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Semistable Reduction in Characteristic Zero for Families of Surfaces and Threefolds

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Abstract. We consider the problem of extending the semistable reduction theorem of [KKMS] from the case of one-parameter families of varieties to families over a base of arbitrary dimension. Following [KKMS], semistable reduction of such families can be reduced to a problem in the combinatorics of polyhedral complexes [AK]. In this paper we solve it in the case when the relative dimension of the morphism is at most three, i.e., for families of surfaces and threefolds.

1. Introduction

One of the milestones in algebraic geometry is the semistable reduction theorem proved in [KKMS]:

Theorem 1.1 [KKMS]. *Let $f\colon X \to C$ be a flat morphism from a variety X onto a nonsingular curve C, defined over an algebraically closed field k of characteristic zero. Assume that $0 \in C$ is a point and the restriction $f\colon X \setminus f^{-1}(0) \to C \setminus \{0\}$ is smooth. Then there exist a finite morphism $\pi\colon C' \to C$, with $\pi^{-1}(0) = \{0'\}$, and a proper birational morphism* (*in fact, a blowup with center lying in the special fiber*) $p\colon X' \to X \times_C C'$,

$$\begin{array}{ccccc} X' & \xrightarrow{p} & X \times_C C' & \rightarrow & X \\ & & \downarrow & & \downarrow \\ & & C' & \xrightarrow{\pi} & C \end{array}$$

so that the induced morphism $f'\colon X' \to C'$ is semistable; *i.e.*,

(i) *both X' and C' are nonsingular, and*
(ii) *the special fiber $f'^{-1}(0')$ is a reduced divisor with nonsingular components crossing normally.*

To prove the theorem, Kempf et al. [KKMS] invented the theory of toroidal embeddings and reduced the geometric problem to the following purely combinatorial problem:

Theorem 1.2 [KKMS]. *Let $P \subset \mathbb{R}^n$ be an n-dimensional polytope with vertices lying in the integral points $\mathbb{Z}^n \subset \mathbb{R}^n$. Then there exists an integer M and a projective subdivision $\{P_\alpha\}_\alpha$ of P such that every P_α has vertices in $(1/M)\mathbb{Z}^n$ and the volume of P_α (in the usual metric) is the minimal possible: $vol(P_\alpha) = 1/M^n n!$.*

Here a subdivision is called projective (or coherent) if it is defined by a continuous piecewise linear convex function.

The main goal of [AK] was to extend the semistable reduction theorem to the case where the base variety has arbitrary dimension. The problem can then be formulated as follows:

Conjecture 1.3. *Let $f: X \to B$ be a surjective morphism of projective varieties with geometrically integral generic fiber, defined over an algebraically closed field of characteristic zero. There exist a proper surjective generically finite morphism $B' \to B$ and a proper birational morphism $X' \to X \times_B B'$ such that $X' \to B'$ is semistable; i.e., for any closed point $x' \in X'$ and $b' = f'(x') \in B'$ one can find formal coordinates $x_1, \dots, x_n$ at x' and $t_1, \dots, t_m$ at b so that the morphism f is given by*

$$t_i = \prod_{j=l_{i-1}+1}^{l_i} x_j$$

for some $0 = l_0 < l_1 < \cdots < l_m \leq n$.

Using the theory of toroidal embeddings, the geometric problem of semistable reduction can again be reduced to a combinatorial problem involving conical polyhedral complexes. The aim of this paper is to solve the combinatorial problem for the case when f has low relative dimension. First, we recall the definition of polyhedral complexes and morphisms.

1.1. *Polyhedral Complexes*

We consider (rational, conical) polyhedral complexes $\Delta = (|\Delta|, (\{\sigma, N_\sigma\})$ consisting of a finite collection of lattices $N_\sigma \cong \mathbb{Z}^n$ and rational full cones $\sigma \subset N_\sigma \otimes \mathbb{R}$ with a vertex. The cones σ are glued together to form the space $|\Delta|$ so that the usual axioms of polyhedral complexes hold:

1. If $\sigma \in \Delta$ is a cone, then every face σ' of σ is also in Δ, and $N_{\sigma'} = N_\sigma|_{Span(\sigma')}$.
2. The intersection of two cones $\sigma_1 \cap \sigma_2$ is a face of both of them.

A morphism $f_\Delta: \Delta_X \to \Delta_B$ of polyhedral complexes $\Delta_X = (|\Delta_X|, \{\sigma, N_\sigma\})$ and $\Delta_B = (|\Delta_B|, \{\tau, N_\tau\})$ is a compatible collection of linear maps $f_\sigma: (\sigma, N_\sigma) \to (\tau, N_\tau)$; i.e., if σ' is a face of σ, then $f_{\sigma'}$ is the restriction of f_σ. We only consider morphisms $f: \Delta_X \to \Delta_B$ such that $f_\sigma^{-1}(0) \cap \sigma = \{0\}$ for all $\sigma \in \Delta_X$.

Polyhedral complexes arise naturally in the theory of toroidal embeddings [KKMS]. They generalize the notion of fans of toric varieties. An open embedding of varieties $U_X \subset X$ is said to be toroidal if it is locally formally isomorphic to a torus embedding $T \subset X_\sigma$; a morphism of toroidal embeddings is a morphism of varieties that locally formally comes from a toric morphism. To a toroidal embedding one associates a polyhedral complex, and a morphism of toroidal embeddings gives rise to a morphism of polyhedral complexes. The condition of semistability, when applied to a toroidal embedding, translates into the following condition on the associated morphism of polyhedral complexes.

Definition 1.4. A surjective morphism $f_\Delta\colon \Delta_X \to \Delta_B$ such that $f^{-1}(0) = \{0\}$ is **semistable** if:

1. Δ_X and Δ_B are nonsingular.
2. For any cone $\sigma \in \Delta_X$, we have $f(\sigma) \in \Delta_B$ and $f(N_\sigma) = N_{f(\sigma)}$.

We say that f is **weakly semistable** if it satisfies the two conditions except that Δ_X may be singular.

The following two operations are allowed on Δ_X and Δ_B:

1. Projective subdivisions Δ'_X of Δ_X and Δ'_B of Δ_B such that f induces a morphism $f'\colon \Delta'_X \to \Delta'_B$.
2. Lattice alterations: let $\Delta'_X = (|\Delta_X|, \{\sigma, N'_\sigma\})$, $\Delta'_B = (|\Delta_B|, \{\tau, N'_\tau\})$, for some compatible collection of sublattices $N'_\tau \subset N_\tau$, $N'_\sigma = f^{-1}(N'_\tau) \cap N_\sigma$, and let $f'\colon \Delta'_X \to \Delta'_B$ be the morphism induced by f.

Conjecture 1.5 (Combinatorial Semistable Reduction). *Given a surjective morphism $f\colon \Delta_X \to \Delta_B$, such that $f^{-1}(0) = \{0\}$, there exists a projective subdivision $f'\colon \Delta'_X \to \Delta'_B$ followed by a lattice alteration $f''\colon \Delta''_X \to \Delta''_B$ so that f'' is semistable.*

$$\begin{array}{ccccc} \Delta_{X''} & \to & \Delta_{X'} & \to & \Delta_X \\ \downarrow f'' & & \downarrow f' & & \downarrow f \\ \Delta_{B''} & \to & \Delta_{B'} & \to & \Delta_B \end{array}$$

The importance of Conjecture 1.5 lies in the fact that it implies Conjecture 1.3 [AK]. Although we are concerned with the combinatorial version of semistable reduction in this paper, we indicate briefly how the two conjectures are related. It is shown in [AK] that a morphism $f\colon X \to B$ as in Conjecture 1.3 can be modified to a toroidal morphism, and so we get a morphism of polyhedral complexes $f_\Delta\colon \Delta_X \to \Delta_B$. Then one checks that if f_Δ is semistable according to Definition 1.4, then f is semistable as defined in Conjecture 1.3. It remains to show that the two combinatorial operations on $f_\Delta\colon \Delta_X \to \Delta_B$ have geometric analogues for $f\colon X \to B$. Indeed, subdivisions of Δ_X and Δ_B correspond to birational morphisms (see [KKMS]), and a lattice alteration corresponds to a finite base change (see [AK]).

In the case when $\dim(\Delta_B) = 1$, Conjecture 1.5 reduces to the combinatorial version of the semistable reduction theorem proved in [KKMS]. In [AK] the conjecture was proved with semistable replaced by weakly semistable. The main result of this paper is

Theorem 1.6. *Conjecture* 1.5 *is true if* f_Δ *has relative dimension* ≤ 3. *Hence, Conjecture* 1.3 *is true if* f *has relative dimension* ≤ 3.

The relative dimension of a linear map $f_\sigma\colon \sigma \to \tau$ of cones σ, τ is $\dim(\sigma) - \dim(f(\sigma))$. The relative dimension of $f_\Delta\colon \Delta_X \to \Delta_B$ is by definition the maximum of the relative dimensions of $f_\sigma\colon \sigma \to \tau$ over all $\sigma \in \Delta_X$. To see that the second statement of the theorem follows from the first, consider a surjective morphism of affine toric varieties $f\colon X_\sigma \to X_\tau$ defined by a linear map of cones and lattices $f_\Delta\colon (\sigma, N_\sigma) \to (\tau, N_\tau)$. A general fiber of this morphism has dimension equal to the rank of the kernel of $f_\Delta\colon N_\sigma \to N_\tau$, and this is at least the relative dimension of $f_\Delta\colon \sigma \to \tau$. Therefore, if a toroidal morphism $f\colon X \to B$ has relative dimension $\leq d$, then the associated morphism of polyhedral complexes $f_\Delta\colon \Delta_X \to \Delta_B$ also has relative dimension $\leq d$.

We remark that semistable reduction for families of curves over a base of an arbitrary dimension was proved by de Jong [dJ]. Thus, the new result of Theorem 1.6 is semistable reduction for families of surfaces and threefolds.

The rest of the paper is organized as follows. In Section 2 we use the construction of [KKMS] to make f semistable over the edges of Δ_B without increasing the multiplicity of Δ_X. In Section 3 we modify the barycentric subdivision of Δ_X so that we get a morphism to the barycentric subdivision of Δ_B. It is shown in Section 4 that in certain situations we can choose a modified barycentric subdivision that decreases the multiplicity of Δ_X. The conditions when this happens are then verified for relative dimension ≤ 3 in Section 5.

2. Notation and Preliminaries

2.1. *Notation*

We use notation from [KKMS] and [F]. For a cone $\sigma \in N \otimes \mathbb{R}$ we write $\sigma = \langle v_1, \ldots, v_n \rangle$ if the points $v_1, \ldots, v_n$ lie on the one-dimensional edges of σ and generate the cone. If v_i are the first lattice points along the edges we call them primitive points of σ. An n-dimensional cone is simplicial if it has exactly n primitive points. For a simplicial cone σ with primitive points $v_1, \ldots, v_n$, the multiplicity of σ is

$$m(\sigma, N_\sigma) = [N_\sigma\colon \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_n].$$

A polyhedral complex Δ is nonsingular if and only if $m(\sigma, N_\sigma) = 1$ for all $\sigma \in \Delta$. To compute the multiplicity of σ we can count the representatives $w \in N_\sigma$ of classes of $N_\sigma / \mathbb{Z}v_1 \oplus \ldots \oplus \mathbb{Z}v_n$ of the form

$$w = \sum_i \alpha_i v_i, \qquad 0 \leq \alpha_i < 1.$$

The set of all such points is denoted by $W(\sigma)$. For cones $\sigma_1, \sigma_2 \in \Delta$ we write $\sigma_1 \leq \sigma_2$ if σ_1 is a face of σ_2. Notice that if $\sigma_1 \leq \sigma_2$, then the multiplicity of σ_1 is at most the multiplicity of σ_2. Hence, to compute the multiplicity of a polyhedral complex Δ, it suffices to consider maximal cones only.

Let $f_\Delta\colon \Delta_X \to \Delta_B$ be a morphism of polyhedral complexes, and assume that Δ_B is simplicial. Let $u_1, \ldots, u_m$ be the primitive points of Δ_B, and let $M_1, \ldots, M_m$ be positive integers. By taking the $(M_1, \dots, M_m)$-sublattice at $u_1, \dots, u_m$ we mean the lattice alteration $N'_\tau = \mathbb{Z}\{m_{i_1}u_{i_1}, \dots, m_{i_l}u_{i_l}\}$ for all cones $\tau \in \Delta_B$ with primitive points $u_{i_1}, \dots, u_{i_l}$.

A subdivision Δ' of Δ is called projective if there exists a homogeneous continuous piecewise linear function $\psi\colon |\Delta| \to \mathbb{R}$, convex on each cone $\sigma \in \Delta$, and taking rational values on the lattice points N_σ such that the maximal cones of Δ' are exactly the maximal pieces in which ψ is linear.

2.2. *Applying the Result of* [*KKMS*]

Let $\sigma_1 \subset \mathbb{R}^{n_1}$ and $\sigma_2 \subset \mathbb{R}^{n_2}$ be two cones. We consider $\sigma_1 \times \sigma_2$ as a cone in $\mathbb{R}^{n_1+n_2}$. If $\{\sigma_{1,\alpha}\}_\alpha$ is a subdivision of σ_1, and $\{\sigma_{2,\beta}\}_\beta$ is a subdivision of σ_2, then $\{\sigma_{1,\alpha} \times \sigma_{2,\beta}\}_{\alpha,\beta}$ gives us a subdivision of $\sigma_1 \times \sigma_2$.

If Δ_X and Δ_B are simplicial, we say that $f\colon \Delta_X \to \Delta_B$ is simplicial if $f(\sigma) \in \Delta_B$ for all $\sigma \in \Delta_X$. Assume that $f_\Delta\colon \Delta_X \to \Delta_B$ is a simplicial map of simplicial complexes. Let $u_i, i = 1, \dots, m$, be the primitive points of Δ_B, and let v_{ij}, $i = 1, \dots, m$, $j = 1, \dots, J_i$, be the primitive points of Δ_X such that v_{ij} is mapped to an integer multiple of u_i. For each $i = 1, \dots, m$ we denote by $\Delta_{X,i}$ the subcomplex of Δ_X lying over the cone $\langle u_i \rangle$ of Δ_B:

$$\Delta_{X,i} = f_\Delta^{-1}(\langle u_i \rangle).$$

Note that if we forget the lattices of Δ_X, then by the assumption that $f_\Delta^{-1}(0) = \{0\}$ we get that $\Delta_X = \Delta_{X,1} \times \cdots \times \Delta_{X,m}$. If $\Delta'_{X,i}$ are subdivisions of $\Delta_{X,i}$, we get a subdivison Δ'_X of Δ_X by taking the product

$$\Delta'_X = \Delta'_{X,1} \times \cdots \times \Delta'_{X,m}.$$

Lemma 2.1. *If $\Delta'_{X,i}$ are projective subdivisions of $\Delta_{X,i}$, then Δ'_X is a projective subdivision of Δ_X.*

Proof. Let ψ_i be a convex piecewise linear function defining the subdivison $|\Delta'_{X,i}|$. Extend ψ_i linearly to the entire $|\Delta_X|$ by setting $\psi_i(|\Delta_{X,j}|) = 0$ for $j \neq i$. Clearly, $\psi = \sum_i \psi_i$ is a convex piecewise linear function defining the subdivision Δ'_X. □

Consider the restriction $f_\Delta|_{\Delta_{X,i}}\colon \Delta_{X,i} \to \mathbb{R}_+ u_i$. By the Main Theorem of Chapter 2 in [KKMS] there exist a subdivision $\Delta'_{X,i}$ of $\Delta_{X,i}$ and a positive integer M_i such that after taking the M_i-sublattice at u_i the induced morphism $f'_\Delta|_{\Delta'_{X,i}}$ is semistable. We let Δ'_X be the product of the subdivisions $\Delta'_{X,i}$, and we take the $(M_1, \dots, M_m)$-sublattice at $(u_1, \dots, u_n)$. Then $f'_\Delta\colon \Delta'_X \to \Delta'_B$ is a simplicial map and $f'_\Delta|_{\Delta'_{X,i}}$ is semistable for all i.

Lemma 2.2. *The multiplicity of Δ'_X is not greater than the multiplicity of Δ_X.*

Proof. Let $\sigma \in \Delta_X$ have primitive points v_{ij} and let $\sigma' \subset \sigma$ be a maximal cone in the subdivision with primitive points v'_{ij}. The multiplicity of σ' is the number of points in

$W(\sigma')$. We show that $W(\sigma')$ can be mapped injectively to $W(\sigma)$, hence the multiplicity of σ' is not greater than the multiplicity of σ.

If $w' \in W(\sigma')$, we write

$$w' = \sum_{i,j} (\beta_{ij} + b_{ij}) v_{ij}, \qquad 0 \le \beta_{ij} < 1, \quad b_{ij} \in \mathbb{Z}_+.$$

Then $w = \sum_{ij} \beta_{ij} v_{ij} \in N_\sigma$ is in $W(\sigma)$. If two points $w'_1, w'_2 \in W(\sigma')$ give the same w, then their difference $w'_1 - w'_2$ is an integral linear combination of v_{ij}. However, then $w'_1 - w'_2$ is also an integral linear combination of v'_{ij} because $\mathbb{Z}\{v'_{ij}\}_{i,j} = \mathbb{Z}\{v_{ij}\}_{i,j} \cap N_{\sigma'}$. Hence $w'_1 - w'_2 = 0$. □

3. Modified Barycentric Subdivisions

Let $f_\Delta \colon \Delta_X \to \Delta_B$ be a simplicial morphism. Consider the barycentric subdivision $BS(\Delta_B)$ of Δ_B. The one-dimensional cones of $BS(\Delta_B)$ are of the form $\mathbb{R}_+\hat{\tau}$ where $\hat{\tau} = \sum u_i$ is the barycenter of a cone $\tau \in \Delta_B$ with primitive points $u_1, \ldots, u_m$. A cone $\tau' \in BS(\Delta_B)$ is spanned by $\hat{\tau}_1, \ldots, \hat{\tau}_k$, where $\tau_1 \le \tau_2 \le \cdots \le \tau_k$ is a chain of cones in Δ_B.

In general, f_Δ does not induce a morphism $BS(\Delta_X) \to BS(\Delta_B)$. For example, if $\sigma = \langle v_{11}, v_{12}, v_{21}\rangle$, $\tau = \langle u_1, u_2\rangle$, and $f_\Delta \colon v_{ij} \mapsto u_i$, then f_Δ does not induce a morphism of barycentric subdivisions. To get a morphism we need to modify the barycenters $\hat{\sigma}$ of cones $\sigma \in \Delta_X$ so that they map to (multiples of) barycenters of Δ_B.

Definition 3.1. The data of **modified barycenters** consists of:

1. A subset of cones $\tilde{\Delta}_X \subset \Delta_X$.
2. For each cone $\sigma \in \tilde{\Delta}_X$ a lattice point $b_\sigma \in \operatorname{int}(\sigma) \cap N_\sigma$ such that $f_\Delta(b_\sigma) \in \mathbb{R}_+\hat{\tau}$ for some $\tau \in \Delta_B$.

Recall that for any total order $\prec$ on the set of cones in Δ_X refining the partial order $\le$, the barycentric subdivision $BS(\Delta_X)$ can be realized as a sequence of star subdivisions at the barycenters $\hat{\sigma}$ for all cones $\sigma \in \Delta_X$ in the descending order $\prec$.

Definition 3.2. Given modified barycenters $(\tilde{\Delta}_X, \{b_\sigma\})$ and a total order $\prec$ on Δ_X refining the partial order $\le$, the **modified barycentric subdivision** $MBS_{\tilde{\Delta}_X, \{b_\sigma\}, \prec}(\Delta_X)$ is the sequence of star subdivisions at b_σ for all $\sigma \in \tilde{\Delta}_X$ in the descending order $\prec$.

Example 3.3. Let $f_\Delta \colon \langle v_{11}, v_{12}, v_{21}\rangle \to \langle u_1, u_2\rangle$ be the morphism defined by $f_\Delta \colon v_{ij} \mapsto u_i$. Let $\tilde{\Delta}_X$ consist of the two cones $\tilde{\Delta}_X = \{\langle v_{11}, v_{21}\rangle, \langle v_{12}, v_{21}\rangle\}$, and let the modified barycenters be $\{b_\sigma\} = \{v_{11} + v_{21}, v_{12} + v_{21}\}$. Depending on whether $\langle v_{11}, v_{21}\rangle \prec \langle v_{12}, v_{21}\rangle$ or vice versa, we get two modified barycentric subdivisions as shown in Fig. 1.

To simplify notation, we write $MBS_{\tilde{\Delta}_X}(\Delta_X)$ or simply $MBS(\Delta_X)$ instead of $MBS_{\tilde{\Delta}_X, \{b_\sigma\}, \prec}(\Delta_X)$. By definition, $MBS(\Delta_X)$ is a projective simplicial subdivision of

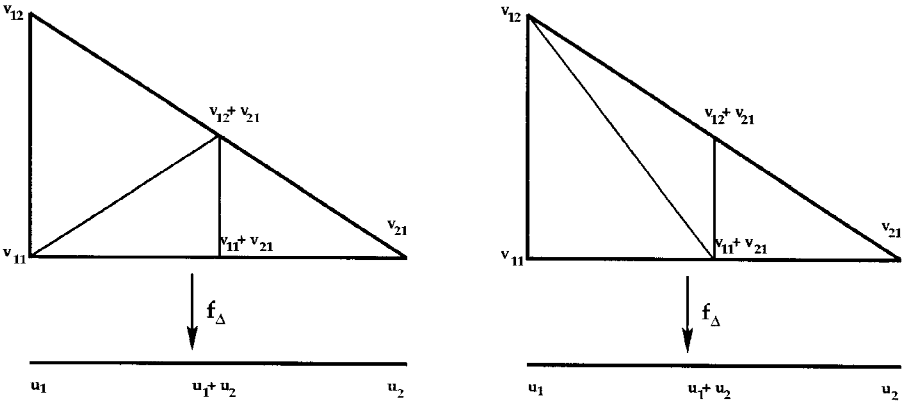

Fig. 1. Two modified barycentric subdivisions from Example 3.3

Δ_X. Next, we show that, as in the case of the ordinary barycentric subdivision, the cones of $MBS(\Delta_X)$ can be characterized by chains of cones in Δ_X. We may assume that the zero- and one-dimensional cones of Δ_X are all in $\tilde{\Delta}_X$, and they precede all other cones in the order $\prec$. For a cone $\sigma \in \Delta_X$ let $\tilde{\sigma}$ be the maximal face of σ (with respect to $\prec$) in $\tilde{\Delta}_X$. Given a chain of cones $\sigma_1 \leq \cdots \leq \sigma_k$ in Δ_X, the cone $\langle b_{\tilde{\sigma}_1}, \ldots, b_{\tilde{\sigma}_k} \rangle$ is a subcone of σ_k. Let $C(\Delta_X)$ be the set of all such cones corresponding to chains $\sigma_1 \leq \cdots \leq \sigma_k$ in Δ_X.

Proposition 3.4. $C(\Delta_X) = MBS(\Delta_X)$.

Proof. We do induction on the number of cones in $\tilde{\Delta}_X$ of dimension at least 2. If $\tilde{\Delta}_X$ contains only zero- or one-dimensional cones, then the statement is trivial. So, assume that $\tilde{\Delta}_X = \tilde{\Delta}_{X,0} \cup \{\sigma_0\}$, where $\sigma \prec \sigma_0$ for any $\sigma \in \tilde{\Delta}_{X,0}$, and assume that the proposition is proved for $\tilde{\Delta}_{X,0}$.

Without loss of generality we may assume that Δ_X consists of cones containing σ_0 and their faces only. We get $MBS_{\tilde{\Delta}_X}(\Delta_X)$ from Δ_X if we first subdivide at b_{σ_0} and then at b_σ for $\sigma \in \tilde{\Delta}_{X,0}$ in the descending order $\prec$. If $\Delta_{X,0}$ is the subcomplex of Δ_X consisting of cones *not* containing σ_0, then the star subdivision of Δ_X at b_{σ_0} is $\Delta_{X,0} \times \langle b_{\sigma_0} \rangle$. Since σ_0 is greater than any $\sigma \in \tilde{\Delta}_{X,0}$ with respect to $\prec$, all $b_\sigma \in \Delta_{X,0}$, and we see that

$$MBS_{\tilde{\Delta}_X}(\Delta_X) = MBS_{\tilde{\Delta}_{X,0}}(\Delta_{X,0}) \times \langle b_{\sigma_0} \rangle.$$

A cone in $MBS_{\tilde{\Delta}_{X,0}}(\Delta_{X,0}) \times \langle b_{\sigma_0} \rangle$ is of the form $\sigma \times \rho$, where ρ is a face of $\langle b_{\sigma_0} \rangle$, i.e. either $\{0\}$ or $\langle b_{\sigma_0} \rangle$ itself, and where σ is a cone in $MBS_{\tilde{\Delta}_{X,0}}(\Delta_{X,0})$. Applying induction hypothesis to $MBS_{\tilde{\Delta}_{X,0}}(\Delta_{X,0})$, we get that $\sigma = \langle b_{\tilde{\sigma}_1}, \ldots, b_{\tilde{\sigma}_l} \rangle$ for a chain of cones $\sigma_1 \leq \cdots \leq \sigma_l$ in $\Delta_{X,0}$. Now if $\rho = \{0\}$, then $\sigma \times \rho = \langle b_{\tilde{\sigma}_1}, \ldots, b_{\tilde{\sigma}_l} \rangle \in C(\Delta_X)$. If $\rho = \langle b_{\sigma_0} \rangle$, we let σ_{l+1} be a cone in Δ_X that contains both σ_l and σ_0. Then $\tilde{\sigma}_{l+1} = \sigma_0$, and $\sigma \times \rho = \langle b_{\tilde{\sigma}_1}, \ldots, b_{\tilde{\sigma}_l}, b_{\tilde{\sigma}_{l+1}} \rangle \in C(\Delta_X)$.

Conversely, let $\langle b_{\tilde{\sigma}_1}, \ldots, b_{\tilde{\sigma}_l} \rangle$ be a cone in $C(\Delta_X)$ for some chain $\sigma_1 \leq \cdots \leq \sigma_l$ in Δ_X. Then for some $k \leq l$ we have that $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_k \in \tilde{\Delta}_{X,0}$, and $\tilde{\sigma}_{k+1} = \cdots = \tilde{\sigma}_l = \sigma_0$.

By induction hypothesis, the cone $\langle b_{\tilde{\sigma}_1}, \ldots, b_{\tilde{\sigma}_k}\rangle$ coming from the chain $\sigma_1 \leq \cdots \leq \sigma_k$ in $\Delta_{X,0}$ is in $MBS_{\tilde{\Delta}_{X,0}}(\Delta_{X,0})$. Hence the cone $\langle b_{\tilde{\sigma}_1}, \ldots, b_{\tilde{\sigma}_l}\rangle$ is of the form $\sigma \times \rho$, where $\sigma = \langle b_{\tilde{\sigma}_1}, \ldots, b_{\tilde{\sigma}_k}\rangle \in MBS_{\tilde{\Delta}_{X,0}}(\Delta_{X,0})$, and $\rho = \langle b_{\sigma_0}\rangle$ if $k < l$, and $\rho = \{0\}$ if $k = l$. □

Corollary 3.5. *Assume that $f_\Delta\colon \Delta_X \to \Delta_B$ is a simplicial morphism. If $f_\Delta(\tilde{\sigma}) = f_\Delta(\sigma)$ for all $\sigma \in \Delta_X$, then f_Δ induces a simplicial morphism $f'_\Delta\colon MBS(\Delta_X) \to BS(\Delta_B)$.*

Proof. Let $\sigma' \in MBS(\Delta_X)$ correspond to a chain $\sigma_1 \leq \cdots \leq \sigma_k$ in Δ_X. Since f_Δ is simplicial, we have a chain of cones $f_\Delta(\sigma_1) \leq \cdots \leq f_\Delta(\sigma_k)$ in Δ_B. Recall that $b_{\tilde{\sigma}_i}$ is mapped to a multiple of a barycenter: $f_\Delta(b_{\tilde{\sigma}_i}) = \mathbb{R}_+\hat{\tau}$ for some $\tau \in \Delta_B$. The assumption that $f_\Delta(\tilde{\sigma}_i) = f_\Delta(\sigma_i)$ implies that $f_\Delta(b_{\tilde{\sigma}_i}) \in \mathbb{R}_+\widehat{f_\Delta(\sigma_i)}$, hence the cone $\langle b_{\tilde{\sigma}_1}, \ldots, b_{\tilde{\sigma}_k}\rangle$ is mapped onto the cone $\langle \widehat{f_\Delta(\sigma_1)}, \ldots, \widehat{f_\Delta(\sigma_k)}\rangle \in BS(\Delta_B)$. □

Example 3.6. Assume that $f_\Delta\colon \Delta_X \to \Delta_B$ is a simplicial morphism taking primitive points of Δ_X to primitive points of Δ_B. Then for a cone $\sigma \in \Delta_X$ such that $f_\Delta\colon \sigma \xrightarrow{\simeq} \tau$ for some $\tau \in \Delta_B$, we have $f_\Delta(\hat{\sigma}) = \hat{\tau}$.

Let $\overline{\Delta}_X = \{\sigma \in \Delta_X\colon\ f_\Delta|_\sigma \text{ is injective}\}$, $b_\sigma = \hat{\sigma}$. Clearly, the hypothesis of the lemma is satisfied, and we have a simplicial morphism $f'_\Delta\colon MBS_{\overline{\Delta}_X}(\Delta_X) \to BS(\Delta_B)$.

Conversely, if $(\tilde{\Delta}_X, \{b_\sigma\})$ is the data of modified barycenters such that f_Δ induces a morphism $f'_\Delta\colon MBS_{\tilde{\Delta}_X}(\Delta_X) \to BS(\Delta_B)$, then $\overline{\Delta}_X \subset \tilde{\Delta}_X$. Thus, we may always assume that $\overline{\Delta}_X \subset \tilde{\Delta}_X$.

4. Reducing the Multiplicity of Δ_X

Proposition 4.1. *Let $f_\Delta\colon \Delta_X \to \Delta_B$ be a simplicial morphism taking primitive points to primitive points. Assume that Δ_B is nonsingular, Δ_X is singular, and every singular cone $\sigma \in \Delta_X$ contains a point $w \in W(\sigma)\backslash\{0\}$ mapping to a barycenter in Δ_B. Then there exists a modified barycentric subdivision $MBS(\Delta_X)$ of Δ_X having multiplicity strictly less than the multiplicity of Δ_X such that f_Δ induces a simplicial morphism $f'_\Delta\colon MBS(\Delta_X) \to BS(\Delta_B)$.*

Proof. For every singular cone $\sigma \in \Delta_X$ we choose a point w_σ as follows. By assumption, there exists a point $w \in W(\sigma)\backslash\{0\}$ mapping to a barycenter of Δ_B: $f_\Delta(w) = \hat{\tau}$. Then for a unique cone $\tau_0 \in \Delta_B$ we have $f_\Delta(\sigma) = \tau \times \tau_0$. We choose a face $\sigma_0 \leq \sigma$ such that $f_\Delta\colon \sigma_0 \xrightarrow{\simeq} \tau_0$. Set $w_\sigma = w + \hat{\sigma}_0$; then

$$f_\Delta(w_\sigma) = f_\Delta(w) + f_\Delta(\hat{\sigma}_0) = \hat{\tau} + \hat{\tau}_0 = \widehat{f_\Delta(\sigma)}.$$

Having chosen the set $\{w_\sigma\}$, we may remove some of the points w_σ if necessary so that every simplex $\rho \in \Delta_X$ contains at most one w_σ in its interior. With $\overline{\Delta}_X$ as in Example 3.6, let $\tilde{\Delta}_X = \overline{\Delta}_X \cup \{\rho \in \Delta_X | w_\sigma \in \operatorname{int}(\rho) \text{ for some singular } \sigma\}$, $b_\rho = \hat{\rho}$ if $\rho \in \overline{\Delta}_X$, and $b_\rho = w_\sigma$ if $w_\sigma \in \operatorname{int}(\rho)$.

Next we specify the order $\prec$. We refine the partial order $\leq$ as follows: for two faces σ_1 and σ_2 of a cone $\sigma \in \Delta_X$ we set $\sigma_1 \prec_0 \sigma_2$ if $\dim f_\Delta(\sigma_1) < \dim f_\Delta(\sigma_2)$. Since $\overline{\Delta}_X \subset \tilde{\Delta}_X$, this ensures that the condition $f_\Delta(\tilde{\sigma}) = f_\Delta(\sigma)$ of Corollary 3.5 is satisfied for any refinement of $\prec_0$. Now if σ is singular, then the point w_σ constructed above lies in the interior of a face ρ_σ such that $f_\Delta(\rho_\sigma) = f_\Delta(\sigma)$. We further refine the order $\prec_0$ by setting $\sigma_1 \prec_0 \rho_\sigma$ for any face σ_1 of the singular cone σ. Then $\tilde{\sigma} = \rho_\sigma$ whenever σ is singular. Finally we let $\prec$ be any refinement of $\prec_0$ to a total order.

Let $\sigma \in \Delta_X$ be a cone, and let a maximal cone $\sigma' \in MBS(\Delta_X)$ be given by a maximal chain of faces of σ: $\sigma_1 \leq \cdots \leq \sigma_n$. We have to show that $m(\sigma', N_{\sigma'}) \leq m(\sigma, N_\sigma)$, and if σ is singular, then the inequality is strict.

We can order the primitive points $v_1, \ldots, v_n$ of σ so that $\sigma_1 = \langle v_1 \rangle, \sigma_2 = \langle v_1, v_2 \rangle, \ldots,$ $\sigma_n = \langle v_1, \ldots, v_n \rangle$. Since $b_{\tilde{\sigma}_i} \in \sigma_i$, the primitive points of $\sigma' = \langle b_{\tilde{\sigma}_1}, \ldots, b_{\tilde{\sigma}_n} \rangle$ can be written as

$$
\begin{aligned}
v_1' &= \frac{1}{\mu_1} b_{\tilde{\sigma}_1} = a_{11} v_1, \\
v_2' &= \frac{1}{\mu_2} b_{\tilde{\sigma}_2} = a_{21} v_1 + a_{22} v_2, \\
&\cdots \\
v_n' &= \frac{1}{\mu_n} b_{\tilde{\sigma}_n} = a_{n1} v_1 + \cdots + a_{nn} v_n
\end{aligned}
$$

for some $a_{ij} \geq 0$ and integers $\mu_i \geq 1$. The multiplicity of σ' is $a_{11} \cdot a_{22} \cdots a_{nn}$ times the multiplicity of σ. By the choice of b_ρ above, all $a_{ii} \leq 1$, hence $m(\sigma', N_{\sigma'}) \leq m(\sigma, N_\sigma)$. If σ is singular, let i be the smallest index such that the face σ_i is singular. Then, with notation as above, $b_{\tilde{\sigma}_i} = w + \hat{\sigma}_0$ for some $w \in W(\sigma_i) \backslash \{0\}$, and $\sigma_0 \leq \sigma_i$. Now if $a_{ii} = 1$, then $w \in \langle v_1, \ldots, v_{i-1} \rangle$, and this gives a contradiction with the choice of i. Hence $a_{ii} < 1$ and $m(\sigma', N_{\sigma'}) < m(\sigma, N_\sigma)$. □

5. Families of Surfaces and Threefolds

Proof of Theorem 1.6. Let $f_\Delta \colon \Delta_X \to \Delta_B$ be a surjective morphism of polyhedral complexes such that $f_\Delta^{-1}(0) = \{0\}$. It is shown in [AK] that there exist projective simplicial subdivisions Δ_X' of Δ_X and Δ_B' of Δ_B such that Δ_B is nonsingular and f_Δ induces a simplicial morphism $f_\Delta' \colon \Delta_X' \to \Delta_B'$. To obtain these subdivisions, one first subdivides Δ_B such that the image of every cone in Δ_X is a union of cones in Δ_B'. The convex piecewise linear function defining the subdivision Δ_B' can then be composed with f_Δ to give a subdivision of Δ_X. A sequence of star subdivisions centered at the one-dimensional edges yields the required simplicial subdivision Δ_X'. Thus, we may assume that Δ_X is simplicial, Δ_B is nonsingular, and $f_\Delta \colon \Delta_X \to \Delta_B$ is a simplicial map.

Applying the construction of [KKMS] over the edges of Δ_B (Section 2.2), we can make $f_\Delta|_{\Delta_{X,i}}$ semistable without increasing the multiplicity of Δ_X. We show below that every singular simplex $\sigma \in \Delta_X$ contains a point $w \in W(\sigma) \backslash \{0\}$ mapping to a barycenter of Δ_B. By Proposition 4.1, there exists a modified barycentric subdivision such that f_Δ induces a simplicial morphism $f_\Delta' \colon MBS(\Delta_X) \to BS(\Delta_B)$, with multiplicity of

$MBS(\Delta_X)$ strictly less than the multiplicity of Δ_X. Since f'_Δ is simplicial and $BS(\Delta_B)$ nonsingular, the proof is completed by induction on the multiplicity of Δ_X.

Consider the restriction of f_Δ to a singular simplex $f_\Delta\colon \sigma \to \tau$, where τ has primitive points $u_1, \ldots, u_m$, σ has primitive points v_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, J_i$, and $f_\Delta(v_{ij}) = u_i$. Since σ is singular, it contains a point $w \in W(\sigma)\backslash\{0\}$:

$$w = \sum_{i,j} \alpha_{ij} v_{ij}, \qquad 0 \le \alpha_{ij} < 1, \qquad \sum \alpha_{ij} > 0.$$

Considering a face of σ if necessary, we may assume that w lies in the interior of σ, hence $0 < \alpha_{ij}$. Since $f_\Delta(w) \in N_\tau$, it follows that $\sum_j \alpha_{ij} \in \mathbb{Z}$ for all i. In particular, if $J_{i_0} = 1$ for some i_0, then $\alpha_{i_0 1} = 0$, and w lies in a face of σ. So we may assume that $J_i > 1$ for all i. Since the relative dimension of f_Δ is $\sum_i (J_i - 1)$, we have to consider all possible decompositions $\sum_i (J_i - 1) \le 3$, where $J_i > 1$ for all i.

The cases when the relative dimension of f_Δ is 0 or 1 are trivial and left to the reader.

If the relative dimension of f_Δ is 2, then either $J_1 = 3$ or $J_1 = J_2 = 2$. In the first case we have that $\langle v_{11}, v_{12}, v_{13}\rangle$ is singular, contradicting the semistability of $f_\Delta|_{\Delta_{X,1}}$. In the second case, $\alpha_{11} + \alpha_{12}, \alpha_{21} + \alpha_{22} \in \mathbb{Z}$ and $0 < \alpha_{ij} < 1$ imply that $\alpha_{11} + \alpha_{12} = \alpha_{21} + \alpha_{22} = 1$. Hence $f_\Delta(w) = u_1 + u_2$ is a barycenter.

In relative dimension 3, either $J_1 = 4$, or $J_1 = 3$, $J_2 = 2$, or $J_1 = J_2 = J_3 = 2$. In the first case we get a contradiction with the semistability of $f_\Delta|_{\Delta_{X,1}}$; the third case gives $\alpha_{11} + \alpha_{12} = \alpha_{21} + \alpha_{22} = \alpha_{31} + \alpha_{32} = 1$ as for relative dimension 2. In the second case, either $\alpha_{11} + \alpha_{12} + \alpha_{13} = \alpha_{21} + \alpha_{22} = 1$ and w maps to a barycenter, or $\alpha_{11} + \alpha_{12} + \alpha_{13} = 2, \alpha_{21} + \alpha_{22} = 1$ and $(\sum v_{ij}) - w$ maps to a barycenter. □

Example 5.1. We show by an example that the previous construction of modified barycentric subdivisions does not work in relative dimension ≥ 4. Let $\tau = \langle u_1, u_2\rangle$ and $\sigma = \langle v_{11}, v_{12}, v_{13}, v_{14}, v_{21}, v_{22}\rangle$, with lattices $N_\tau = \mathbb{Z}\{u_1, u_2\}$ and $N_\sigma = \mathbb{Z}\{v_{11}, \ldots, v_{22}, \frac{1}{2}(v_{11} + \cdots + v_{22})\}$. Then $W(\sigma)\backslash\{0\}$ consists of a single point $w = \frac{1}{2}(v_{11} + \cdots + v_{22})$, and if f_Δ maps v_{ij} to u_i, then w is mapped to $2u_1 + u_2$, which is not a barycenter.

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