right to get $P_{m-1}^{\leq t}$ and $Q_{m-1}^{\leq t}$.

If N_m ends with D, then as before,

$$P_{m-1}^{\leq t}(D - A) = P_m^{\leq t} - \overline{Q}_m^{\leq t} = h_{m-1}(D - A) + A l_{m-2}(D - A),$$

$$Q_{m-1}^{\leq t}(D - A) = Q_m^{\leq t} - \overline{P}_m^{\leq t} = h_{m-1}(D - A) + D l_{m-2}(D - A).$$

□

Let m be such that $M = M_m M_{n-m}$, where M_m, M_{n-m} are **cd**-monomials of degree $m, n-m$, respectively. The lemma then implies:

(1) If M_m ends with **c** and $m > 1$, then $f_{m-1} + A g_{m-2}$ is obtained by contracting $f_m + A g_{m-1}$ with **c** from the right.

(2) If M_m ends with **c** and $m = 1$, then P_0 is obtained by contracting $f_1 + \mathbf{c} g_0$ with **c** from the right and Q_0 is obtained by contracting f_1 with **c** from the right.

(3) If M_m ends with $\mathbf{d}$, then $f_{m-2} + Ag_{m-3}$ is obtained by contracting $f_m + Ag_{m-1}$ with $\mathbf{d}$ from the right.

It follows from this that if $P_n^{\leq t} = f_n + Ag_{n-1}$, then $P_0^{\leq t}$ is obtained from $f_n + \mathbf{c}g_{n-1}$ by contracting with M. Thus, $P_0^{\leq t} = |T_M(u,v)_{\leq t}|$ is the coefficient of M in $f_n + \mathbf{c}g_{n-1}$. Similarly, $Q_0^{\leq t}$ is obtained from f_n by contracting with M, hence $Q_0^{\leq t} = |\overline{T}_M(u,v)_{\leq t}|$ is the coefficient of M in f_n. □

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