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Hard Lefschetz theorem for simple polytopes

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Abstract McMullen's proof of the Hard Lefschetz Theorem for simple polytopes is studied, and a new proof of this theorem that uses conewise polynomial functions on a simplicial fan is provided.

Keywords Simple polytopes · Simplicial fans · Hard Lefschetz theorem · Hodge–Riemann–Minkowski bilinear relations

1 Introduction

Let K be a simple convex polytope corresponding to a projective toric variety X_K with an ample line bundle L on it. Then the linear map in the cohomology of X_K defined by multiplication with the first Chern class of L satisfies the Hard Lefschetz theorem. Peter McMullen [7] gave a combinatorial proof of this theorem using the polytope algebra to represent cohomology. Since the publication of McMullen's proof, a simpler description of the cohomology ring of a toric variety has been found, using conewise polynomial functions on a fan. In this article we give another proof of McMullen's result using conewise polynomial functions.

Theorem 1.1 (Hard Lefschetz) *Let l be a strictly convex conewise linear function on a complete simplicial fan Σ. Let $\mathcal{A}(\Sigma)$ be the ring of continuous conewise polynomial functions on Σ, and let $H(\Sigma)$ be its quotient by the ideal generated by global linear functions, graded by degree. Then l defines a Lefschetz operation on $H(\Sigma)$;*

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that is, multiplication by

$$l^{n-2k} : H^k(\Sigma) \to H^{n-k}(\Sigma)$$

is an isomorphism for each k.

The Hard Lefschetz theorem for simple polytopes was proved previously by Stanley [9] by applying the geometric Hard Lefschetz theorem to the associated toric varieties. This theorem forms the necessity part of the conjecture of McMullen [6] about the possible face numbers of simple polytopes. The sufficiency was proved by Billera and Lee [2].

The proof we give here is elementary and essentially self-contained. We follow the main steps of McMullen's proof, but we hope that the proof using conewise polynomial functions simplifies several steps and makes the main ideas more visible. There have already appeared several simplifications to McMullen's original proof. In [8] McMullen studies the algebra of weight spaces on polytopes and shows that this is an equivalent replacement for the polytope algebra. In [10] Timorin replaces the polytope algebra with an algebra of differential operators. One advantage of the ring of conewise polynomial functions over the polytope algebra and its variants is that it can be defined for a nonprojective fan that does not come from a polytope, and one may even hope to use the same steps to prove the Hard Lefschetz theorem for such fans. However, since the conewise linear function l is then not convex, the Hodge–Riemann–Minkowski bilinear relations (see below) that are essential for McMullen's proof do not usually hold in this situation, and one needs an appropriate replacement for these.

Let us describe the main ideas of the proof. The first step is to replace Theorem 1.1 by a stronger statement, that of the Hodge–Riemann–Minkowski bilinear relations. Let $\langle\cdot\rangle : H^n(\Sigma) \xrightarrow{\simeq} \mathbb{R}$ be the "evaluation map" and consider the quadratic form $Q_l(h) = \langle l^{n-2k} h \cdot h\rangle$ on $H^k(\Sigma)$. Define the primitive cohomology

$$PH^k(\Sigma) = \ker\bigl(l^{n-2k+1} : H^k(\Sigma) \to H^{n-k+1}(\Sigma)\bigr).$$

Theorem 1.2 (Hodge–Riemann–Minkowski bilinear relations) *Let* l *be a strictly convex conewise linear function on a complete simplicial fan* Σ. *Then for each* $k \le \frac{1}{2}n$, *the quadratic form* $(-1)^k Q_l$ *is positive definite on* $PH^k(\Sigma)$.

An equivalent statement of the Hodge–Riemann–Minkowski relations is that the quadratic form Q_l on $H^k(\Sigma)$ is nondegenerate and has the signature

$$\sum_{0\le i\le k} (-1)^i g_i,$$

where $g_i = \dim H^i(\Sigma) - \dim H^{i-1}(\Sigma)$, which is (if Q_l is nondegenerate) the dimension of $PH^i(\Sigma)$.

The next step in McMullen's proof is to show that in fact the Hodge–Riemann–Minkowski relations for all pairs of complete simplicial fans and convex functions in dimension $n-1$ imply the Hard Lefschetz theorem for the same in dimension n.

Thus, by induction we may assume that a convex function l defines a Lefschetz operation on $H(\Sigma)$ or, equivalently, that the form Q_l is nondegenerate. This implies that for a continuous family of convex functions l_t on Σ, the signature of Q_{l_t} is the same for all t, and therefore if the Hodge–Riemann–Minkowski relations hold for one function, they hold for every function in the family. The last step of McMullen's proof is to study the change in the signature as a fan undergoes an "elementary flip." Any complete simplicial Σ with convex function l can be transformed to the normal fan of a simplex (for which direct calculation is possible) by a sequence of such flips, with continuous deformations of associated convex functions. So the proof is finished by showing that the Hodge–Riemann–Minkowski relations hold on one side of the flip if and only if they hold on the other side. This is done by explicitly relating the signatures of the forms Q_{l_t}.

2 Preliminaries

We give here the necessary background for the definition of cohomology using conewise polynomial functions. Further details and more advanced results about the space of conewise polynomial functions on a fan may be found in [1, 3, 4].

2.1 Fans and polytopes

We use the notation from [5] for fans and polytopes, except that we do not consider lattices. Fans and polytopes are not assumed to be rational. We briefly recall some of this notation.

By a *cone* σ we mean a pointed polyhedral cone in a real vector space V of dimension $\dim V = n$; that is, σ is the set of nonnegative linear combinations of vectors in some generating set, and 0 is the largest subspace contained in σ. The dimension of σ is the dimension of the subspace of V which it generates. A *face* of σ is its intersection with the kernel of a support function, meaning a linear function on V which is nonnegative on σ. A ray is a face of dimension one, and a facet is a face of codimension one. A cone is *simplicial* if it has a generating set of linearly independent vectors or, equivalently, if its rays are equal in number to its dimension. If $v_1, \ldots, v_r$ are linearly independent vectors in V, we write $\langle v_1, \ldots, v_r\rangle$ for the simplicial cone they generate.

A *fan* Σ is a collection of cones in V satisfying the following conditions: first, if σ is a cone in Σ, then so is every face of σ; and second, if σ and σ' are cones in Σ, then their intersection is a face of each. The *support* $|\Sigma|$ of Σ is the union of its cones in V. The fan Σ is called simplicial if every cone of Σ is simplicial, and *complete* if its support is equal to V.

We write

$$\operatorname{star}(\tau) = \{\sigma \in \Sigma \mid \tau \text{ is a face of } \sigma\}$$

and

$$\overline{\operatorname{star}}(\tau) = \{\gamma \in \Sigma \mid \gamma \text{ is a face of } \sigma \text{ for some } \sigma \text{ containing } \tau\}$$

for the open and closed stars of a cone τ. Thus $\overline{\mathrm{star}}(\tau)$ is the smallest subfan of Σ containing $\mathrm{star}(\tau)$ in V, consisting of the maximal cones which contain τ and the faces of these cones, and its support is the closure in V of the support of $\mathrm{star}(\tau)$.

Denote by A the ring of all polynomial functions on V. A (continuous) conewise polynomial function on Σ is a continuous function on $|\Sigma|$ which restricts to a polynomial on each cone of Σ. The set of all such functions is denoted $\mathcal{A}(\Sigma)$. This is an A-module, graded by degree. For a function $f \in \mathcal{A}(\Sigma)$ and cone σ of dimension n, write f^σ for the global polynomial which agrees with f on σ. A *convex* function l on Σ is a function $l \in \mathcal{A}^1(\Sigma)$ such that $l^\sigma(x) < l(x)$ for each maximal cone σ and every $x \notin \sigma$. (That is, the l^σ are distinct, and the region above the graph of l in $V \times \mathbb{R}$ is convex. Such an l is more commonly called a *strictly convex conewise linear* function.) A concave function is the negative of a convex function.

A polytope K is the convex hull of a finite number of points in V^*. The faces of K are again its intersections with supporting hyperplanes. To K we associate its *normal fan* together with a convex function. Define a function on V by

$$l(x) = \max v(x),$$

where the maximum is taken over vertices v of K. The maximal sets on which the restriction of l is the restriction of a global linear function are taken as the maximal cones of Σ. It is immediate that l is convex on Σ. Equivalently, Σ is the fan over a polytope dual to K, and the function l is given on maximal cones by the dual vertices. Conversely, a fan Σ and convex function determine a polytope K for which Σ is the normal fan.

Not all fans are normal fans of polytopes and hence may have no convex functions. Those which do are called *projective*, and we assume everywhere hereafter that Σ is projective.

2.2 Conewise polynomial functions

Let Σ be a complete simplicial projective fan in V. Recall that A denotes the ring of polynomial functions on V and $\mathcal{A}(\Sigma)$ denotes the ring of conewise polynomial functions on Σ.

For a cone σ of Σ, a *characteristic function* (also known as a Courant function [4]) is a conewise polynomial χ_σ, homogeneous of degree equal to the number of rays of σ, which is positive on the relative interior of σ and which is supported on $\mathrm{star}(\sigma)$. For a ray ρ, a characteristic function χ is conewise linear, zero on every other ray of Σ, and determined by a choice of scale on ρ. For a cone σ, a characteristic function is a product of characteristic functions of the rays of σ.

We can show that $\mathcal{A}(\Sigma)$ is a free A-module by *shelling* the fan. Let K be a polytope with normal fan Σ, and let h be a linear function on V^* which takes different values at the vertices of K. The vertices, and hence the maximal cones of Σ, are then ordered by the values of h. For each maximal cone σ, we take a function g_σ of minimal degree which is nonzero on σ but vanishes on each earlier cone. For concreteness, the characteristic function of the face of σ generated by the rays of σ which do not lie in any earlier cones will do. Given a function $g \in \mathcal{A}(\Sigma)$, we can successively subtract multiples of the g_σ in the same ordering to arrive at the zero function. This expresses g as a combination of the g_σ with coefficients in A.

Denote by $\mathbf{m}$ the ideal of A generated by degree-one functions. We define

$$H(\Sigma) = \mathcal{A}(\Sigma)/\mathbf{m}\mathcal{A}(\Sigma).$$

This inherits a grading from $\mathcal{A}(\Sigma)$. Using the generators g_σ constructed in the last paragraph, which give a basis for $H(\Sigma)$ as a vector space, we can establish a few basic facts. First, $H^k(\Sigma) = 0$ for $k > n$. Next,

$$\dim H^k(\Sigma) = \dim H^{n-k}(\Sigma)$$

for all k. We can see this by constructing another basis of functions g'_σ through the ordering function $-h$. For then g_σ is of degree k exactly when g'_σ is of degree $n-k$, as $g_\sigma g'_\sigma$ is a characteristic function for σ. Finally,

$$\dim H^0(\Sigma) = \dim H^n(\Sigma) = 1.$$

In the next subsection we describe an explicit isomorphism $H^n(\Sigma) \to \mathbb{R}$.

We note that $l \in \mathcal{A}^1(\Sigma)$ is convex if and only if $l + f$ is convex for each $f \in A^1$, so that it makes sense to say a class in $H^1(\Sigma)$ is convex or not.

2.3 Poincaré pairing

Let Σ be a complete simplicial projective fan in V. We recall from [4] an isomorphism

$$\langle \cdot \rangle_\Sigma : H^n(\Sigma) \to \mathbb{R}$$

called the *evaluation map*. We will omit the subscript where it is clear from the context.

This map is constructed as follows. Fix a metric on $\bigwedge^n V^*$. For each maximal cone $\sigma \in \Sigma$, choose support functions for the various facets of σ, scaled so that their wedge product has length 1, and let Φ_σ be their ordinary product in A. Thus Φ_σ is a global polynomial function restricting to a characteristic function on σ, with scale determined by the chosen metric. For $f \in \mathcal{A}(\Sigma)$, recall that f^σ is the global polynomial which restricts to f on σ. Set

$$\langle f \rangle = \sum_{\substack{\sigma \in \Sigma \\ \dim \sigma = n}} \frac{f^\sigma}{\Phi_\sigma}.$$

In [4] it is shown that this defines a degree $-n$ map $\mathcal{A}(\Sigma) \to A$. (It suffices to check that the right side has no poles along facets in Σ. Such a facet is the intersection of two maximal cones, and by choosing coordinates one can check that the poles of the corresponding terms cancel.)

Since $\langle \cdot \rangle$ is a map of A-modules, it descends to a map of cohomology $H(\Sigma) \to A/\mathbf{m} = \mathbb{R}$. The evaluation map is then the restriction $H^n(\Sigma) \to \mathbb{R}$ to degree n. This is not the zero map, as $\langle \chi_\sigma \rangle > 0$ for each maximal cone σ, so is an isomorphism, as $\dim H^n(\Sigma) = 1$.

Proposition 2.1 *The bilinear pairing*

$$\begin{aligned} H^k(\Sigma) \times H^{n-k}(\Sigma) &\to \mathbb{R}, \\ \big([f],[g]\big) &\mapsto \langle f \cdot g \rangle \end{aligned}$$

is nondegenerate for each k.

We call this pairing the Poincaré pairing for $H(\Sigma)$. In the notation of the previous section, the matrix with entries $\langle g_\sigma g'_\tau \rangle$, where the g'_τ are listed in reverse order, is upper-triangular, with diagonal entries $\langle g_\sigma g'_\sigma \rangle = \langle \chi_\sigma \rangle \neq 0$. This proves the proposition.

2.4 Fans with boundary

Let $\tau \in \Sigma$ be a cone and consider the subfan $\Delta = \overline{\text{star}}(\tau)$. Let π be the linear projection from the span of τ. The image of Δ under π is a complete fan Λ. Pullback of conewise linear functions gives a linear map

$$\mathcal{A}(\Lambda) \to \mathcal{A}(\Delta).$$

We claim that the induced map in cohomology is an isomorphism. This can be seen by choosing a shelling of Σ that starts with maximal cones in Δ and hence gives a shelling of Δ. The same argument as for complete fans gives generators for the free A-module $\mathcal{A}(\Delta)$. Since maximal cones of Δ correspond with maximal cones in Λ, the shelling of $\mathcal{A}(\Delta)$ induces a shelling of Λ. The generators of $\mathcal{A}(\Lambda)$ that one gets from this shelling pull back to generators of $\mathcal{A}(\Delta)$.

Let $\mathcal{A}(\Delta, \partial\Delta) \subset \mathcal{A}(\Delta)$ be the space of functions vanishing on the boundary. This is again a free A-module. A set of generators can be found by choosing a shelling of Σ that ends with maximal cones in Δ. The last generators then form a set of generators for $\mathcal{A}(\Delta, \partial\Delta)$. One can also see either from these generators, or directly from the definition, that

$$\mathcal{A}(\Delta, \partial\Delta) = \chi_\tau \mathcal{A}(\Delta).$$

In particular, if we define $H(\Delta, \partial\Delta) = \mathcal{A}(\Delta, \partial\Delta)/\mathbf{m}\mathcal{A}(\Delta, \partial\Delta)$, then

$$H^k(\Delta, \partial\Delta) = \chi_\tau H^{k-\dim\tau}(\Delta).$$

Let $\Gamma = \Sigma \setminus \text{star}(\tau)$. This is again a fan with boundary, and we can consider conewise polynomial functions on Γ and functions vanishing on the boundary $\partial\Gamma$. The same shelling argument as for Δ proves that these spaces are free A-modules. Restrictions of functions give exact sequences

$$0 \to \mathcal{A}(\Gamma, \partial\Gamma) \to \mathcal{A}(\Sigma) \to \mathcal{A}(\Delta) \to 0,$$
$$0 \to \mathcal{A}(\Delta, \partial\Delta) \to \mathcal{A}(\Sigma) \to \mathcal{A}(\Gamma) \to 0,$$

inducing exact sequences in cohomology.

Finally, let us discuss the evaluation maps in $H(\Delta, \partial\Delta)$ and $H(\Lambda)$. If σ is a maximal cone in Δ, then $\chi_\sigma = \chi_\tau \pi^*(\chi_\lambda)$ for some maximal cone $\lambda \in \Lambda$. It now follows from the definition of the evaluation maps, constructed with measures of volume compatible with π, that for $f \in \mathcal{A}(\Lambda)$,

$$\left\langle \chi_\tau \pi^*(f) \right\rangle_\Sigma = \langle f \rangle_\Lambda .$$

3 Flips

We explain here how to transform a complete simplicial projective fan to the normal fan of a simplex by elementary operations called flips.

Let K be a simple convex polytope, and let $h(x)$ be a linear function on the polytope that takes different values on the vertices. Consider the family of polytopes

$$K_t = K \cap \left\{ x \mid h(x) \geq t \right\}.$$

When varying t, the polytope K_t undergoes a combinatorial change at the values t_i for which a vertex of K lies in the hyperplane $h(x) = t_i$. The change from $K_{t_i-\varepsilon}$ to $K_{t_i+\varepsilon}$ is called a flip. Note that both $K_{t_i-\varepsilon}$ and $K_{t_i+\varepsilon}$ are simple polytopes, while K_{t_i} may not be simple. This construction gives a sequence of flips that transforms any polytope K to the simplex and then to the empty polytope.

The family K_t of polytopes defines a family of fans Σ_t with convex functions l_t. As t varies between t_i and t_{i+1}, the fans Σ_t do not change; only l_t changes. But as t crosses t_i, the fans also change by a flip. To describe a flip of a fan, let $\sigma = \langle v_1, \ldots, v_{n+1} \rangle$ be the simplicial $(n+1)$-dimensional cone in $\mathbb{R}^{n+1}$ generated by some linearly independent vectors $v_1, \ldots, v_{n+1}$. Fix $1 \leq m \leq n$ and write the boundary of σ as

$$\partial \sigma = \overline{\mathrm{star}} \langle v_1, \ldots, v_m \rangle \cup \overline{\mathrm{star}} \langle v_{m+1}, \ldots, v_{n+1} \rangle,$$

where the closed stars are taken in $\partial\sigma$. There exists a projection $\pi : \mathbb{R}^{n+1} \to V$ that is injective on both $\overline{\mathrm{star}}\langle v_1, \ldots, v_m \rangle$ and $\overline{\mathrm{star}}\langle v_{m+1}, \ldots, v_{n+1} \rangle$. A flip replaces the subfan $\pi(\overline{\mathrm{star}}\langle v_1, \ldots, v_m \rangle)$ in $\Sigma_{t_i-\varepsilon}$ with $\pi(\overline{\mathrm{star}}\langle v_{m+1}, \ldots, v_{n+1} \rangle)$ in $\Sigma_{t_i+\varepsilon}$. To indicate the choice of m, this flip is called an m-flip.

Note that an m-flip and an $(n+1-m)$-flip are inverse operations. The definition makes sense also for $m = 0$ and $m = n+1$—these flips pass between the empty fan and the normal fan of a simplex—but we will not have need of these cases. The cases $m = 1$ and $m = n$ pass between a simplicial cone and a star subdivision thereof. These are the only flips which change the number of rays of the fan.

In the following sections we consider a single flip only. To simplify notation, let us say that this flip occurs at $t = 0$, changing a fan Σ_{-1} to Σ_1. We also assume that $1 \leq m \leq \frac{1}{2}(n+1)$ (otherwise, exchange the roles of Σ_{-1} and Σ_1). The goal then is to prove that $Q_{l_{-1}}$ satisfies the Hodge–Riemann–Minkowski relations on Σ_{-1} if and only if Q_{l_1} satisfies the relations on Σ_1. The sequence of flips constructed above then reduces proving the Hodge–Riemann–Minkowski relations for an arbitrary Σ to the case of the normal fan of a simplex. This can be proved directly.

Lemma 3.1 *Let Π be the normal fan of a simplex, and let l be any convex function on Π. Then Theorem 1.2 holds for (Π, l).*

Proof By shelling Π we see that $\dim H^k(\Pi) = 1$ for each $0 \le k \le n$. It suffices to show that $\langle l^n \rangle > 0$, as this implies that l generates $H(\Pi)$, so that $PH(\Pi) = H^0(\Pi)$, and asserts that Q_l is positive definite on $H^0(\Pi)$. Now in $H^1(\Pi)$ the positive multiples of l are the convex classes (and the negative multiples are the concave classes). Since a characteristic function of any ray ρ of Π is convex, there is a choice of scale so that $l = \chi_\rho$ in $H^1(\Pi)$. Choosing n distinct rays of Π, we find $l^n = \chi_\sigma$ for the cone σ of dimension n they generate. Therefore $\langle l^n \rangle = \langle \chi_\sigma \rangle > 0$, as required. □

In [4] it is shown that in fact $\langle l^n \rangle_\Sigma = n! \operatorname{vol}(K)$ for any convex polytope K with corresponding (Σ, l), with the volume on V^* agreeing with the choice made in the construction of the evaluation map.

4 Flips and cohomology

Consider one m-flip that changes Σ_{-1} to Σ_1 by replacing $\overline{\text{star}}(\tau) = \Delta_{-1}$ with $\overline{\text{star}}(\tau') = \Delta_1$.

4.1 Cohomology of $\Delta_{\pm 1}$

Note that, combinatorially,

$$\Delta_{-1} \simeq [\tau] \times \Pi^{n-m},$$

where $[\tau]$ is the fan consisting of τ and all its faces, and Π^{n-m} is the normal fan of an $(n-m)$-dimensional simplex. Then

$$H(\Delta_{-1}) \simeq H\big(\Pi^{n-m}\big),$$

which has dimension 1 in degrees $k = 0, \dots, n-m$. Also

$$H^k(\Delta_{-1}, \partial\Delta_{-1}) \simeq H^{k-m}\big(\Pi^{n-m}\big),$$

and this has dimension 1 in degrees $k = m, \dots, n$.

Similarly,

$$H(\Delta_1) \simeq H\big(\Pi^{m-1}\big),$$

which has dimension 1 in degrees $k = 0, \dots, m-1$, and

$$H^k(\Delta_1, \partial\Delta_1) \simeq H^{k-(n-m+1)}\big(\Pi^{m-1}\big)$$

with dimension 1 in degrees $k = n-m+1, \dots, n$.

4.2 Decomposition of cohomology

Let $\Gamma = \Sigma_{-1} \setminus \text{star}(\tau) = \Sigma_1 \setminus \text{star}(\tau')$, and consider the exact sequences

$$0 \to H(\Delta_{\pm 1}, \partial\Delta_{\pm 1}) \to H(\Sigma_{\pm 1}) \to H(\Gamma) \to 0 \tag{1}$$

and

$$0 \to H(\Gamma, \partial\Gamma) \to H(\Sigma_{\pm 1}) \to H(\Delta_{\pm 1}) \to 0. \tag{2}$$

We construct an inclusion

$$H(\Sigma_1) \hookrightarrow H(\Sigma_{-1})$$

using maps from the exact sequences. In degrees $k = 0, \dots, m-1$, sequences (1) give isomorphisms

$$H^k(\Sigma_1) \xrightarrow{\simeq} H^k(\Gamma) \xleftarrow{\simeq} H^k(\Sigma_{-1}).$$

In degrees $k = m, \dots, n$, sequences (2) give maps

$$H^k(\Sigma_1) \xleftarrow{\simeq} H^k(\Gamma, \partial\Gamma) \hookrightarrow H^k(\Sigma_{-1}).$$

These two compositions define the required inclusion $H(\Sigma_1) \hookrightarrow H(\Sigma_{-1})$. This inclusion can also be viewed as the unique linear map that commutes with projections to $H(\Gamma)$ and with inclusions from $H(\Gamma, \partial\Gamma)$.

From the construction we can identify the cokernel of the inclusion, which lies in degrees $k = m, \dots, n-m$. For $k \geq m$, we have an exact sequence

$$0 \to H^k(\Sigma_1) \to H^k(\Sigma_{-1}) \to H^k(\Delta_{-1}) \to 0.$$

We can find a splitting of this sequence by noticing that the composition

$$H^k(\Delta_{-1}, \partial\Delta_{-1}) \to H^k(\Sigma_{-1}) \to H^k(\Delta_{-1})$$

is an isomorphism in degrees $k = m, \dots, n-m$. Define

$$\mathcal{K} = \bigoplus_{k=m}^{n-m} H^k(\Delta_{-1}, \partial\Delta_{-1}).$$

Then

$$H(\Sigma_{-1}) = H(\Sigma_1) \oplus \mathcal{K}.$$

Lemma 4.1 *The decomposition*

$$H(\Sigma_{-1}) = H(\Sigma_1) \oplus \mathcal{K}$$

is orthogonal with respect to the Poincaré pairing.

Proof Since $\mathcal{K}$ vanishes for degrees $k > n-m$, it suffices to consider pairings with elements of $H^k(\Sigma_1)$ for $k \geq m$. Now, as

$$H(\Gamma, \partial\Gamma) \cdot H(\Delta_{-1}, \partial\Delta_{-1}) = 0$$

as subspaces of $H(\Sigma_{-1})$, the claim follows by the construction of the decomposition. □

For $t=0$, the function l_0 is conewise linear on both of $\Sigma_{\pm 1}$; hence the quadratic form Q_{l_0} is defined on both of $H(\Sigma_{\pm 1})$. Moreover, l_0 lies in the kernel of the restriction $H(\Sigma_0) \to H(\Delta_0)$, since $H(\Delta_0)$ is nonzero only in degree 0, so l_0 determines a class in $H^1(\Gamma, \partial\Gamma)$.

We note that, while the inclusion $H(\Sigma_1) \hookrightarrow H(\Sigma_{-1})$ is not a homomorphism of rings in general (i.e., for $m \geq 2$), by construction it does commute with multiplication by elements of $H(\Gamma, \partial\Gamma)$.

Lemma 4.2 *The form Q_{l_0} on $H(\Sigma_{\pm 1})$ commutes with the inclusion $H(\Sigma_1) \hookrightarrow H(\Sigma_{-1})$.*

Proof Note that we have the same isomorphism $\langle\cdot\rangle_{\Sigma_{\pm 1}} : H^n(\Gamma, \partial\Gamma) \xrightarrow{\simeq} \mathbb{R}$ from the two evaluation maps. So it suffices to see that, for $f \in H^k(\Sigma_1)$, multiplication by $l_0^{n-2k} f$ commutes with the inclusion. If $n-2k>0$, this is clear, as $l_0 \in H(\Gamma, \partial\Gamma)$, while if $k = \frac{1}{2}n$, then again $f \in H^k(\Sigma_1) = H^k(\Gamma, \partial\Gamma)$. □

Lemma 4.3 *The decomposition*

$$H(\Sigma_{-1}) = H(\Sigma_1) \oplus \mathcal{K}$$

is orthogonal with respect to the quadratic form Q_{l_t} for every $t \leq 0$.

Proof Fix any $t \leq 0$ and let $k \leq \frac{1}{2}n$. Note that l_t^{n-2k} maps $\mathcal{K}^k$ into $\mathcal{K}^{n-k}$, vacuously for $k<m$ and otherwise because $H(\Delta_{-1}, \partial\Delta_{-1})$ is an ideal of $H(\Sigma_{-1})$. So, for $f \in H^k(\Sigma_1)$ and $g \in \mathcal{K}^k$, by Lemma 4.1 we have $\langle l_t^{n-2k} g \cdot f\rangle = 0$. Therefore $Q_{l_t}(f+g) = Q_{l_t}(f) + Q_{l_t}(g)$. □

Lemma 4.4 *The form $(-1)^m Q_{l_t}$ is positive definite on $\mathcal{K}^m$ for every $t<0$.*

Proof Fix $t<0$. Our goal is to evaluate Q_{l_t} on the subspace

$$\mathcal{K}^m = H^m(\Delta_{-1}, \partial\Delta_{-1}) = \chi_\tau H^0(\Delta_{-1})$$

through the projection $\Delta_{-1} \to \Pi = \Pi^{n-m}$ along the face τ onto the normal fan of a simplex of dimension $n-m$.

Let ρ be a ray of τ, with characteristic function χ. We show that χ is concave on Δ_{-1}. Pick also a ray ρ' of τ'. The remaining rays of Δ_{-1} generate a facet γ in $\partial\Delta_{\pm 1}$ (because they generate such a face in the simplicial cone that projects to $\Delta_{\pm 1}$). Let h be a support function of γ (and hence of the convex set $\Delta_{\pm 1}$) which equals χ on ρ. Then $h-\chi$ is a characteristic function of ρ'. These are pullbacks of characteristic functions of a ray of Π, so are convex. So χ is concave.

It follows that for each such ρ, there is a choice of characteristic function χ_ρ so that $l_t|_{\Delta_{-1}} = -\chi_\rho$ in $H^1(\Delta_{-1})$, for as $\dim H^1(\Delta_{-1}) = 1$, the convex classes are exactly the positive multiples of the restriction of l_t to Δ_{-1}. So we may write $(-l_t)^m = \chi_\tau$.

We therefore have an isomorphism $H(\Pi) \to H(\Delta_{-1}) \xrightarrow{(-l_t)^m} H(\Delta_{-1}, \partial\Delta_{-1})$. Let the restriction of l_t to Δ_{-1} be the pullback of a convex function l on Π. Under this isomorphism, the form $(-1)^m Q_{l_t}$ on $\mathcal{K}^m$ corresponds to the form Q_l on $H^0(\Pi)$, since, for $f \in H^{n-m}(\Pi)$,

$$(-1)^m \langle l_t^m \pi^*(f) \rangle_{\Sigma_{-1}} = \langle \chi_\tau \pi^*(f) \rangle_{\Sigma_{-1}} = \langle f \rangle_\Pi.$$

The Hodge–Riemann–Minkowski relations for Π now imply that $(-1)^m Q_{l_t}$ is positive definite on $\mathcal{K}^m$. □

5 Proof of the main theorem

We keep the notation set as in the last two sections. To establish Theorem 1.2 we track the rank and signature of the form Q_{l_t} across an m-flip. The rank is easy to give: induction on dimension establishes that Q_{l_t} is nondegenerate in most cases because the Hodge–Riemann–Minkowski relations in one dimension imply the Hard Lefschetz theorem in the next. This next lemma is the main tool for the induction.

Lemma 5.1 *Let Σ be a complete simplicial fan with a conewise linear function l_Σ such that $l_\Sigma > 0$ on $V \setminus 0$. For a ray ρ of Σ, write $\Delta = \overline{\text{star}}(\rho)$, let π be the projection along ρ, and write $\Lambda = \pi(\Delta)$. The class of $l_\Sigma|_\Delta \in H^1(\Delta)$ is the pullback of some $l \in H^1(\Lambda)$. Assume for every ray ρ of Σ that the Λ and l so obtained satisfy the Hodge–Riemann–Minkowski relations. Then $H(\Sigma)$ and l_Σ satisfy the Hard Lefschetz theorem.*

Proof By Poincaré duality it suffices to show that multiplication by $l_\Sigma^{n-2k} : H^k(\Sigma) \to H^{n-k}(\Sigma)$ is injective. Suppose that $f \in H^k(\Sigma)$ is such that $l_\Sigma^{n-2k} f = 0$; we need to show that $f = 0$.

By assumption we can write $l_\Sigma = \sum_\rho \chi_\rho$ by scaling the characteristic functions to agree with l_Σ on ρ. For each ρ, $f|_\Delta$ is the pullback of some $g \in H^k(\Lambda)$ which is primitive with respect to l, for $l^{(n-1)-2k+1} g = l^{n-2k} g$ maps to zero under the correspondence $H(\Lambda) \simeq H(\Delta)$. Consequently, applying the Hodge–Riemann–Minkowski relations for each Λ,

$$0 = (-1)^k \langle l_\Sigma^{n-2k} f^2 \rangle_\Sigma = \sum_\rho (-1)^k \langle \chi_\rho l_\Sigma^{n-1-2k} f^2 \rangle_\Sigma = \sum_\rho (-1)^k \langle l^{(n-1)-2k} g^2 \rangle_\Lambda \geq 0,$$

with equality at the last stage if and only if each $g = 0$. Therefore each $f|_\Delta \in H(\Delta)$ is zero.

This implies that $\chi_\rho f = 0$ in $H(\Sigma)$ for every ray ρ. Since the characteristic functions generate $H^{n-k}(\Sigma)$, the nondegeneracy of the Poincaré pairing now implies that in fact $f = 0$ in $H(\Sigma)$, as required. □

The class of a convex function on Σ has a representative which satisfies the first hypothesis of this lemma, as does the class of l_0 on both of $\Sigma_{\pm 1}$.

Parts (a) of the next two results are the main theorems; the other parts are auxiliary results used in the induction.

Theorem 5.2 *The conewise linear function l defines a Lefschetz operation on $H(\Sigma)$ in the following cases*:

(a) *l is a convex function on a complete simplicial fan Σ*,
(b) *$l = l_0$ on Σ_1, and*
(c) *$l = l_0$ on Σ_{-1} when $m = \frac{1}{2}n$.*

Theorem 5.3 *The conewise linear function l satisfies the Hodge–Riemann–Minkowski relations on $H(\Sigma)$ in the following cases*:

(a) *l is a convex function on a complete simplicial fan Σ*,
(b) *$l = l_0$ on Σ_1, and*
(c) *$l = l_0$ on Σ_{-1} when $m = \frac{1}{2}n$.*

Proof We first show that Theorem 5.2 in dimension n follows from Theorem 5.3 in dimension $n - 1$. We consider the parts in turn. Theorem 5.2(a) is immediate from Theorem 5.3(a) by application of Lemma 5.1.

For Theorem 5.2(b), we will again invoke Lemma 5.1, this time using all three parts of Theorem 5.3. First, the star of a ray not in Δ_1 is simplicial, and for these, Theorem 5.3(a) applies.

Next we consider a ray in Δ_1. Note that restriction to the star of a ray in Δ_t and projection yields a family of fans Λ_t and convex functions so that the fan Λ_1 is obtained from Λ_{-1} by a flip, which is either an m-flip or an $(m-1)$-flip, if the ray belongs to τ' or to τ, respectively. Indeed, the projection of a simplicial cone of dimension $n+1$ to a star in $\Lambda_{\pm 1}$ so described, through a projection onto $\Delta_{\pm 1}$ and then along a ray of $\Delta_{\pm 1}$, is the same as the projection along one of its rays onto a simplicial cone of dimension n and then onto the star in $\Lambda_{\pm 1}$.

Now for $m < \frac{1}{2}(n+1)$, and for $m = \frac{1}{2}(n+1)$ when the ray of Δ_1 is in τ, we may therefore use Theorem 5.3(b). In the case $m = 1$ note that τ consists of a single ray and does not appear in Σ_1, as the flip from Σ_1 to Σ_{-1} is the star subdivision at this ray, so that the case $m = 0$ does not occur in the induction. Last is the case where $m = \frac{1}{2}(n+1)$ and we consider the star of a ray in τ'. Here the family Λ_t has Λ_1 obtained from Λ_{-1} by an m-flip with $m = \frac{1}{2}((n-1)+2) > \frac{1}{2}((n-1)+1)$, contrary to our convention on labelling fans $\Sigma_{\pm 1}$. This flip, however, is the inverse of a flip with $m' = (n-1) - m + 1 = \frac{1}{2}(n-1)$, and with this reversed labelling Theorem 5.3(c) applies. This establishes Theorem 5.2(b).

Finally, Theorem 5.2(c) is immediate from Theorem 5.2(a–b) through the decomposition $H(\Sigma_{-1}) = H(\Sigma_1) \oplus \mathcal{K}$. Multiplication by $l_0 \in H(\Gamma, \partial\Gamma)$ commutes with the inclusion $H(\Sigma_1) \hookrightarrow H(\Sigma_{-1})$, so Theorem 5.2(b) applies to the first summand. The second is nonzero only in the middle degree $m = \frac{1}{2}n$ where the claim is vacuous.

It remains now to establish Theorem 5.3 in dimension n, assuming that Theorem 5.2 holds in dimension n. We again consider each part in turn.

Theorem 5.2(a–b) imply that Q_{l_t} is nondegenerate on $H(\Sigma_1)$ for $t \geq 0$. Therefore its signature is constant for $t \geq 0$ as well. By Lemma 4.2 the form Q_{l_0} is the same on $H(\Sigma_1)$ as on the first summand of the decomposition $H(\Sigma_{-1}) = H(\Sigma_1) \oplus \mathcal{K}$, so that Q_{l_0} has the same rank and signature on this summand as on $H(\Sigma_1)$. By Theorem 5.2(a) Q_{l_t} is nondegenerate on the first summand for $t > 0$, so that by continuity

$Q_{l_{-1}}$ has this same rank and signature on the first summand as well. Lemma 4.4 gives the signature of Q_{l_t} on $\mathcal{K}$ for $t < 0$. Adding these up and using the orthogonality of Lemma 4.3, we see that the Hodge–Riemann–Minkowski relations hold for Σ_1 and l_1 if and only if they hold for Σ_{-1} and l_{-1}.

For an arbitrary fan Σ and convex function l, choosing a sequence of flips to deform Σ to the normal fan of a simplex and invoking Lemma 3.1 now gives Theorem 5.3(a).

We have already observed that the rank and signature of Q_{l_t} on $H(\Sigma_1)$ are constant for $t \geq 0$. Therefore Theorem 5.3(a) implies Theorem 5.3(b).

Finally, Theorem 5.3(c) follows from Theorem 5.3(b) and Lemma 4.4 through the decomposition $H(\Sigma_{-1}) = H(\Sigma_1) \oplus \mathcal{K}$. This completes the proof. □

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