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SEMISTABLE REDUCTION IN CHARACTERISTIC ZERO

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ABSTRACT

Let $f : X \to B$ be a family of varieties. We consider the problem of replacing the family f with a new family $f' : X' \to B'$ such that all fibers of f' are as nice as possible. The correct definition of "as nice as possible" is given in terms of toric geometry, and a morphism f' satisfying it is called semistable. The semistable reduction problem then asks to find a generically finite proper base change $B' \to B$ and a proper birational morphism $X' \to X \times_B B'$ such that the induced morphism $f' : X' \to B'$ is semistable.

Semistable reduction was proved by Kempf, Knudsen, Mumford and Saint-Donat in the case when the base B is a curve, and by de Jong in the case when the relative dimension of f is one. We extend these results by proving semistable reduction for families of surfaces and three-folds, and a slightly weaker version of semistable reduction for an arbitrary morphism. As an application of weak semistable reduction, we show that, assuming the minimal model program in dimension $n + 1$, one can compactify the moduli space of n-dimensional nonsingular stable varieties with a given Hilbert function by adding a finite number of families of stable (singular) n-folds on the boundary.

Contents

CHAPTER 1

Introduction

One of the milestones in algebraic geometry is the semistable reduction theorem proved in [**KKMS**]:

THEOREM 0.1. *([**KKMS**]) Let $f : X \to C$ be a morphism from a variety X onto a nonsingular curve C, defined over an algebraically closed field k of characteristic zero. Assume that $0 \in C$ is a point and the restriction $f : X \setminus f^{-1}(0) \to C \setminus \{0\}$ is smooth. Then there exist a finite morphism $\pi : C' \to C$, $\pi^{-1}(0) = \{0'\}$, and a proper birational morphism (in fact, a blowup with center lying in the special fiber) $p : X' \to X \times_C C'$*

$$\begin{array}{ccccc} X' & \xrightarrow{p} & X \times_C C' & \to & X \\ & & \downarrow & & \downarrow \\ & & C' & \xrightarrow{\pi} & C \end{array}$$

so that the induced morphism $f' : X' \to C'$ is semistable; i.e.

(i) *Both X' and C' are nonsingular.*

(ii) *The special fiber ${f'}^{-1}(0')$ is a reduced divisor with nonsingular components crossing normally.*

The goal of this thesis is to extend the semistable reduction theorem to the case of a morphism $f : X \to B$ where the base B may have arbitrary dimension. So, we may formulate the problem as follows:

PROBLEM 0.2. Let $f : X \to B$ be a surjective morphism of projective varieties defined over an algebraically closed field k of characteristic zero. Find a generically finite proper surjective morphism (=alteration) $B' \to B$ and a proper birational morphism (=modification) $X' \to X \times_B B'$ such that the induced morphism $f' : X' \to B'$ is semistable.

First of all, we need to specify what we mean by a morphism $f' : X' \to B'$ being semistable. Recall that conditions (i) and (ii) in Theorem 0.1 imply that in suitably chosen local (formal or analytic) coordinates $x_1, \dots, x_n$ at $x \in X'$ and t at $0' \in C'$, the morphism f' is given by $t = x_1 \cdots x_l$ for some $l \leq n$. (Here we ignore the strictness condition; i.e. that the components of the special fiber have to be normal.) As a preliminary definition, we say that a morphism $f : X \to B$ is semistable if locally it is a product of semistable morphisms as above:

Definition 0.3. (Preliminary definition of semistability) A surjective flat morphism $f : X \to B$ of projective varieties is **semistable** if

(i) Both X and B are nonsingular.
(ii) At every point $x \in X$ and $b = f(x) \in B$ we can find formal coordinates $x_1, \dots, x_n$ at x and $t_1, \dots, t_m$ at b such that f is given by
$$t_i = \prod_{j=l_{i-1}+1}^{l_i} x_j$$
for some $0 = l_0 < l_1 < \ldots < l_m \leq n$.

A more precise definition of semistability will be given in next chapter in terms of toroidal embeddings. It follows from Definition 0.3 that if f is semistable then all fibers of f are reduced and have the same dimension. However, unless B is a curve, or the relative dimension of f is at most one, the components of fibers do not intersect transversely. For example, if $f : \mathbb{A}^4 \to \mathbb{A}^2$ is a two-parameter family of surfaces given by $t_1 = x_1x_2$, $t_2 = x_3x_4$, then the fiber of f over the point $t_1 = t_2 = 0$ has four components that do not intersect transversely. For this particular example, we will show in Section 1 of Chapter 2 that there does not exist an alteration $B' \to B$ and a modification $X' \to X \times_B B'$ such that the components of all fibers of the induced morphism $f' : X' \to B'$ intersect transversely. Thus, one may think of semistability as the best possible condition one can achieve using the two operations described above: alterations of the base B and modifications of X.

Before the proof of Theorem 0.1 in [**KKMS**], semistable reduction of a one-parameter family of curves was known for a long time. The problem of semistable reduction over an arbitrary base was first formulated in the introduction of [**KKMS**].

One may also view Problem 0.2 with the help of moduli space theory. Assume that we have a complete moduli space of certain "nice" n-folds. Now given any family of n-folds with generic fiber nice, one can try to replace this family with a new one having all fibers nice, essentially by pulling back a universal n-fold. Using moduli spaces of stable k-pointed curves, de Jong [**dJ**] proved the semistable reduction theorem for families of curves over an arbitrary base. This result will be our main tool in converting an arbitrary morphism into a toroidal morphism (Section 2 in Chapter 2). The proof of the existence of complete moduli spaces for stable surfaces has also been completed recently [**KSB, Ale1**]. Using these moduli spaces, one can replace a family of surfaces (having smooth generic fiber) with a family where all fibers are semi-log-canonical surfaces.

The rest of the thesis is organized as follows. In the next chapter we define semistability in terms of toroidal embeddings, and we solve a slightly weaker problem of "weak semistable reduction". We also formulate the semistable reduction problem in terms of combinatorics of polyhedral complexes. This combinatorial problem is solved in Chapter 3 for low relative dimensions of f, namely, for families of surfaces and three-folds. The last chapter gives an application of weak semistable reduction. We show that assuming the minimal model program in dimension $n+1$ (more precisely, we assume the existence of relative canonical models in the semistable case, the so-called semistable MMP), one can complete the moduli space of smooth n-folds with ample canonical bundle and a given Hilbert polynomial. The points at the boundary of the moduli space correspond to varieties with semi-log-canonical sigularities.

CHAPTER 2

Weak semistable reduction

We work over an algebraically closed field k of characteristic zero.

In this chapter we define weak semistability of a morphism and solve the problem of weak semistable reduction[1]. To motivate the definition of weak semistability, consider a toric morphism of affine toric varieties $f : X_\sigma \to X_\tau$ given by a linear map of cones and lattices $f_\Delta : (\sigma, N_\sigma) \to (\tau, N_\tau)$ (we refer to the next section for notation). It is easy to see that Definition 0.3 of Chapter 1 applied to the morphism f is equivalent to the following two conditions on the linear map f_Δ: (i) both σ and τ are nonsingular; and (ii) f_Δ maps primitive points of σ to primitive points of τ. Assuming that the image cone τ is nonsingular, condition (ii) can be restated as: f_Δ maps faces of σ onto faces of τ, and the lattice N_σ restricted to a face of σ onto the lattice N_τ restricted to the image face of τ. We will see below that the condition of "faces mapping onto faces" corresponds to the geometric condition of f being equidimensional, and the condition of "lattices mapping onto lattices" geometrically means that the fibers of f are reduced. In the definition of weak semistability of a toroidal (i.e. locally toric) morphism we want to keep these two conditions and give up nonsingularity of X.

[1]This chapter describes joint work with Dan Abramovich. The results have been submitted for publication in [ℵ-**K**].

Definition 0.4. A surjective toroidal morphism $f : X \to B$ without horizontal toroidal divisors is called **weakly semistable** if

1. B is nonsingular.
2. f is equidimensional;
3. all fibers of f are reduced;

If also X is nonsingular, we say that the morphism $f : X \to B$ is **semistable**.

We define toroidal embeddings and toroidal morphisms in the next section. A morphism of toroidal embeddings $f : (U_X \subset X) \to (U_B \subset B)$ is said to have no horizontal toroidal divisors if $f^{-1}(U_B) = U_X$. Now we can formulate the main problem more precisely:

Conjecture 0.5 (Semistable reduction). Let $f : X \to B$ be a surjective morphism of projective varieties. There exist a generically finite proper surjective morphism (=alteration) $B' \to B$ and a proper birational morphism (=modification) $X' \to X \times_B B'$ such that the induced morphism $f' : X' \to B'$ is semistable.

This conjecture, although proved for special cases in Chapter 3, remains open in general. The main theorem of this chapter is:

Theorem 0.6 (Weak semistable reduction). *Let $X \to B$ be a surjective morphism of projective varieties. There exist an alteration $B' \to B$ and a modification $X' \to X \times_B B'$, such that $X' \to B'$ is weakly semistable.*

Notation 0.7. Note that the fiber product $X' \to X \times_B B'$ in the statement of the theorem may be reducible. However, it contains a unique component that dominates both X and B'. In the following, we will use $X \times_B B'$ to denote this dominant component.

The proof of the theorem consists of several steps. At each step we improve the morphism f so that it will satisfy one more condition of Definition 0.4. After preliminary results on toroidal morphisms we show in Section 2 that f can be replaced by a toroidal morphism. The proof of this fact is a slightly modified version of a similar proof in [ℵ-**dJ**], and its main ingredient is an inductive application of de Jong's semistable reduction theorem for families of curves.

In Section 3 we show how to remove horizontal divisors from the toroidal structure of X so that the morphism f remains toroidal. This technical condition is needed in Section 8 to show that X has at most rational Gorenstein singularities. We will discuss in Section 9 what happens if we allow certain "good" horizontal divisors.

The next step is to make f equidimensional. We show in Section 4 that equidimensionality of f corresponds to a combinatorial condition on the associated map of polyhedral complexes $f_\Delta : \Delta_X \to \Delta_B$; namely, f_Δ has to map cones onto cones. This requirement is easy to fulfil by subdividing Δ_X and Δ_B.

In Section 5 we attempt to make all fibers of f reduced. First we prove that the condition of all fibers of f being reduced corresponds to the combinatorial condition of f_Δ mapping lattices onto lattices. The combinatorial way to improve f is to replace lattices in Δ_B with sublattices. The question "does this correspond to any geometric operation on the varieties?" is answered positively by **Kawamata's trick**.

The steps in Sections 2-5 prove the weak semistable reduction theorem. We go on to formulate the semistable reduction conjecture purely in combinatorial terms. The resulting combinatorial problem is solved for some special cases in the next chapter.

1. Toroidal morphisms

We recall the definitions of toric varieties, toric morphisms and toroidal embeddings. Everything in this section is classical and can be found for example in [**KKMS, Oda, F**], except the definition of a toroidal morphism and its properties.

1.1. Toric varieties. We start with a brief introduction into affine toric varieties and toric morphisms. A toric variety, or a torus embedding $T \subset X$, is defined to be a normal varieties X containing the algebraic torus $T = \mathbb{G}_m^n$ as an open dense subset, such that the action of T on itself extends to an action on X. What makes toric varieties relatively simple to study, compared to arbitrary algebraic varieties, is that they admit a combinatorial description in terms of cones and lattices. Many geometric properties of toric varieties can then be translated into combinatorial (often linear algebraic) properties of the associated cones and lattices.

Let $N \cong \mathbb{Z}^n$ be a lattice. A rational polyhedral cone $\sigma \subset N \otimes \mathbb{R}$ is a convex cone generated by a finite number of points $v_1, \ldots, v_m \in N$:

$$\sigma = \langle v_1, \ldots, v_m \rangle = \mathbb{R}_{\geq 0} v_1 + \ldots + \mathbb{R}_{\geq 0} v_m.$$

We say that a cone is strictly convex if it has a vertex. Equivalently, a strictly convex cone does not contain any linear subspaces of $N \otimes \mathbb{R}$.

Let $M = \mathrm{Hom}(N, \mathbb{Z})$ be the dual lattice of N, and consider the dual cone:

$$\sigma^\vee = \{u \in M \otimes \mathbb{R} : u(v) \geq 0 \quad \text{for all} \quad v \in \sigma\}.$$

The semigroup algebra $k[\sigma^\vee \cap M]$ is by definition the set of all finite formal linear combinations

$$\sum a_i \chi^{u_i}, \qquad a_i \in k, \qquad u_i \in \sigma^\vee \cap M,$$

where χ is a formal symbol, and the product is defined by

$$(\sum_i a_i \chi^{u_i}) \cdot (\sum_j b_j \chi^{t_j}) = \sum_{i,j} a_i b_j \chi^{u_i + t_j}.$$

The image of $\sigma^\vee \cap M$ in $k[\sigma^\vee \cap M]$ is called the set of monomials. These form a k-basis for $k[\sigma^\vee \cap M]$.

Definition 1.1. The **affine toric variety** X_σ associated to a lattice N and a rational polyhedral strictly convex cone $\sigma \subset N \otimes \mathbb{R}$ is

$$X_\sigma = \text{Spec } k[\sigma^\vee \cap M].$$

Example 1.2.

1. Let $N = \mathbb{Z}^n$ with basis $v_1, \ldots, v_n$, and let σ be generated by a part of the basis, say $v_1, \ldots, v_m$ for some $m \leq n$. Then $\sigma^\vee$ is generated by $v_1^*, \ldots, v_m^*, v_{m+1}^*, -v_{m+1}^*, \ldots, v_n^*, -v_n^*$, where v_i^* is the dual of v_i. Therefore,

$$X_\sigma = \text{Spec } k[x_1, \ldots, x_m, x_{m+1}, x_{m+1}^{-1}, \ldots, x_n, x_n^{-1}] \cong \mathbb{A}^m \times \mathbb{G}_m^{n-m}.$$

   One can also see that all nonsingular affine toric varieties are of this form ([**F**], page 29).

2. For a singular example, let $N = \mathbb{Z}^3$ with basis v_1, v_2, v_3, and let σ be generated by $v_1, v_2, v_3, v_1 + v_2 - v_3$. The dual cone $\sigma^\vee$ consists of points of the form $\sum a_i v_i^*$, where $a_1, a_2, a_3 \geq 0$ and $a_1 + a_2 - a_3 \geq 0$. Then $\sigma^\vee$ is generated by $v_1^*, v_2^*, v_1^* + v_3^*, v_2^* + v_3^*$, with the relation $(v_1^* + v_3^*) + v_2^* = (v_2^* + v_3^*) + v_1^*$, and X_σ is

$$X_\sigma = \text{Spec } k[x, y, z, w]/(zy - wx).$$

$\square$

The k-valued points of the toric variety X_σ are given by k-algebra homomorphisms $k[\sigma^\vee \cap M] \to k$, and such an homomorphism is determined by a semigroup homomorphism $\mu : \sigma^\vee \cap M \to k$, where k is considered as a multiplicative semigroup. Let $\sigma^\perp \subset \sigma^\vee$ be the set of elements $v \in \sigma^\vee$ such that $v(u) = 0$ for all $u \in \sigma$. One sees that $\sigma^\perp$ is the largest linear subspace in $\sigma^\vee$. We define the **special point** x_σ of X_σ to be the point given by the following semigroup homomorphism:

$$v \mapsto \begin{cases} 0 & \text{if } v \notin \sigma^\perp, \\ 1 & \text{if } v \in \sigma^\perp. \end{cases}$$

Let $\sigma \subset N \otimes \mathbb{R}$ be a strictly convex rational cone, and let $v_1, \ldots, v_m$ be a minimal set of generators of σ. If v_i are the first lattice points along the rays $\mathbb{R}_{\geq 0} v_i$, we call them the **primitive points** of σ. The toric variety X_σ is nonsingular if and only if the primitive points of σ form a part of the basis of N. In this case we say that σ is **nonsingular**. The cone σ is **simplicial** if it is generated by $\dim(\sigma)$ lattice points; i.e. $v_1, \ldots, v_m$ are linearly independent.

Let us verify that the toric variety X_σ is normal and contains the algebraic torus $T = \mathbb{G}^n$ as an open dense subset such that the action of T on itself extends to an action on X_σ. First, note that X_σ is irreducible, i.e., its coordinate ring $k[\sigma^\vee \cap M] \subset k[M]$ is a domain.

We have that $\mathbb{G}_m^n = \operatorname{Spec} k[M]$, and the inclusion $\sigma^\vee \cap M \hookrightarrow M$ induces an inclusion of $k[\sigma^\vee \cap M] \hookrightarrow k[M]$, hence a dominant morphism $\mathbb{G}_m^n \to X_\sigma$. Since σ is strictly convex, it follows that the dual cone $\sigma^\vee$ has dimension n. Therefore, inverting a finite number of monomials in $k[\sigma^\vee \cap M]$ gives us $k[M]$. Hence the morphism $\mathbb{G}_m^n \to X_\sigma$ is an open embedding.

The action of $\mathbb{G}_m^n$ on X_σ is induced by the diagonal map

$$\sigma^\vee \cap M \to (\sigma^\vee \cap M) \oplus M,$$

and this extends the action of the torus on itself induced by the diagonal map $M \to M \oplus M$.

To see that X_σ is normal, we note that $\mathbb{G}_m^n$ is normal, so it suffices to show that the extension $k[\sigma^\vee \cap M] \hookrightarrow k[M]$ is integrally closed. The ring $k[M]$ is graded by M, so if an element of $k[M]$ satisfies a polynomial equation, we may assume that that this element is in fact a monomial, and all coefficients of the polynomial are also monomials. This way we get that a multiple of an element $m \in M$ is in $\sigma^\vee \cap M$, and we have to show that $m \in \sigma^\vee \cap M$. Since $\sigma^\vee$ is a cone, this is clear.

As a consequence of the action of $T = \mathbb{G}_m^n$ on X_σ, we get that X_σ is a union of orbits. If $x \in X_\sigma(k)$ is given by $\mu_x : \sigma^\vee \cap M \to k$, and $t \in T(k)$ is given by $\mu_t : M \to k$, then x translated by t is defined by $\mu_t \cdot \mu_x : \sigma^\vee \cap M \to k$. We see that two points $x, y \in X_\sigma$ lie in the same orbit if and only if the kernels (i.e. the inverse images of $0 \in k$) of μ_x and μ_y are the same. Since the kernel of a semigroup homomorphism is a face of $\sigma^\vee$ intersected with M, the orbits are in one-to-one correspondence with the faces of $\sigma^\vee$, hence also with the faces of σ. This correspondence reverses dimensions as follows: a d-dimensional face σ_1 of σ corresponds to the $(n-d)$-dimensional face $\sigma_1^\perp$ of $\sigma^\vee$, and the latter corresponds to the $(n-d)$-dimensional orbit whose closure is defined by the vanishing of all monomials that do not belong to $\sigma_1^\perp$. In particular, the 1-dimensional faces $\langle v_i \rangle$ of σ are in one-to-one correspondence with the codimension-1 orbits. We call the closures of these orbits **toric divisors**. If σ is a full-dimensional cone, i.e. $\dim(\sigma) = n$, then the orbit corresponding to σ itself is 0-dimensional, and consists of the special point $\{x_\sigma\}$.

1.2. Toric morphisms. To distinguish between the lattices and tori of different toric varieties we use subscripts: $N_\sigma, M_\sigma, T_\sigma$. Let X_σ and X_τ be two affine toric varieties given by cones $\sigma \subset N_\sigma \otimes \mathbb{R}$ and $\tau \subset N_\tau \otimes \mathbb{R}$. If $f_\Delta : N_\sigma \to N_\tau$ is a group homomorphism whose scalar extension $f_\Delta : N_\sigma \otimes \mathbb{R} \to N_\tau \otimes \mathbb{R}$ maps σ into τ, denoted by $f_\Delta : (N_\sigma, \sigma) \to (N_\tau, \tau)$, then f_Δ induces a morphism of varieties $f : X_\sigma \to X_\tau$ as follows. The dual of f_Δ maps $f_\Delta^\vee : (M_\tau, \tau^\vee) \to (M_\sigma, \sigma^\vee)$, which yields a morphism of semigroup algebras $k[\tau^\vee \cap M_\tau] \to k[\sigma^\vee \cap M_\sigma]$, and this defines a morphism of varieties $f : X_\sigma \to X_\tau$, called a **toric morphism**.

From the description of the torus action, we see that a toric morphism $f : X_\sigma \to X_\tau$ maps the torus T_σ to the torus T_τ, and the morphism respects the actions of the two tori: $f(t \cdot x) = f(t) \cdot f(x)$ for closed points $t \in T_\sigma$ and $x \in X_\sigma$. In terms of orbits, f maps an orbit corresponding to a face σ_1 of σ to the orbit corresponding to the smallest face τ_1 of τ such that $f_\Delta(\sigma_1) \subset \tau_1$. In particular, if $f(\sigma)$ does not lie in a proper face of τ then f maps the unique closed orbit in X_σ corresponding to σ itself to the unique closed orbit in X_τ corresponding to τ, and moreover $f(x_\sigma) = x_\tau$.

In the definition of a toroidal morphism below we are going to consider dominant toric morphisms only. For such a morphism, the torus T_σ is mapped onto the torus T_τ, hence $f_\Delta^\vee : M_\tau \to M_\sigma$ is injective, and $f_\Delta : N_\sigma \to N_\tau$ has finite cokernel.

1.3. Toroidal embeddings.

Definition 1.3. Given a normal variety X and an open subset $U_X \subset X$, the embedding $U_X \subset X$ is called **toroidal** if for every closed point $x \in X$ there exist a toric variety X_σ, a point $s \in X_\sigma$, and an isomorphism of complete local k-algebras

$$\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{X_\sigma,s}$$

such that the ideal of $X \setminus U_X$ maps isomorphically to the ideal of $X_\sigma \setminus T_\sigma$.

A pair (X_σ, s) in Definition 1.3 is called a **local model** at $x \in X$. By restricting X_σ if necessary, we may assume that the orbit of s is the unique closed orbit in X_σ, and the point s is the special point $x_\sigma \in X_\sigma$.

Definition 1.4. A dominant morphism $f : (U_X \subset X) \to (U_B \subset B)$ of toroidal embeddings is called **toroidal** if for every closed point $x \in X$ there exist local models (X_σ, s) at x, (X_τ, t) at $f(x)$ and a toric morphism $g : X_\sigma \to X_\tau$ so that the following diagram commutes

$$\begin{array}{ccc} \hat{\mathcal{O}}_{X,x} & \xrightarrow{\cong} & \hat{\mathcal{O}}_{X_\sigma,s} \\ {\scriptstyle \hat{f}^\#}\uparrow & & \uparrow{\scriptstyle \hat{g}^\#} \\ \hat{\mathcal{O}}_{B,f(x)} & \xrightarrow{\cong} & \hat{\mathcal{O}}_{X_\tau,t} \end{array}$$

where $\hat{f}^\#$ and $\hat{g}^\#$ are the homomorphisms induced by f and g.

We will often suppress the notation $(U_X \subset X)$, $(U_B \subset B)$, and simply write a toroidal morphism as $f : X \to B$. We also use the same letter f for the morphism $X_\sigma \to X_\tau$ of local models. Without loss of generality, we may again assume that the points $s \in X_\sigma$ and $t \in X_\tau$ are the special points x_σ and x_τ.

Lemma 1.5. *Let $f : X \to B$ be a toroidal morphism, $x \in X$, $b = f(x) \in B$. Then there exist local models X_σ at x and X_τ at b such that*

$$\begin{aligned} X_\sigma &= X_{\sigma'} \times \mathbb{G}_m^l \\ X_\tau &= X_{\tau'} \times \mathbb{G}_m^l \end{aligned}$$

where the closed orbit in $X_{\tau'}$ consists of the point $x_{\tau'}$, and the morphism f in the local models is given by a product of toric morphisms

$$f = f' \times id : X_{\sigma'} \times \mathbb{G}_m^l \to X_{\tau'} \times \mathbb{G}_m^l$$

Proof. Let X_σ and X_τ be local models at $x \in X$ and $b = f(x) \in B$, respectively, and let f also denote the toric morphism of the local models defined by

$$f_\Delta : (N_\sigma, \sigma) \to (N_\tau, \tau).$$

We choose a splitting

$$\begin{aligned} N_\tau &\cong N_{\tau,1} \oplus N_{\tau,2} \\ \tau &\cong \tau' \oplus \{0\} \\ X_\tau &\cong X_{\tau'} \times \mathbb{G}_m^l \end{aligned}$$

with $\tau \cong \tau' \subset N_{\tau,1} \otimes \mathbb{R}$ a full dimensional cone.

Since f is dominant, $\tilde{N}_{\tau,2} = f_\Delta(f_\Delta^{-1}(N_{\tau,2}))$ is a sublattice of finite index in $N_{\tau,2}$. The inclusion $\tilde{N}_{\tau,2} \subset N_{\tau,2}$ corresponds to an étale cover of $\mathbb{G}_m^l$. Since the completed local rings are isomorphic, we may replace the local model by its étale cover and assume that $f_\Delta^{-1}(N_{\tau,2})$ surjects onto $N_{\tau,2}$.

Let $N_{\sigma,1} = f_\Delta^{-1}(N_{\tau,1}) \subset N_\sigma$ and let $N_{\sigma,2} \subset N_\sigma$ be a complementary lattice such that the restriction $f_\Delta : N_{\sigma,2} \to N_{\tau,2}$ is an isomorphism. This gives us a splitting

$$\begin{aligned} N_\sigma &\cong N_{\sigma,1} \oplus N_{\sigma,2} \\ \sigma &\cong \sigma' \oplus \{0\} \\ X_\sigma &\cong X_{\sigma'} \times \mathbb{G}_m^l \end{aligned}$$

Moreover, f_Δ maps $N_{\sigma,1}$ to $N_{\tau,1}$ and identifies $N_{\sigma,2}$ with $N_{\tau,2}$. Thus, the toric morphism f is a product $f = f' \times id$ where $f' : X_{\sigma'} \to X_{\tau'}$ is the toric morphism induced by the restriction of f_Δ to $N_{\sigma,1}$. $\square$

Proposition 1.6. *Let $f : X \to B$ be a toroidal morphism, $x \in X$, $b = f(x) \in B$. If (X_τ, t) is any local model at b then there exists a local model (X_σ, s) at x so that f in the local models is given by a toric morphism $g : X_\sigma \to X_\tau$.*

Proof. By the definition of toroidal morphism, there exist local models $(X_{\sigma'}, x_{\sigma'})$, $(X_{\tau'}, x_{\tau'})$ at x and b so that f is given by a toric morphism $h : X_{\sigma'} \to X_{\tau'}$. Consider the commutative diagram

$$\begin{tikzcd}
\hat{\mathcal{O}}_{X,x} \arrow[r, "\cong"] & \hat{\mathcal{O}}_{X_{\sigma'},x_{\sigma'}} \\
\hat{\mathcal{O}}_{B,b} \arrow[u, "\hat{f}^{\#}"] \arrow[r, "\cong"] & \hat{\mathcal{O}}_{X_{\tau'},x_{\tau'}} \arrow[u, "\hat{h}^{\#}"'] \\
\hat{\mathcal{O}}_{B,b} \arrow[u, "id"] \arrow[r, "\cong"] & \hat{\mathcal{O}}_{X_{\tau},x_{\tau}} \arrow[u, "\hat{\alpha}^{\#}"']
\end{tikzcd}$$

where $\hat{\alpha}^{\#}$ is defined by tracing the other three sides of the lower square.

Let the coordinate rings of the tori in $X_{\sigma'}$, $X_{\tau'}$ and X_{τ} be $k[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, $k[y_1, y_1^{-1}, \ldots, y_m, y_m^{-1}]$ and $k[z_1, z_1^{-1}, \ldots, z_m, z_m^{-1}]$, respectively, and let the toric morphism h be defined by

$$y_i = \prod_{j=1}^{n} x_j^{a_{ij}}$$

for some $a_{ij} \in \mathbb{Z}$. The matrix of the exponents (a_{ij}) is the matrix of the linear map $h_{\Delta}^{\vee} : M_{\tau'} \to M_{\sigma'}$, and has maximal rank $(= m)$ by the assumption that h is dominant. We may choose the bases for $M_{\tau'}$ and $M_{\sigma'}$ so that the matrix (a_{ij}) is diagonal ([**Hun**], page 339). Hence, we may assume that h is given by

$$y_i = x_i^{d_i} \qquad i = 1, \ldots, m$$

for some integers $d_i > 0$.

Assume first that $\{x_\tau\}$ is the closed orbit in X_τ. Since z_i define Cartier divisors supported in $X_\tau \setminus T_\tau$, and these are mapped to Cartier divisors supported in $X_{\tau'} \setminus T_{\tau'}$, defined by monomials in y_j, we get that $\hat{\alpha}^{\#}$ is given by

$$z_i = u_i \prod_{j=1}^{m} y_j^{b_{ij}}$$

where u_i are units in $\hat{\mathcal{O}}_{X_{\tau'},x_{\tau'}}$, and $b_{ij} \in \mathbb{Z}$. Now the two groups of Cartier divisors are isomorphic, so the matrix (b_{ij}) is invertible, and again by changing the bases of $M_{\tau'}$ and M_τ if necessary, we may assume that $b_{ij} = \delta_{ij}$:

$$z_i = u_i y_i.$$

If we now choose a root $h^\#(u_i)^{\frac{1}{d_i}}$ for every $i = 1, \dots, m$ and set

$$\tilde{x}_i = h^\#(u_i)^{\frac{1}{d_i}} x_i$$

then in these new coordinates the composition $\hat{h}^\# \circ \hat{\alpha}^\#$ is induced by a toric morphism defined by

$$z_i = \tilde{x}_i{}^{d_i}.$$

If $\{x_\tau\}$ is not a closed orbit in X_τ, that means, the point $b \in B$ is not cut out by toroidal divisors, we can reduce to the previous case by adding new divisors through b to the toroidal structure of B. (We refer to Section 3 for further discussion on adding and removing toroidal divisors.)

Consider the morphism of formal germs:

$$\begin{array}{ll} X_{\sigma'} & = X'_{\sigma'} \times \mathbb{G}_m^l \\ \downarrow & h = h' \times id \\ X_{\tau'} & = X'_{\tau'} \times \mathbb{G}_m^l \\ \downarrow & \alpha \\ X_\tau & = X'_\tau \times \mathbb{G}_m^l \end{array}$$

To add new divisors to the toroidal structure of B, we replace the local model $X'_\tau \times \mathbb{G}_m^l$ by $X'_\tau \times \mathbb{A}^l$. If m is a monomial in the coordinate ring of X_τ defining a new divisor $X'_\tau \times \mathbb{A}^{l-1} \times \{0\} \subset X'_\tau \times \mathbb{A}^l$, then $\alpha^\#(m)$ restricts to a local parameter on $\{x'_{\tau'}\} \times \mathbb{G}_m^l \subset X'_{\tau'} \times \mathbb{G}_m^l$, hence $h^\#\alpha^\#(m)$ restricts to a local parameter on $\{x'_{\sigma'}\} \times \mathbb{G}_m^l \subset X'_{\sigma'} \times \mathbb{G}_m^l$.

Now we may change the local models $X_{\sigma'}$ and $X_{\tau'}$ so that $\alpha^{\#}(m)$ and $h^{\#}\alpha^{\#}(m)$ are monomials, hence the composition $\alpha \circ h$ is toroidal in the new local models. □

Corollary 1.7. *If $f : X \to Y$ and $g : Y \to Z$ are toroidal morphisms then $g \circ f : X \to Z$ is also toroidal.*

Proof. Choose local models in which g is toric. By the proposition, there exists a local model of X in which f, hence also $g \circ f$, is toric. □

1.4. Polyhedral complexes. Next we review the construction of a polyhedral complex Δ_X associated to a toroidal embedding $U_X \subset X$. We start by considering a toric variety X_σ and ask how much of the combinatorial structure (N_σ, σ) can be recovered from knowing the torus-invariant divisors in X_σ.

Let X_σ be an affine toric variety defined by a cone $\sigma \subset N_\sigma \otimes \mathbb{R}$, and let the union of the torus-invariant divisors in X_σ be $X_\sigma \setminus T_\sigma = D$. A monomial in $k[M_\sigma]$ defines a Cartier divisor supported in D, so we get a homomorphism from M_σ to the group of T_σ-invariant Cartier divisors of X_σ. This homomorphism is surjective, and its kernel consists of invertible monomials $\sigma^\perp \cap M_\sigma$. Therefore, we can recover $M_\sigma/(\sigma^\perp \cap M_\sigma)$ as the group of T_σ-invariant Cartier divisors in X_σ and the cone $\sigma^\vee/\sigma^\perp$ generated by the effective Cartier divisors. Since Span $(\sigma) \cap N_\sigma$ is dual to $M_\sigma/\sigma^\perp \cap M_\sigma$, we also get $\sigma \subset (\text{Span } (\sigma) \cap N_\sigma) \otimes \mathbb{R}$ from the data of T_σ-invariant Cartier divisors of X_σ.

It is also easy to see that the stratification of the toric variety X_σ into orbits of the torus action determines the face-relations of the cone σ in the following sense. If $s \in X_\sigma$ lies in the orbit corresponding to a face σ_1 of σ, we consider the torus-invariant

Cartier divisors of X_σ passing through s. In other words, we consider the torus-invariant Cartier divisors in the toric variety X_{σ_1} defined by the cone $\sigma_1 \subset N_\sigma \otimes \mathbb{R}$. As before, we recover the cone σ_1 and the lattice Span $(\sigma_1) \cap N_\sigma$.

Now if $U_X \subset X$ is a toroidal embedding, then at every closed point $x \in X$ we have a local model X_σ defined by a cone $\sigma \subset N_\sigma \otimes \mathbb{R}$. Since the cone σ and the lattice Span $(\sigma) \cap N_\sigma$ are determined by Cartier divisors in (the formal germ of) $X \setminus U_X$, we see that they do not depend on the local model chosen. Next we define a stratification of X, which in a local model corresponds to the stratification by orbits. Then it follows that the cone σ and the lattice Span $(\sigma) \cap N_\sigma$ are the same for all points in the same stratum; the polyhedral complex Δ_X is obtained by gluing together the cones and lattices corresponding to different strata.

Let $X \setminus U_X = \cup_{i \in I} E_i$, where E_i is an irreducible Weil divisor for all $i \in I$. We assume that E_i are normal, that is, $U_X \subset X$ is a toroidal embedding **without self-intersection**. For a subset $J \subset I$ we define

$$E_J = \begin{cases} \emptyset & \text{if } \cap_{j \in J} E_j = \emptyset, \\ \cap_{j \in J} E_j \setminus \cup_{J \subsetneq K} E_K & \text{otherwise.} \end{cases}$$

Now X can be written as a disjoint union $X = \cup_{J \subset I} E_J$. If $x \in X$ is a closed then in a local model X_σ at x the locally closed subsets E_J corresponds to the orbits in X_σ, hence E_J are nonsingular. We say that the components Y of E_J form a stratification of X. If Y is a stratum, we let Star(Y) be the union of those strata that contain Y in their closure.

To a stratum Y in X one associates:

1. M^Y: – the group of Cartier divisors in Star(Y) supported in Star$(Y) \setminus U_X$
2. N^Y: – Hom$(M^Y, \mathbb{Z})$

3. $M^Y_+ \subset M^Y$: – effective Cartier divisors
4. $\sigma^Y \subset N^Y \otimes \mathbb{R}$: – the dual of M^Y_+

It follows from the previous discussion that if (X_σ, s) is a local model at $x \in X$ in the stratum Y, then

$$\begin{aligned} M^Y &\cong M_\sigma/\sigma^\perp \\ N^Y &\cong N_\sigma \cap \mathrm{Span}\ (\sigma) \\ \sigma^Y &\cong \sigma \end{aligned}$$

Moreover, if Y lies in the closure of another stratum Z then σ^Z is a face of σ^Y and $N^Z = N^Y|_{\mathrm{Span}\ (\sigma^Z)}$. Thus we may glue together the cones σ^Y and lattices N^Y to get a polyhedral complex Δ_X:

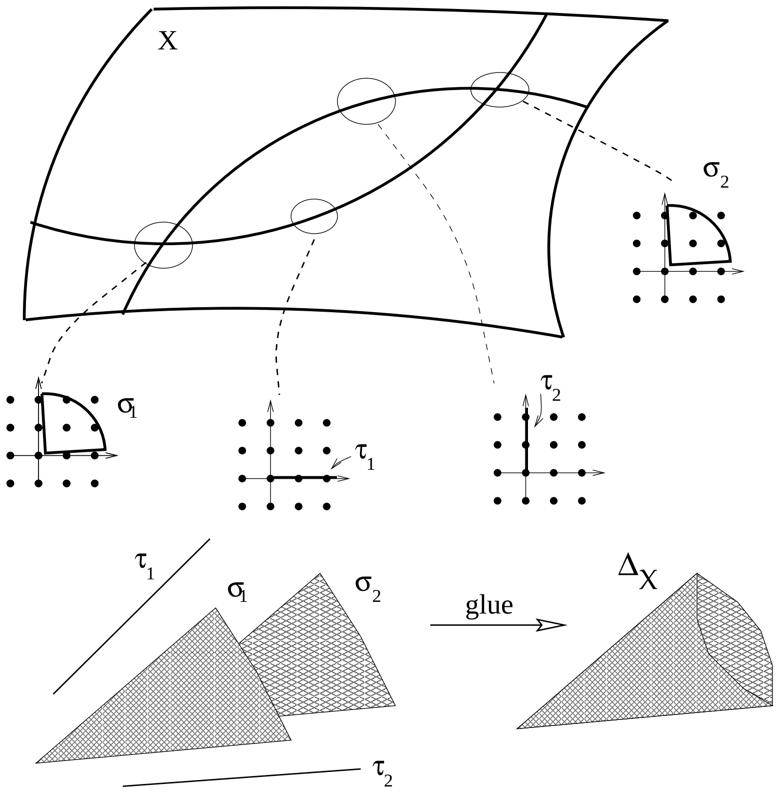

Figure 1. Polyhedral complex associated to a toroidal embedding

Definition 1.8. A rational conical **polyhedral complex** $\Delta = (|\Delta|, \{(N_\sigma, \sigma)\})$ consists of a finite collection of lattices $N_\sigma \cong \mathbb{Z}^{n_\sigma}$ and rational polyhedral strictly convex full dimensional cones $\sigma \subset N_\sigma \otimes \mathbb{R}$. The cones σ are glued together to form the space $|\Delta|$ so that the usual axioms of polyhedral complexes hold:

1. If $\sigma \in \Delta$ is a cone then every face σ' of σ is also in Δ, and $N_{\sigma'} = N_\sigma|_{\text{Span }(\sigma')}$.
2. The intersection of two cones $\sigma_1 \cap \sigma_2$ is a closed union of faces of both of them.

A morphism $f_\Delta : \Delta_X \to \Delta_B$ of polyhedral complexes $\Delta_X = (|\Delta_X|, \{(N_\sigma, \sigma)\})$ and $\Delta_B = (|\Delta_B|, \{(N_\tau, \tau)\})$ is a compatible collection of linear maps $f_\sigma : (N_\sigma, \sigma) \to (N_\tau, \tau)$; i.e. if σ' is a face of σ then $f_{\sigma'}$ is the restriction of f_σ.

Lemma 1.9. *A toroidal morphism $f : X \to B$ induces a morphism of polyhedral complexes $f_\Delta : \Delta_X \to \Delta_B$. If $x \in X$ lies in stratum Y and $f(x) \in B$ lies in stratum Z, let $g : X_\sigma \to X_\tau$ be the representation of f in some local models at x and $f(x)$, defined by a linear map $g_\Delta : (N_\sigma, \sigma) \to (N_\tau, \tau)$. Then via the identification above, $f_\Delta : (N^Y, \sigma^Y) \to (N^Z, \sigma^Z)$ is the restriction of g_Δ to the span of σ.*

Proof. We define the maps $f_\Delta : (N^Y, \sigma^Y) \to (N^Z, \sigma^Z)$ as the restrictions of $g_\Delta : (N_\sigma, \sigma) \to (N_\tau, \tau)$ for some representation of f in local models $g : X_\sigma \to X_\tau$ at some $x \in X$ and $f(x) \in B$. This does not depend on the point x or the local models chosen because the group homomorphism $g_\Delta^\vee : M_\tau/\tau^\perp \to M_\sigma/\sigma^\perp$ is defined by pulling back torus-invariant Cartier divisors, hence $f_\Delta^\vee : M^Z \to M^Y$ is defined by pulling back Cartier divisors in $\text{Star}(Z)$ supported on $\text{Star}(Z) \setminus U_B$ and restricting to $\text{Star}(Y)$. By the same reason, the linear maps $f_\Delta|_{\sigma^Y}$ glue together to yield a morphism of polyhedral complexes $f_\Delta : \Delta_X \to \Delta_B$. $\square$

Remark 1.10.

We will use the same notation (N_σ, σ) for lattices and cones both in a local model and in the polyhedral complex associated to a toroidal embedding, with the understanding that if N_σ is the lattice of the local model, the corresponding lattice of the polyhedral complex is obtained by restricting N_σ to the span of σ.

REMARK 1.11. Note that the polyhedral morphism $f_\Delta : \Delta_X \to \Delta_B$ is well defined even if f is not toroidal, as long as $f(U_X) \subset U_B$. We use this in the following example to show that a semistable morphism cannot be improved any further, as promised in the introduction.

EXAMPLE 1.12. Let $f : \mathbb{A}^4 \to \mathbb{A}^2$ be a morphism from the affine 4-plane $\mathbb{A}^4$ to the affine 2-plane $\mathbb{A}^2$ given by $t_1 = x_1x_2$, $t_2 = x_3x_4$. Then f is semistable according to Definition 0.4. The fiber $f^{-1}(0)$ consists of four components that do not intersect transversely. We show that one cannot improve the morphism f to a semistable morphism f' such that the components of all fibers of f' cross transversely.

Suppose that, using an alteration B' of B and a modification X' of $X \times_B B'$, we can replace f by a family of surfaces $f' : X' \to B'$ in which both X' and B' are nonsingular, and all fibers of f' consist of nonsingular components crossing transversely.

$$\begin{array}{ccc} X' & \xrightarrow{q} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{p} & B \end{array}$$

The morphism $f : (U_X \subset X) \to (U_B \subset B)$ is toroidal (in fact toric), with $U_B = B \setminus (D_1 \cup D_2)$ and $U_X = X \setminus (E_1 \cup \ldots \cup E_4)$, where D_i is the divisor defined by $t_i = 0$ and E_j is the divisor defined by $x_j = 0$. By assumption, the morphism $f' : (U_{X'} \subset X') \to (U_{B'} \subset B')$ is also toroidal. Since f is not smooth over $D_1 \cup D_2$, we see that q maps $U_{B'}$ into U_B and p maps $U_{X'}$ into U_X. Let $\Delta_X, \Delta_B, \Delta_{X'}, \Delta_{B'}$ be the associated polyhedral complexes. Even though p and q need not be toroidal, by the remark above we get a commutative diagram of morphisms of polyhedral complexes.

$$\begin{array}{ccc} \Delta_{X'} & \xrightarrow{q_\Delta} & \Delta_X \\ {\scriptstyle f'_\Delta}\downarrow & & \downarrow{\scriptstyle f_\Delta} \\ \Delta_{B'} & \xrightarrow{p_\Delta} & \Delta_B \end{array}$$

Here Δ_X consists of the cone $\sigma = \langle v_1, v_2, v_3, v_4\rangle$ and all its faces; and Δ_B consists of the cone $\tau = \langle u_1, u_2\rangle$ and all its faces. The morphism f_Δ maps $v_1, v_2 \mapsto u_1$ and $v_3, v_4 \mapsto u_2$. If we assume that the components of all fibers of f' intersect transversely, we can find local coordinates $y_1, \ldots, y_4$ at $x' \in X'$ and s_1, s_2 at $f'(x') \in B'$ in which f' is given by $s_1 = y_1$, $s_2 = y_2 \cdots y_i$, where $i = 2, 3$ or 4. In terms of polyhedral complexes this means that if $\sigma' = \langle v'_1, v'_2, v'_3, v'_4\rangle \in \Delta_{X'}$ is any four-dimensional cone mapping to $\langle u'_1, u'_2\rangle \in \Delta_{B'}$, then f'_Δ maps $v'_1 \mapsto u'_1$ and $v'_2, v'_3, v'_4 \mapsto u'_2$. It follows that the image of σ' in σ is at most three-dimensional. Hence, almost all points of $N_\sigma \cap \sigma$ do not lie in the image of q_Δ. We show that this contradicts the properness of q.

For a point $v = \sum_i a_i v_i \in N_\sigma \cap \sigma$, consider the morphism of germs $\phi : (C, 0) \to (X, x)$ from a nonsingular curve $C = \mathrm{Spec}\ k[z]$, given by $x_i = z^{a_i}$. We can choose the point v so that it does not lie in the image of q_Δ, and q is étale over $\phi(C \setminus \{0\})$ (recall that q is generically finite). Using this, and the properness of q we can lift a finite cover C' of C to X' so that the following diagram commutes:

$$\begin{array}{ccc} (C', 0') & \xrightarrow{\phi'} & X' \\ {\scriptstyle z=(z')^m}\downarrow & & \downarrow{\scriptstyle q} \\ (C, 0) & \xrightarrow{\phi} & X \end{array}$$

Let Y be the stratum of X' containing x'. Then the morphism ϕ' defines a linear map $M^Y \to \mathbb{Z}$, hence an element $v' \in N^Y$, as follows: if E' is a Cartier divisor through x' locally defined by $g = 0$, we map E' to the order of vanishing of $g \circ \phi'$ at $0' \in C'$. Since $q^\vee_\Delta$ is defined by pulling back Cartier divisors, using the projection formula [**Har2**], we see that q_Δ maps v' to mv. Now since $v \notin q_\Delta(\Delta_X)$, then also $mv \notin q_\Delta(\Delta_X)$, and we have a contradiction. $\square$

1.5. Subdivisions. Given a toroidal embedding $U_X \subset X$ with the polyhedral complex Δ_X, and a subdivision Δ'_X of Δ_X, one constructs a new toroidal embedding $U_{X'} \subset X'$ with the polyhedral complex Δ'_X, and a birational toroidal morphism $f' : X' \to X$ such that the induced map of the polyhedral complexes $\Delta'_X \to \Delta_X$ is the given subdivision. If Y is a stratum in X corresponding to the cone $\sigma^Y \in \Delta_X$, and if $\sigma' \subset \sigma^Y$ is a cone in the subdivision, define

$$V_{\sigma'} = \mathcal{S}\mathrm{pec}_{\ \mathrm{Star}(Y)} \sum_{D \in \sigma'^\vee \cap M^Y} \mathcal{O}_{\mathrm{Star}(Y)}(-D),$$

where the sum is taken inside the field of rational functions of $\mathrm{Star}(Y)$. Then X' is formed by gluing together the open sets $V_{\sigma'}$.

A subdivision Δ'_X of Δ_X is called **projective** if there exists a continuous function $\psi : |\Delta_X| \to \mathbb{R}$, taking rational values on $\sigma \cap N_\sigma$, which is convex and piecewise linear on each cone $\sigma \in \Delta_X$, and the largest pieces in σ where ψ is linear are the cones of the subdivision. Such ψ is called a **good function** (or *lifting function*, or *order function*), for the subdivision Δ'_X of Δ_X. A projective subdivision corresponds to a projective modification $X' \to X$.

LEMMA 1.13. *Let $f : X \to B$ be a toroidal morphism, $f_\Delta : \Delta_X \to \Delta_B$ the associated morphism of polyhedral complexes. Let $X' \to X$ and $B' \to B$ be toroidal modifications, with associated subdivisions $\Delta_{X'}$ and $\Delta_{B'}$. Then f lifts to a morphism $f' : X' \to B'$ if and only if for each cone $\sigma' \in \Delta_{X'}$, there exists a cone $\tau' \in \Delta_{B'}$ such that $f_\Delta(\sigma') \subset \tau'$.*

Proof. Let $\sigma' \subset \sigma^Y \in \Delta_X$ and $\tau' \subset \tau^Z \in \Delta_B$ be cones in the subdivisions such that $f_\Delta(\sigma') \subset \tau'$. The homomorphism $\mathcal{O}_{\mathrm{Star}(Z)} \to \mathcal{O}_{\mathrm{Star}(Y)}$ extends to

$$\sum_{D \in \tau'^\vee \cap M^Z} \mathcal{O}_{\mathrm{Star}(Z)}(-D) \to \sum_{E \in \sigma'^\vee \cap M^Y} \mathcal{O}_{\mathrm{Star}(Y)}(-E)$$

because $f_\Delta^\vee(\tau'^\vee) \subset \sigma'^\vee$. This shows that the rational map f' is a morphism on each $V_{\sigma'}$. Since $V_{\sigma'}$ cover X', f' is a morphism.

Conversely, suppose f' is a morphism. Let $x' \in X', f'(x') = b' \in B'$. Since the three morphisms $X' \to X$, $X \to B$ and $B' \to B$ are toroidal, we have a toric morphism $g : X_{\sigma'} \to X_{\tau'}$ where $(X_{\sigma'}, s')$ and $(X_{\tau'}, t')$ are local models at x' and b'. Thus $g_\Delta(\sigma') \subset \tau'$. But $g_\Delta|_{\sigma'} = f_\Delta|_{\sigma'}$, hence $f_\Delta(\sigma') \subset \tau'$. Since this is true for x' in any stratum of X', for any cone $\sigma' \in \Delta_{X'}$ we get that $f_\Delta(\sigma') \subset \tau'$ for some $\tau' \in \Delta_{B'}$. $\square$

2. Toroidal reduction

The purpose of this section is to modify any family of varieties into a toroidal morphism.

THEOREM 2.1. *Let $f : X \to B$ be a surjective morphism of projective varieties, and let $Z \subset X$ be a proper closed subscheme. There exists a diagram*

$$\begin{array}{ccccc} U_X & \subset & X' & \overset{m_X}{\to} & X \\ \downarrow & & \downarrow f' & & \downarrow f \\ U_B & \subset & B' & \overset{m_B}{\to} & B \end{array}$$

where m_B and m_X are modifications, X' and B' are nonsingular, the inclusions on the left are toroidal embeddings, and such that

1. *f' is toroidal.*
2. *Let $Z' = m_X^{-1} Z$. Then $Z' \subset X' \setminus U_{X'}$.*

Proof. The theorem was proved in [ℵ-**dJ**] for B a point. The proof of the relative version ($\dim B \geq 0$) given here is the same, only the first induction step has to be changed.

We proceed by induction on the relative dimension of f.

2.1. Relative dimension 0. Assume that the relative dimension of f is zero. Since f is surjective, it is generically finite. Let L be a normalization of the function field $K(X)$ of X over the function field $K(B)$ of B. Then $L/K(B)$ is a finite Galois extension with Galois group G. We choose a projective model $\tilde{X}$ of L such that G acts on $\tilde{X}$. If we fix an embedding $K(X) \subset L$ then for each $g \in G$ we get a rational map $\phi_g : \tilde{X} \to X$ corresponding to the embedding $gK(X) \subset L$. We let

$$\overline{\Gamma} \subset \tilde{X} \times_B \prod_{g \in G} X$$

be the closure of the graph of $\prod_{g \in G} \phi_g$. Then the group G acts on $\overline{\Gamma}$, and projection to one of the factors X gives us a morphism $\overline{\Gamma} \to X$. So, we may replace $\tilde{X}$ by $\overline{\Gamma}$ and assume that the rational map $\tilde{X} \to X$ is a morphism.

Consider the quotient variety $\tilde{X}/G$. Since G fixes the the field $K(B)$, we have a birational morphism $p : \tilde{X}/G \to B$, hence a rational map $p^{-1} \circ f : X \to \tilde{X}/G$. Let $X_0 \subset X \times_B \tilde{X}/G$ be the closure of the graph of this rational map, and note that the projection $\tilde{X} \to \tilde{X}/G$ factors through X_0 (we think of $\tilde{X}$ as the graph of the projection $\tilde{X} \to \tilde{X}/G$ in $\tilde{X} \times_B \tilde{X}/G$, and the latter maps to $X \times_B \tilde{X}/G$):

$$\begin{array}{ccc} & & \tilde{X} \\ & \swarrow & \downarrow \\ X_0 & \to & X \\ f_0 \downarrow & & \downarrow f \\ \tilde{X}/G & \to & B \end{array}$$

Since $\tilde{X} \to \tilde{X}/G$ is finite, so is $f_0 : X_0 \to \tilde{X}/G$. Let $D \in \tilde{X}/G$ be the branch locus of f_0. We let $B' \to \tilde{X}/G$ be a resolution of singularities such that the inverse image of $D \cup f(Z)$ is a divisor of normal crossing, and we let X' be the normalization of $X_0 \times_{\tilde{X}/G} B'$. Then the projection $f' : X' \to B'$ is a finite morphism that ramifies over a divisor of normal crossing. By a lemma of Abhyankar, such a morphism is toroidal.

LEMMA 2.2. *(Abhyankar, cf.* [**GM**]*) Let X be a normal variety, and $f : X \to B$ a surjective finite morphism onto a nonsingular variety, ramifying over a divisor D of normal crossing. Then $(B \setminus D \subset B)$ and $(X \setminus f^{-1}(D) \subset X)$ are toroidal embeddings, and f is a toroidal morphism.* □

Assume now that we have proved the result for morphisms of relative dimension $n-1$, and consider the case $\mathrm{rel.\,dim}\, f = n$.

2.2. Preliminary reduction steps. First, we may replace X by its normalization, therefore we may assume X normal, and by blowing up Z in X we may assume Z a Cartier divisor. Let $\eta \in B$ be the generic point of B. By the projectivity assumption we have $X \subset \mathbb{P}^N_B$ for some N. Choosing a generic projection $\mathbb{P}^N_\eta \dashrightarrow \mathbb{P}^{n-1}_\eta$ we get a rational map $X_\eta \dashrightarrow \mathbb{P}^{n-1}_\eta$. Replacing X by the closure of the graph of this map, we may assume that we have a morphism $X \to \mathbb{P}^{n-1}_B$. Let $X \xrightarrow{g} P \to \mathbb{P}^{n-1}_B$ be the Stein factorization, where $g : X \to P$ is a morphism of relative dimension 1 with connected fibers.

DEFINITION 2.3. Let $\alpha : X_1 \to X$ be an alteration, and $Z \subset X$ an irreducible divisor. The **strict altered transform** $Z_1 \subset X_1$ is the closure of $\alpha^{-1}(\eta)$ in X_1, where η is the generic point of Z.

2.3. Semistable reduction of a family of curves. By [**dJ2**], Theorem 2.4, we have a diagram as follows:

$$\begin{array}{ccc} X_1 & \stackrel{\alpha}{\to} & X \\ \downarrow g_1 & & \downarrow g \\ P_1 & \stackrel{a}{\to} & P \\ & & \downarrow \\ & & B \end{array}$$

and a finite group $G \subset \mathrm{Aut}_P P_1$, with the following properties:

1. The morphism $a : P_1 \to P$ is a Galois alteration with Galois group G (i.e. $P_1/G \to P$ is birational).
2. The action of G lifts to $\mathrm{Aut}_X X_1$, and $\alpha : X_1 \to X$ is a Galois alteration with Galois group G.
3. There are n disjoint sections $\sigma_i : P_1 \to X_1$ such that the strict altered transform $Z_1 \subset X_1$ of Z is the union of their images, and G permutes the sections σ_i.
4. The morphism $g_1 : X_1 \to P_1$ is a nodal family of curves, and $\sigma_i(P_1)$ is disjoint from $\mathrm{Sing}\, g_1$.

We may replace X, P and Z by X_1/G and P_1/G, and $\alpha^{-1}Z$. Note that the image of the complement $\alpha^{-1}(Z) \setminus Z_1$ in X has codimension at least two, hence it lies over a proper closed subset of P.

2.4. Using the inductive hypothesis. Let $\Delta \subset P$ be the union of the loci over which Z, P_1 or X_1 are not smooth. We apply the inductive assumption to $\Delta \subset P \to B$, and obtain a diagram

$$\begin{array}{ccccc} U_{P'} & \hookrightarrow & P' & \stackrel{m}{\to} & P \\ \downarrow & & \downarrow & & \downarrow \\ U_{B'} & \hookrightarrow & B' & \to & B \end{array}$$

such that P', B' are nonsingular, $P' \to P$ and $B' \to B$ are modifications, the left square is a toroidal morphism, and $m^{-1}\Delta$ is a divisor of strict normal crossings contained in $P' \setminus U_{P'}$.

We may again replace P, B by P', B', and further we may replace X, X_1, P_1, Z and σ_i by their pullback to P'. In particular $P \to B$ has a toroidal structure, and $P_1 \to P$ is unramified over U_P. By Lemma 2.2, since P_1 is normal, it inherits a toroidal structure given by $U_{P_1} = a^{-1}U_P$ as well, so that $P_1 \to P$ is a toroidal finite morphism.

2.5. Conclusion of proof. Now $X_1 \to P_1$ is a nodal family which is smooth over U_{P_1}, therefore it as well inherits a toroidal structure $U_{X_1} \subset X_1$, where $U_{X_1} = ({g_1}^{-1}U_{P_1}) \setminus (\cup\sigma_i(P_1))$; e.g. local equations around a node are of the form $uv = f(t)$, where $f(t)$ is a monomial on P_1. By construction, $\alpha^{-1}Z$ is a divisor contained in $X_1 \setminus U_{X_1}$.

Definition 2.4. Let $U_Y \subset Y$ be a toroidal embedding and G a finite group acting on the embedding $U_Y \subset Y$. We say that G **acts toroidally** if the stabilizer G_y of every point $y \in Y$ can be identified with a subgroup of the torus T in a local model at y.

Lemma 2.5. *Let $Y \to P$ be a toroidal morphism, and $G \subset \mathrm{Aut}_P Y$ a finite group acting toroidally on Y. Then Y/G can be given a toroidal structure in which $Y/G \to P$ is toroidal.*

Proof. It suffices to consider a toric morphism of local models $g : X_\sigma \to X_\tau$, and a finite subgroup of the torus $G \subset T_\sigma$. The group G acts on the coordinate ring $k[\sigma^\vee \cap M_\sigma]$ by multiplying the monomials with roots of unity. A basis for the quotient algebra is given by the invariant monomials, and these form a sublattice $M_\sigma^G \subset M_\sigma$. Thus, the quotient variety X_σ/G is toric, given by the cone σ and lattice $(M_\sigma^G)^\vee$. Since G acts over X_τ, the image of M_τ lies in the sublattice M_σ^G, hence the morphism $X_\sigma/G \to X_\tau$ is toric. $\square$

By the previous lemma, it remains to modify X_1 so that G acts toroidally on it. Then X_1/G is a toroidal embedding, birational to X, and $X_1/G \to P_1/G = P$ is a toroidal morphism. Finally, a toroidal resolution of singularities completes the proof of the theorem.

In order to find such a modification of X_1, we apply the procedure of [**ℵ-dJ**], Section 1.4. Recall that G acts on the nodal family $g_1 : X_1 \to P_1$ permuting the sections $\sigma_i : P_1 \to X_1$. After blowing up the nodes, i.e. the locus $sing(g_1)$, one sees that G acts toroidally at the nodes and the marked points σ_i of a fiber. The points x where G does not act toroidally lie on the smooth locus of g_1, and we can choose coordinates so that G_x acts via a character ψ_x on a parameter u of the smooth component of the fiber through x. Now there exists a canonical G-invariant ideal I_G such that the blowup $\tilde{X}_1 = Bl_{I_G} X_1$ can be given a toroidal structure in which G acts toroidally and the morphism $\tilde{X}_1 \to P_1$ is toroidal. If X_σ is a local model at $x \in X_1$ such that the parameter u on which G acts via the character ψ_x is a monomial, then the ideal I_G in this local model is generated by all monomials on which G acts via the same character ψ_x. $\square$

3. Adding and removing divisors

Let $U_X \subset X$ be a toroidal embedding. We call the complement $D_X = X \setminus U_X$ the **toroidal divisor** of the embedding. If D is a divisor in X we may **add D to**

the toroidal structure, provided that $D_X \cup D$ is the complement of a toroidal embedding. Similarly, if D lies in D_X, we may **remove it from the toroidal structure** if the closure of $D_X \setminus D$ in X is the complement of a toroidal embedding.

Now if $f : X \to B$ is a toroidal morphism, we consider situations in which we can add or remove divisors in the toroidal structures of X and B so that the morphism f remains toroidal. We first show that if X is nonsingular, we can remove all **horizontal divisors** – components of D_X that dominate B. The idea is that f is locally given by monomials $t_i = x_1^{k_1} \cdots x_r^{k_r}$ in which the variables defining horizontal divisors cannot appear. The absence of horizontal divisors will not be used until Section 8 where we study properties of weakly semistable morphisms.

Proposition 3.1. *Let $f : (U_X \subset X) \to (U_B \subset B)$ be a toroidal morphism. Assume that X is nonsingular. Then, denoting $U'_X = f^{-1}(U_B)$, we have that $U'_X \subset X$ is a toroidal embedding, and $f : (U'_X \subset X) \to (U_B \subset B)$ is a toroidal morphism.*

Proof. Let f be given in local models at $x \in X$ and $f(x) \in B$ by a toric morphism $f : X_\sigma \to X_\tau$ defined by $f_\Delta : (N_\sigma, \sigma) \to (N_\tau, \tau)$. Let $N_{\sigma_1} = N_\sigma \cap \operatorname{Span} \sigma \cap Ker(f_\Delta)$, and let N_{σ_2} be a complementary lattice containing all primitive points of σ not in $Ker(f_\Delta)$. Then we have a splitting of the morphism f_Δ and the cone σ:

$$\begin{array}{rcl} N_\sigma \cong N_{\sigma_1} \oplus N_{\sigma_2} & \stackrel{0 \oplus f'_\Delta}{\to} & N_\tau \\ \sigma \cong \sigma_1 \oplus \sigma_2 & \to & \tau \end{array}$$

and f factors through the projection

$$f : X_\sigma \cong \mathbb{A}^l \times X_{\sigma_2} \stackrel{p_2}{\to} X_{\sigma_2} \stackrel{f'}{\to} X_\tau$$

The horizontal divisors are the coordinate hyperplanes in $\mathbb{A}^l$ pulled back to X_σ. We can remove all of them by taking a new local model

$$f : X'_\sigma \cong \mathbb{G}_m^l \times X_{\sigma_2} \xrightarrow{p_2} X_{\sigma_2} \xrightarrow{f'} X_\tau$$

□

Proposition 3.2. *Let $f : X \to B$ be a toroidal morphism. Assume that H is a divisor in B, and at every point $b \in H$ there exists a local model X_τ of B of the form*

$$\begin{aligned} X_\tau &\cong X'_\tau \times \mathbb{G}_m \\ H &\cong X'_\tau \times \{1\}. \end{aligned}$$

Then one can add H and $f^{-1}(H)$ to the toroidal structures of B and X so that the morphism f remains toroidal.

The above conditions are satisfied if B is quasi-projective and H a general hyperplane section.

Proof. As in Lemma 1.5 and Proposition 1.6 we choose a local model $X_\sigma \cong X'_\sigma \times \mathbb{G}_m$ at $x \in X$ where $f(x) = b$ such that

$$f = f' \times id : X'_\sigma \times \mathbb{G}_m \to X'_\tau \times \mathbb{G}_m.$$

To add H and $f^{-1}(H)$ we take new local models

$$f = f' \times id : X'_\sigma \times \mathbb{A}^1 \to X'_\tau \times \mathbb{A}^1.$$

If B is quasi-projective then using Bertini's theorem ([**Har2**], page 179), we may choose the hyperplane section H so that it intersects all (nonsingular) strata of B transversely. At any point $b \in H$ we can find a local model $X_\tau \cong X'_\tau \times \mathbb{G}_m^l$ with $l > 0$ because H does not meet any 0-dimensional strata of B. If $Y = \{x'_\tau\} \times \mathbb{G}_m^l$ is the unique closed orbit in X_τ then H intersects Y transversely, and we can choose new coordinates so that

$$\begin{aligned} X_\tau &\cong X'_\tau \times \mathbb{G}_m^{k-1} \times \mathbb{G}_m \\ H &\cong X'_\tau \times \mathbb{G}_m^{k-1} \times \{1\} \end{aligned}$$

□

PROPOSITION 3.3. *Let $f : X \to B$ be a toroidal morphism and let H be a component of $B \setminus U_B$. Assume that at every point $x \in X$ such that $b = f(x) \in H$ one can find local models $X_\sigma \cong X'_\sigma \times \mathbb{A}^1$ at x and $X_\tau \cong X'_\tau \times \mathbb{A}^1$ at b such that $H \cong X'_\tau \times \{0\} \subset X'_\tau \times \mathbb{A}^1$ and f is a product of toric morphisms:*

$$f = f' \times id : X'_\sigma \times \mathbb{A}^1 \to X'_\tau \times \mathbb{A}^1.$$

Then the morphism f remains toroidal after removing H and $f^{-1}(H)$ from the toroidal structures of B and X.

Proof. We choose new local models

$$f = f' \times id : X'_\sigma \times \mathbb{G}_m \to X'_\tau \times \mathbb{G}_m.$$

□

4. Making the morphism equidimensional

The goal of this section is to perform modifications on B and X, after which the morphism becomes equidimensional. First a lemma which characterizes equidimensional toroidal morphisms:

LEMMA 4.1. *Let $f : X \to B$ be a surjective toroidal morphism, $f_\Delta : \Delta_X \to \Delta_B$ the associated morphism of polyhedral complexes. Then f is equidimensional if and* *only if for any cone $\sigma \in \Delta_X$, we have $f_\Delta(\sigma) \in \Delta_B$. That is, the image of a cone of Δ_X is a cone of Δ_B.*

Proof. Consider the morphism f in a local model $f : X_\sigma \to X_\tau$. By splitting the local models and the morphism as in Lemma 1.5, we may assume that the closed orbit in X_τ is a point $\{x_\tau\}$. The generic fiber of f has dimension $\mathrm{Rank}(N_\sigma) - \mathrm{Rank}(N_\tau)$. The fiber $f^{-1}(x_\tau)$ is the union of orbits in X_σ corresponding to the faces of σ whose image under f_Δ intersects the interior of τ. If σ' is such a face of σ then the orbit corresponding to σ' has dimension $\mathrm{Rank}(N_\sigma) - \dim(\sigma')$. For f to be equidimensional, $\mathrm{Rank}(N_\sigma) - \dim(\sigma') \leq \mathrm{Rank}(N_\sigma) - \mathrm{Rank}(N_\tau)$, i.e. $\dim(\sigma') \geq \mathrm{Rank}(N_\tau) = \dim(\tau)$. Thus, f is equidimensional if and only if f_Δ maps every cone of dimension l to a cone of dimension $\leq l$, which is equivalent to the statement that the image of a cone is a cone. $\square$

REMARK 4.2. In case Δ_B is simplicial, the condition of the lemma is that all 1-dimensional cones of Δ_X map to 0 or 1-dimensional cones of Δ_B.

Let $X \to B$ is a toroidal morphism with the associated morphism of polyhedral complexes $f_\Delta : \Delta_X \to \Delta_B$. If Δ'_X and Δ'_B are subdivisions of Δ_X and Δ_B so that the induced morphism $f' : \Delta_X \to \Delta_B$ maps cones onto cones, then the corresponding morphism of varieties (Lemma 1.13) $X' \to B'$ is equidimensional by the previous proposition. To prove that there exist such subdivisions of Δ_X and Δ_B, we need the following lemma.

LEMMA 4.3. *Let $f_\Delta : \Delta_X \to \Delta_B$ be a morphism of polyhedral complexes. There exists a projective subdivision Δ'_B of Δ_B so that the image of a cone $\sigma \in \Delta_X$ is a union of cones of Δ'_B.*

Proof. First we show that it suffices to find such a subdivision for each cone $\tau \in \Delta_B$. Let $\psi_\tau : \tau \to \mathbb{R}$ is a good function (Section 1.5) defining the subdivision of τ, and consider

$$\psi = \sum_{\tau \in \Delta_B} \bar{\psi}_\tau$$

where $\bar{\psi}_\tau : |\Delta_B| \to \mathbb{R}$ is a good extension of ψ_τ. Since the subdivision of $\tau \in \Delta_B$ defined by ψ is a refinement of the subdivision defined by ψ_τ, we get that ψ defines the required subdivision of Δ_B.

Now assume that Δ_B consists of a single cone τ and all its faces. For any cone $\sigma \in \Delta_X$, $f_\Delta(\sigma)$ is a strictly convex rational subcone of τ. Choose rational linear functions $l_{\sigma,1}, \ldots, l_{\sigma,N}$ on τ such that $f_\Delta(\sigma)$ is defined by $l_{\sigma,i} \geq 0$ for all i. Let

$$\psi_\tau = -\sum_{\sigma,i} |l_{\sigma,i}|$$

Then ψ_τ is a good function defining the required subdivision of τ. $\square$

Proposition 4.4. *Let $X \to B$ be a toroidal morphism with the associated morphism of polyhedral complexes $f_\Delta : \Delta_X \to \Delta_B$. Then there exist projective subdivisions Δ'_X of Δ_X and Δ'_B of Δ_B with Δ'_B nonsingular, such that the induced (a priori rational) map $f' : X' \to B'$ is an equidimensional toroidal morphism.*

If X has no horizontal divisors with respect to f then X' has no horizontal divisors with respect to f'.

Proof. Let Δ_B^0 be a subdivision of Δ_B as in the previous lemma, and let Δ'_B be a nonsingular projective subdivision of Δ_B^0. Then $f_\Delta(\sigma)$ is a union of cones of Δ'_B for every $\sigma \in \Delta_X$.

Let ψ be a good function on $|\Delta_B|$ defining the subdivision Δ'_B. The function $\psi \circ f_\Delta$ is good and defines a projective subdivision Δ'_X of Δ_X whose cones map onto cones of Δ'_B.

If we assume from the beginning that $f^{-1}(U_B) = U_X$, that means $f_\Delta^{-1}(0) \cap |\Delta_X| = \{0\}$, then clearly the same is true for any subdivision $|\Delta'_X| = |\Delta_X|$, hence $(f')^{-1}(U_{B'}) = U_{X'}$. $\square$

Remark 4.5. Given an equidimensional toroidal morphism and the corresponding map of polyhedral complexes $\Delta_X \to \Delta_B$, where Δ_B is simplicial, one can find a projective *simplicial* subdivision of Δ_X so that the induced map is equidimensional. To construct such a subdivision of Δ_X, note that if $\sigma \in \Delta_X$ is any non-simplicial cone, then there exists a 1-dimensional face of σ such that the star subdivision centered at this face has the following properties:

1. The number of codimension-1 faces of any maximal cone of the subdivision is smaller than the number of codimension-1 faces of σ.
2. The 1-dimensional cones of the subdivision are the same as the 1-dimensional faces of σ.

Thus, a finite number of such star subdivisions of Δ_X gives us a simplicial subdivision Δ'_X. Since the 1-dimensional cones of Δ_X and Δ'_X are the same, the induced morphism $\Delta'_X \to \Delta_B$ maps cones onto cones by Remark 4.2.

5. Kawamata's trick and reduced fibers

The goal in this section is to find a finite base change after which all the fibers of the resulting morphism are reduced. The base change we perform will not necessarily be toroidal, but the morphism after base change will still be toroidal.

5.1. The toric picture. To start, we characterize equidimensional toroidal morphisms with reduced fibers in terms of polyhedral complexes. In the proof of the lemma we will use the following simple fact:

FACT 5.1. Let $\sigma \subset M \otimes \mathbb{R}$ be a full dimensional cone ($\dim \sigma = \mathrm{Rank} M$), and let $m_1 \in \mathrm{Int}(\sigma) \cap M$, $m_2 \in M \setminus (\sigma \cap M)$. Then there exist integers $k, l > 0$ such that $km_1 + lm_2$ lies in a proper face of σ.

Proof. The line joining m_1 and m_2 crosses the boundary of σ at a rational point because the cone and the points are rational. Hence $m = \lambda m_1 + (1 - \lambda) m_2$ lies in a proper face of σ for some rational number $0 < \lambda < 1$. Now canceling the denominators proves the claim. □

LEMMA 5.2. *Let $f : X \to B$ be an equidimensional toroidal morphism, $f_\Delta : \Delta_X \to \Delta_B$ the associated morphism of polyhedral complexes. Then f has reduced fibers if and only if for any cone $\sigma \in \Delta_X$ we have*

$$f_\Delta(N_\sigma) = N_{f_\Delta(\sigma)}.$$

Proof. It suffices to consider a toric morphism of local models $f : X_\sigma \to X_\tau$ and the fiber $f^{-1}(x_\tau)$. By Lemma 1.5 we may assume that $\tau \subset N_\tau \otimes \mathbb{R}$ is full dimensional. If $\sigma \subset N_\sigma \otimes \mathbb{R}$ is not full dimensional, we can choose a splitting $N_\sigma \cong N_{\sigma,1} \oplus N_{\sigma,2}$

corresponding to $X_\sigma \cong X'_\sigma \times \mathbb{G}_m^l$. The restriction of f_Δ to $N_{\sigma,1}$ and $N_{\sigma,2}$ gives us toric morphisms $f' : X'_\sigma \to X_\tau$ and $f'' : \mathbb{G}_m^l \to X_\tau$. Consider the morphism

$$g : X_\sigma \overset{pr_1}{\to} X'_\sigma \overset{f'}{\to} X_\tau.$$

We claim that $g^{-1}(x_\tau) \cong f^{-1}(x_\tau)$, hence $f^{-1}(x_\tau) \cong f'^{-1}(x_\tau) \times \mathbb{G}_m^l$, and we may replace X_σ by X'_σ. The ideal of $\{x_\tau\} \subset X_\tau$ in $k[M_\tau \cap \tau^\vee]$ is generated by the monomials $M_\tau \cap \tau^\vee \setminus \{0\}$, and the ideals of the fibers $g^{-1}(x_\tau)$ and $f^{-1}(x_\tau)$ are generated by the monomials $g_\Delta^\vee(M_\tau \cap \tau^\vee \setminus \{0\})$ and $f_\Delta^\vee(M_\tau \cap \tau^\vee \setminus \{0\})$ respectively. For any $m \in M_\tau$ the images $f_\Delta^\vee(m)$ and $g_\Delta^\vee(m)$ differ by an element in $M_{\sigma,2} \subset \sigma^\vee$.

Thus we may consider a toric morphism $f : X_\sigma \to X_\tau$ defined by a linear map $f_\Delta : (N_\sigma, \sigma) \to (N_\tau, \tau)$, where both $\sigma \subset N_\sigma \otimes \mathbb{R}$ and $\tau \subset N_\tau \otimes \mathbb{R}$ are full dimensional cones. It follows from the equidimensionality of f that $f_\Delta : \sigma \to \tau$ is surjective.

Suppose that $f_\Delta(N_\sigma) \neq N_\tau$. Since $N_\tau / f_\Delta(N_\sigma)$ is finite, the image of the dual map $f_\Delta^\vee : M_\tau \to M_\sigma$ is not saturated, and because $\tau \subset N_\tau \otimes \mathbb{R}$ is full dimensional and $f_\Delta^\vee$ is injective, the image of the restriction

$$f_\Delta^\vee : M_\tau \cap \tau^\vee \hookrightarrow M_\sigma \cap \sigma^\vee$$

is not saturated; i.e., there exists $m \in M_\sigma \cap \sigma^\vee$ such that

$$dm = f_\Delta^\vee(m_1)$$

for some $d > 1$, $m_1 \in M_\tau \cap \tau^\vee$, and no such m_1 exists for $d = 1$. Let w be a strictly positive linear function on $\sigma^\vee \setminus \{0\}$. We claim that if m is chosen so that $w(m)$ is minimal possible then the monomial corresponding to m is nilpotent in the coordinate ring of $f^{-1}(x_\tau)$. We have to check that m is not in the ideal of $f^{-1}(x_\tau)$. Suppose that

$$m = f_\Delta^\vee(m'_1) + m'_2$$

for some $m_1' \in M_\tau \cap \tau^\vee \setminus \{0\}$, $m_2' \in M_\sigma \cap \sigma^\vee$. Then we may replace m by m_2' with strictly smaller weight $w(m_2') < w(m)$ as follows. We have

$$dm_2' = f_\Delta^\vee(m_1 - dm_1'),$$

where $m_1 - dm_1' \in M_\tau \cap \tau^\vee$ because $(f_\Delta^\vee)^{-1}(\sigma) = \tau$ by equidimensionality of f. If $m_2' = f_\Delta^\vee(m_1'')$ for some $m_1'' \in M_\tau \cap \tau^\vee$ then $m = f_\Delta^\vee(m_1' + m_1'')$ contradicting the choice of m.

Conversely, suppose that the fiber $f^{-1}(x_\tau)$ is not reduced. Since the coordinate ring of X_σ is graded by M_σ and the ideal of $f^{-1}(x_\tau)$ is graded (generated by monomials), we see that the coordinate ring of $f^{-1}(x_\tau)$ is also M_σ-graded, and we can find a nilpotent monomial; i.e, there exists $m \in M_\sigma \cap \sigma^\vee$ such that

$$dm = f_\Delta^\vee(m_1) + m_2$$

for some $d > 1$, $m_1 \in M_\tau \cap \tau^\vee \setminus \{0\}$, $m_2 \in M_\sigma \cap \sigma^\vee$, and no such m_1, m_2 exist for $d = 1$. Let the data (d, m, m_1, m_2) be chosen so that m_2 lies in a minimal face of $\sigma^\vee$, meaning that if ρ is the face of $\sigma^\vee$ containing m_2 in its interior then no such four-tuple (d', m', m_1', m_2') exists with m_2' lying in a proper face of ρ. Let σ_0 be the face of σ defined by $m_2 = 0$, hence $\sigma_0^\perp = \mathrm{Span}\ (\rho)$. We claim that the image of the map

$$\overline{f}_\Delta^\vee : M_\tau \cap \tau^\vee \to (M_\sigma \cap \sigma_0^\vee)/\sigma_0^\perp$$

is not saturated. Since modulo $\sigma_0^\perp$ we have $d\overline{m} = \overline{f}_\Delta^\vee(m_1)$, it suffices to check that $\overline{m}$ is not in the image. Suppose

$$m = f_\Delta(m_1') + m_2'$$

for some $m_1' \in M_\tau \cap \tau^\vee$, $m_2' \in M_\sigma \cap \sigma_0^\perp = M_\sigma \cap \mathrm{Span}\ (\rho)$. Note that we can always choose $m_1' \neq 0$; if $m \in \sigma_0^\perp$, we can take $m_1' = m_1$. Now $m_2' \notin M_\sigma \cap \sigma^\vee$ by the choice of m, hence $m_2' \in \mathrm{Span}\ (\rho) \setminus \rho$. By the fact proved above we can find integers $k, l > 0$

such that $m_2'' = km_2 + lm_2'$ lies in a proper face of ρ, and

$$(kd + l)m = f_\Delta^\vee(km_1 + lm_1') + m_2''$$

contradicts the choice of (d, m, m_1, m_2).

Let τ_0 be the image of σ_0, so that the restriction of f_Δ factors

$$f_\Delta : (N_\sigma \cap \text{Span } \sigma_0, \sigma_0) \to (N_\tau \cap \text{Span } \tau_0, \tau_0) \hookrightarrow (N_\tau, \tau).$$

The dual map factors as:

$$\overline{f}_\Delta^\vee : M_\tau \cap \tau^\vee \twoheadrightarrow (M_\tau \cap \tau_0^\vee)/\tau_0^\perp \xrightarrow{\phi} (M_\sigma \cap \sigma_0^\vee)/\sigma_0^\perp,$$

where the first map is surjective, hence the image of the second map ϕ is not saturated. Since $f_\Delta : \sigma_0 \to \tau_0$ is surjective, we get that ϕ is injective and therefore the image of $M_\tau/\tau_0^\perp \hookrightarrow M_\sigma/\sigma_0^\perp$ is not saturated. Dualizing again, we get

$$f_\Delta(N_\sigma \cap \text{Span } \sigma_0) \neq N_\tau \cap \text{Span } \tau_0.$$

□

Remark 5.3. If f is equidimensional and Δ_B nonsingular, the condition of the lemma is that primitive points of σ map to primitive points of τ.

Next we consider a toric morphism $f : X_\sigma \to X_\tau$, a toric base change $g : X_{\tau'} \to X_\tau$, and ask when is the normalization of the fiber product $X_{\tau'} \times_{X_\tau} X_\sigma$ a toric variety. The coordinate ring of $X_{\tau'} \times_{X_\tau} X_\sigma$ is

$$\mathcal{O}_{X_{\tau'}} \otimes_{\mathcal{O}_{X_\tau}} \mathcal{O}_{X_\sigma} = k[{\tau'}^\vee \cap M_{\tau'}] \otimes_{k[\tau^\vee \cap M_\tau]} k[\sigma^\vee \cap M_\sigma] = k[\pi]$$

where π is the pushout of $g_\Delta^\vee : \tau^\vee \cap M_\tau \to {\tau'}^\vee \cap M_{\tau'}$ and $f_\Delta^\vee : \tau^\vee \cap M_\tau \to \sigma^\vee \cap M_\sigma$:

$$\pi = ({\tau'}^\vee \cap M_{\tau'}) \times (\sigma^\vee \cap M_\sigma)/ \sim .$$

Here $\sim$ is the equivalence relation $(v_1, w_1) \sim (v_2, w_2)$ whenever there exists $u \in \tau^\vee \cap M_\tau$ such that $(v_1, w_1) = (v_2, w_2) \pm (g_\Delta^\vee(u), -f_\Delta^\vee(u))$.

Let M be the Abelian group

$$M = M_{\tau'} \times M_\sigma / \{(g_\Delta^\vee(u), -f_\Delta^\vee(u)) | u \in M_\tau\}$$

and let $\iota : \pi \to M$ be the natural semigroup homomorphism.

Assume that ι is injective. Then $k[\pi] \subset k[M]$ and the integral closure of $k[\pi]$ in $k[M]$ is the semigroup algebra $k[\tilde{\pi}]$ where $\tilde{\pi}$ is the saturation of π in M. Write $M = F \oplus T$ where F is the free part and T is torsion. Then $T \subset \tilde{\pi}$ because $mT = 0$ for some $m > 0$. For any $f + t \in \tilde{\pi}$ where $f \in F$ and $t \in T$, we have $-t \in T \subset \tilde{\pi}$, hence $f \in \tilde{\pi}$. Thus

$$\tilde{\pi} \cong (\tilde{\pi} \cap F) \oplus T.$$

If t_i are generators of T of order m_i, and if x_i is the image of t_i in $k[\tilde{\pi}]$, then

$$k[\tilde{\pi}] \cong k[\tilde{\pi} \cap F][\dots, x_i, \dots]/(x_i^{m_i} = 1)$$

and we see that Spec $k[\tilde{\pi}]$ splits into $|T|$ components with $x_i = \zeta_i$ where ζ_i are m_i'th roots of unity, each component isomorphic to Spec $k[\tilde{\pi} \cap F]$. Thus, if ι is injective then every component of the normalization of $X_{\tau'} \times_{X_\tau} X_\sigma$ is a toric variety.

Lemma 5.4. *With notation as above, if X_τ is simplicial and $g : X_{\tau'} \to X_\tau$ equidimensional then ι is injective for any X_σ and any $f : X_\sigma \to X_\tau$.*

Proof. Suppose that $\iota(v_1, w_1) = \iota(v_2, w_2)$, where $v_1, v_2 \in {\tau'}^\vee \cap M_{\tau'}$ and $w_1, w_2 \in \sigma^\vee \cap M_\sigma$. Say, $(v_1, w_1) = (v_2, w_2) + (g_\Delta^\vee(u), -f_\Delta^\vee(u))$ for some $u \in M_\tau$.

Extend the set of primitive points of τ to a basis of $N_\tau \otimes \mathbb{R}$ and let $t_1, \dots, t_m \in \tau^\vee$ be the dual basis of $M_\tau \otimes \mathbb{R}$. Write $u = \sum_i a_i t_i$ with $a_i \in \mathbb{R}$. Collecting positive and negative terms, we get $u = u_1 - u_2$ with $u_1, u_2 \in \tau^\vee \cap M_\tau$. The equation $v_1 = v_2 + g_\Delta^\vee(u_1) - g_\Delta^\vee(u_2)$ implies that $v_2 - g_\Delta^\vee(u_2) \in {\tau'}^\vee \cap M_{\tau'}$: since g is equidimensional,

$g_\Delta^\vee(u_1)$ and $g_\Delta^\vee(u_2)$ as functions on τ' are nonzero on disjoint subsets of primitive points of τ'. Then

$$(v_1, w_1) \sim (v_1 - g_\Delta^\vee(u_1), w_1 + f_\Delta^\vee(u_1)) = (v_2 - g_\Delta^\vee(u_2), w_2 + f_\Delta^\vee(u_2)) \sim (v_2, w_2)$$

and ι is injective. $\square$

Our main application of the previous lemma is to the following special case.

LEMMA 5.5. *Let $f : X_\sigma \to X_\tau$ be a toric morphism with X_τ simplicial. Let $X_{\tau'}$ be a toric variety given by $\tau' = \tau$ and $N_{\tau'} \subset N_\tau$ a sublattice of finite index. Then every irreducible component of the normalization of $X_\sigma \times_{X_\tau} X_{\tau'}$ is a toric variety $X_{\sigma'}$ given by the cone $\sigma' = \sigma$ and integral lattice $N_{\sigma'} = N_\sigma \cap f_\Delta^{-1}(N_{\tau'})$.*

Proof. We use notation as above. Since X_τ, $X_{\tau'}$ and $g : X_{\tau'} \to X_\tau$ satisfy the assumption of the previous lemma, ι is injective. By the previous discussion, every component of the normalization of $X_{\tau'} \times_{X_\tau} X_\sigma$ is isomorphic to the toric variety Spec $k[\tilde{\pi} \cap F]$.

The image of the homomorphism $\phi : M \to M_\sigma \otimes \mathbb{R}$ defined by $\phi(v, w) = f_\Delta^\vee(v) + w$ for $v \in M_{\tau'}, w \in M_\sigma$ contains M_σ. Since M and M_σ have the same rank, ϕ embeds F in $M_\sigma \otimes \mathbb{R}$ as a lattice of full rank containing M_σ. Now $f_\Delta^\vee$ takes $\tau^\vee$ into $\sigma^\vee$, and we see that ϕ maps $\tilde{\pi} \cap F$ into $\sigma^\vee$ so that the image contains $\sigma^\vee \cap M_\sigma$. Therefore, Spec $k[\tilde{\pi} \cap F]$ is a toric variety defined by the cone $\sigma' = \sigma$ and integral lattice $N_{\sigma'} = F^\vee$, where $F^\vee$ is the group of homomorphisms from M_σ to $\mathbb{Z}$ taking integral values on $f_\Delta^\vee(M_{\tau'}) \subset M_\sigma \otimes \mathbb{R}$, which is $N_\sigma \cap f_\Delta^{-1}(N_{\tau'})$. $\square$

If $f : X \to B$ and $p : B' \to B$ are toroidal morphisms with local models $X_\sigma, X_\tau, X_{\tau'}$ of X, B, B' as in the lemma, then for X' the normalization of $X \times_B B'$, $U_{X'} \subset X'$ is a toroidal embedding and the projection $f' : X' \to B'$ is a toroidal morphism, where $U_{X'} = f^{-1}(U_{B'})$. To see that $U_{X'} \subset X'$ is without self-intersection, notice that if a component D of $X' \setminus U_{X'}$ corresponds to two different T-invariant divisors in a local model of X', then the image of D in X also corresponds to two different T-invariant divisors in a local model of X; since $U_X \subset X$ has no self-intersection, the image of D would be reducible.

5.2. Kawamata's covering. To apply Lemma 5.5 in the toroidal case, we need a toroidal morphism $B' \to B$ which ramifies along a divisor $D \subset B \setminus U_B$ with a certain index.

DEFINITION 5.6. Let L be a Cartier divisor on B, and let $D \in |mL|$. Choose a rational section s_L of $\mathcal{O}_B(L)$ defining L and a regular section s_D of $\mathcal{O}_B(mL)$ defining D. Consider the rational function $\phi = s_D/s_L^m$ on B. Then the field $K(B)(\sqrt[m]{\phi})$ depends only on D and $\mathcal{O}(L)$. The normalization of B in $K(B)(\sqrt[m]{\phi})$ is called **the cyclic cover ramified along D with index m.**

REMARK 5.7. Another way to define the cyclic cover is as the normalization of

$$\mathcal{S}pec_B \mathrm{Sym}^{\bullet}\left(\mathcal{O}_B(L)^{\vee}\right)/(\{f - s_D(f)\}_f)$$

where we view s_D as a morphism $s_D : \mathcal{O}_B(kL) \to \mathcal{O}_B((k+m)L)$.

When B is nonsingular and D a divisor of normal crossings so that $s_D = x_1 \cdots x_l$ for some local parameters $x_1, \ldots, x_l$, the cyclic cover has a local equation

$$z^m = x_1 \cdots x_l.$$

Suppose that $D \subset B$ is a Cartier divisor and m an integer. We wish to find a degree m cyclic cover of B that ramifies along D with index m, i.e., we have to find a Cartier divisor L in B such that $D \in |mL|$. In general, such an L does not exist.

However, assuming that B is quasi-projective, we can find L and the corresponding cyclic cover that ramifies along D *and* a general hyperplane section H, as follows. Let A be an ample divisor in B. We may replace A by a multiple of itself if necessary, so that $mA - D$ is very ample. Now let H be a general member of $|mA - D|$; then $D + H \in |mA|$, and we can construct a cyclic cover ramified along $D + H$ with degree m. Since H is general, $D + H$ is a divisor of normal crossings. Kawamata's trick consists of choosing not one hyperplane section H but $\dim B$ of them, and considering $\dim B$ cyclic covers ramified along $D + H_j$, $j = 1, \dots, \dim B$. Assuming that both B and D are nonsingular, the normalization of the fiber product of these covers is also nonsingular.

Let $U_B \subset B$ be a nonsingular quasi-projective toroidal embedding, and let $B \setminus U_B = \sum D_i$ be the decomposition of the toroidal divisor into nonsingular irreducible components crossing normally. For each component D_i let m_i be a positive integer, $i = 1, \dots, m$. A **Kawamata covering package** consists of the data (D_i, m_i, H_{ij}) with the following properties:

1. H_{ij} are effective reduced nonsingular divisors in B for $i = 1, \dots, m$, $j = 1, \dots, \dim B$.
2. $\sum_i D_i + \sum_{i,j} H_{ij}$ is a reduced divisor of normal crossings (in particular, H_{ij} are distinct).
3. $D_i + H_{ij} \in |m_i L_i|$ for some Cartier divisor L_i for all i, j.

Such H_{ij} can be chosen as general hyperplane sections. By Proposition 3.2 we can add the divisors H_{ij} to the toroidal structure of B. Denote the new toroidal embedding $\tilde{U}_B \subset B$, where $\tilde{U}_B = B \setminus \bigcup_{ij}(D_i \cup H_{ij})$.

Now let B_{ij} be the cyclic cover ramified along $D_i + H_{ij}$ with index m_i; let B' be the normalization of the product

$$B_{1,1} \underset{B}{\times} \dots \underset{B}{\times} B_{m,\dim\ B}$$

and let $p : B' \to B$ be the projection. The next lemma shows that p is a toroidal morphism of nonsingular toroidal embeddings.

LEMMA 5.8 (Kawamata). *Let $\tilde{U}_{B'} = p^{-1}(\tilde{U}_B)$. Then $\tilde{U}_{B'} \subset B'$ is a nonsingular toroidal embedding, and $p : (\tilde{U}_{B'} \subset B') \to (\tilde{U}_B \subset B)$ is a toroidal morphism.*

If X_τ is a local model of $\tilde{U}_B \subset B$ at $b \in B$ then a local model $X_{\tau'}$ of $\tilde{U}_{B'} \subset B'$ at any $b' \in p^{-1}(b)$ is constructed as follows. Let the primitive points of τ be $v_1, \dots, v_l$, and assume that v_α corresponds to the divisor D_{i_α} or $H_{i_\alpha j}$ for some j. If we extend the primitive points of τ to a basis $v_1, \dots, v_{\dim B}$ of N_τ then $\tau' = \tau$ and $N_{\tau'} \subset N_\tau$ is the sublattice of finite index with basis $m_{i_1}v_1, \dots, m_{i_l}v_l, v_{l+1}, \dots, v_{\dim B}$.

Proof. Let x_i be local equations of D_i, and let y_{ij} be local equations of H_{ij} at $b \in B$. Then B_{ij} is locally given by the equation

$$z_{ij}^{m_i} = x_i y_{ij}$$

Let us fix an i and consider the product $\times_j B_{ij}$. If $b \in D_i$ then since $\bigcap_j H_{ij} \cap D_i = \emptyset$, say $b \notin H_{i1}$, and y_{i1} is a unit in $\mathcal{O}_{B,b}$. In local coordinates we may assume that $y_{i1} = 1$. The normalization of the product $\times_j B_{ij}$ is then locally defined by the equations

$$\begin{aligned} z_{i,1}^{m_i} &= x_i \\ \left(\frac{z_{i,j}}{z_{i,1}}\right)^{m_i} &= y_{ij} \qquad j = 2, \dots \dim B. \end{aligned}$$

Similarly, if $b \notin D_i$, then we may assume that $x_i = 1$ and the local equations are $z_{ij}^{m_i} = y_{ij}$. Now all the statements of the lemma follow from these equations: the formal completion $\hat{\mathcal{O}}_{B',b'}$ is obtained from $\hat{\mathcal{O}}_{B,b}$ by adjoining variables $\tilde{x}_i$, $\tilde{y}_{ij}$ modulo the relations

$$\tilde{x}_i^{m_i} = x_i$$

$$\tilde{y}_{ij}^{m_i} = y_{ij}$$

for all i, j. $\square$

PROPOSITION 5.9. *Let $f : (U_X \subset X) \to (U_B \subset B)$ be a toroidal morphism, and $f_\Delta : \Delta_X \to \Delta_B$ the associated morphism of polyhedral complexes. Assume that B is quasi-projective and nonsingular. For every primitive point u_i of Δ_B let $m_i > 0$ be an integer. Then there exists a finite morphism $p : B' \to B$ so that, if X' is the normalization of $X \times_B B'$ and $f' : X' \to B'$, $p' : X' \to X$ the two projections, we have*

$$\begin{array}{ccc} X' & \xrightarrow{p'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{p} & B \end{array}$$

(i) *$U_{B'} = p^{-1}(U_B) \subset B'$ and $U_{X'} = (p')^{-1}(U_X) \subset X'$ are toroidal embeddings with polyhedral complexes $\Delta_{X'}$ and $\Delta_{B'}$; moreover, f' is a toroidal morphism of these embeddings.*

(ii) *There exist morphisms*

$$p_\Delta \ : \ \Delta_{B'} \to \Delta_B$$

$$p'_\Delta \ : \ \Delta_{X'} \to \Delta_X,$$

and for all cones $\sigma' \in \Delta'_X$ and $\tau' \in \Delta'_B$ such that $f'_\Delta(\sigma) \subset \tau$, there exists a unique pair of cones $\sigma \in \Delta_X$ and $\tau \in \Delta_B$, with $f_\Delta(\sigma) \subset \tau$, satisfying

- *p_Δ maps τ' isomorphically to τ and induces an isomorphism between $N_{\tau'}$ and the sublattice of N_τ generated by $m_i u_i$ where u_i are the primitive points of τ;*

- $\circ$ *p'_Δ maps σ' isomorphically to σ and induces an isomorphism between $N_{\sigma'}$ and the sublattice $N_\sigma \cap f_\Delta^{-1} p_\Delta(N_{\tau'})$ of N_σ;*
- $\circ$ *The following diagrams commute:*

$$\begin{array}{ccc} \sigma' & \xrightarrow[p'_\Delta]{\cong} & \sigma \\ {\scriptstyle f'_\Delta}\downarrow & & \downarrow{\scriptstyle f_\Delta} \\ \tau' & \xrightarrow[p_\Delta]{\cong} & \tau \end{array} \qquad\qquad \begin{array}{ccc} N_{\sigma'} & \xrightarrow[p'_\Delta]{\subset} & N_\sigma \\ {\scriptstyle f'_\Delta}\downarrow & & \downarrow{\scriptstyle f_\Delta} \\ N_{\tau'} & \xrightarrow[p_\Delta]{\subset} & N_\tau \end{array}$$

Proof. The proof of the proposition consists of three steps:

Step 1. Let D_i be the irreducible toroidal divisors in B corresponding to the primitive points u_i. We choose hyperplane sections H_{ij} to get the covering package (D_i, m_i, H_{ij}). Proposition 3.2 allows us to add the divisors H_{ij} and $f^{-1}(H_{ij})$ to the toroidal structures of B and X. We call the new toroidal embeddings $\tilde{U}_B \subset B$ and $\tilde{U}_X \subset X$. The associated polyhedral complexes $\tilde{\Delta}_B$ and $\tilde{\Delta}_X$ are obtained from Δ_B and Δ_X by attaching new cones. In the new local models f is given by

$$f = f' \times id : X_{\sigma'} \times \mathbb{A}^l \to X_{\tau'} \times \mathbb{A}^l,$$

where the coordinate hyperplanes in $\mathbb{A}^l$ correspond to the added divisors.

Step 2. Let $p : (\tilde{U}_{B'} \subset B') \to (\tilde{U}_B \subset B)$ be the finite toroidal morphism defined by the covering package, and let X' be the normalization of the fiber product $B' \times_B X$, with $f' : X' \to B'$ and $p' : X' \to X$ the two projections. By Lemma 5.5 the embedding $\tilde{U}_{X'} = (p')^{-1}(\tilde{U}_X) \subset X'$ is toroidal and so are the morphisms f' and p'. We denote the polyhedral complexes of B' and X' by $\tilde{\Delta}_{B'}$ and $\tilde{\Delta}_{X'}$.

In local models (keeping the notation of Step 1) p is given by

$$p = p_1 \times p_2 : X'_{\tau'} \times \mathbb{A}^l \to X_{\tau'} \times \mathbb{A}^l.$$

Here $X'_{\tau'} \cong X_{\tau'}$, and the toric morphisms p_1 and p_2 are defined by raising the toric coordinates to powers m_i. Thus X' has local models of the form

$$(X'_{\tau'} \times \mathbb{A}^l) \times_{X_{\tau'} \times \mathbb{A}^l} (X_{\sigma'} \times \mathbb{A}^l) \cong (X'_{\tau'} \times_{X_{\tau'}} X_{\sigma'}) \times \mathbb{A}^l$$

in which the morphism $f' : X' \to B'$ in given by

$$f' = pr_1 \times id : (X'_{\tau'} \times_{X_{\tau'}} X_{\sigma'}) \times \mathbb{A}^l \to X'_{\tau'} \times \mathbb{A}^l,$$

where pr_1 is the projection.

Step 3. By Proposition 3.3 and the formula for f' in local models, we can remove the divisors $p^{-1}(H_{ij})$ and $(p \circ f')^{-1}(H_{ij})$ from the toroidal structures of B' and X' so that f' (but not p or p') remains toroidal. In terms of polyhedral complexes, we remove from $\tilde{\Delta}_{B'}$ and $\tilde{\Delta}_{X'}$ cones corresponding to strata contained in $p^{-1}(H_{ij})$ or $(p \circ f')^{-1}(H_{ij})$, and call the polyhedral complexes of the new embeddings $\Delta_{B'}$ and $\Delta_{X'}$. The statements (i) and (ii) now follow from Lemmas 5.5 and 5.8. $\square$

PROPOSITION 5.10. *Let $f : (U_X \subset X) \to (U_B \subset B)$ be an equidimensional toroidal morphism. Assume that B is projective and nonsingular. Then there exists a finite surjective morphism $p : B' \to B$ so that, if we denote by X' the normalization of $X \times_B B'$, the projection $f' : (U_{X'} \subset X') \to (U_{B'} \subset B')$ is an equidimensional toroidal morphism with reduced fibers. Here $U_{B'} = p^{-1}(U_B)$, $U_{X'} = (p')^{-1}(U_X)$ and $p' : X' \to X$ is the projection.*

If we assume that X has no horizontal divisors with respect to f, then X' has no horizontal divisors with respect to f'.

Proof. Let Δ_X, Δ_B be the polyhedral complexes of X, B, and $f_\Delta : \Delta_X \to \Delta_B$. By equidimensionality, f_Δ maps the 1-skeleton $\Delta_X^{(1)}$ to the 1-skeleton $\Delta_B^{(1)}$. If u_i are the primitive points of Δ_B, let v_{ij} be the primitive points of Δ_X such that $f_\Delta(v_{ij}) = m_{ij}u_i$ for some $m_{ij} > 0$. Set $m_i = \mathrm{lcm}_j\{m_{ij}\}$. Then for all i, j, some multiple of v_{ij} maps to $m_i u_i$. Now we use the covering data (D_i, m_i) to define the toroidal morphism $f' : X' \to B'$ as in the previous proposition. Let $\sigma' \in \Delta_{X'}$, $f'_\Delta(\sigma') = \tau' \in \Delta_{B'}$. There exist $\sigma \in \Delta_X$, $\tau \in \Delta_B$ such that $\sigma' \cong \sigma$, $\tau' \cong \tau$. The integral lattice $N_{\tau'}$ is generated by $m_i u_i$ with u_i lying on the edges of τ, and $N_{\sigma'} = N_\sigma \cap f_\Delta^{-1}(N_{\tau'})$. For any edge of σ' spanned by v_{ij}, some multiple $a_{ij}v_{ij}$ maps to $m_i u_i$. Hence $a_{ij}v_{ij} \in N_{\sigma'}$ is primitive and maps to the primitive point $m_i u_i$. This proves that f' has reduced fibers.

The last statement of the proposition is clear from the definition of $U_{X'}$. $\square$

This finishes the proof of Theorem 0.6. Before we describe some of the properties of weak semistable morphisms, we formalize and generalize the previous manipulations with polyhedral complexes, which then enables us to state the problem of semistable reduction in combinatorial terms.

6. The Cartier covering

In this section we generalize Kawamata's trick to projective toroidal embeddings $U_B \subset B$ with quotient singularities, i.e. to the case where Δ_B is simplicial but not necessarily nonsingular.

Let $U_B \subset B$ be a toroidal embedding with quotient singularities, i.e., the associated polyhedral complex Δ_B is simplicial. For each component D_i of $B \setminus U_B$ let $n_i > 0$ be an integer such that $n_i D_i$ is Cartier. We choose general hyperplane sections H_{ij} such that $n_i D_i + H_{ij} \in |n_i L_i|$ for some Cartier divisor L_i, $j = 1, \ldots, \dim B$. As before, let B_{ij} be the cyclic cover ramified along $n_i D_i + H_{ij}$ with index n_i, and let B'

be the normalization of the product

$$B_{1,1} \times_B \ldots \times_B B_{m,\dim\ B}$$

with $p : B' \to B$ the projection.

Recall that $\tilde{U}_B \subset B$ is the toroidal embedding after adding the divisors H_{ij} to the toroidal structure of $U_B \subset B$. The local models of the new toroidal embedding are products $X_{\tau'} \times \mathbb{A}^l$ where the divisors H_{ij} correspond to the hyperplanes in $\mathbb{A}^l$. It follows that $\bigcap_j H_{ij} \cap D_i = \emptyset$ as before.

Lemma 6.1. *Let $\tilde{U}_{B'} = p^{-1}(\tilde{U}_B)$. Then $\tilde{U}_{B'} \subset B'$ is a nonsingular toroidal embedding, and $p : (\tilde{U}_{B'} \subset B') \to (\tilde{U}_B \subset B)$ is a toroidal morphism.*

If X_τ is a local model of $\tilde{U}_B \subset B$ at $b \in B$ then a local model $X_{\tau'}$ of $\tilde{U}_{B'} \subset B'$ at any $b' \in p^{-1}(b)$ is constructed as follows. Let $v_1, \ldots, v_l$ be the primitive points of τ, and assume that for all $1 \leq \alpha \leq k$, the primitive point v_α corresponds to the divisor $H_{i_\alpha j}$ for some j. If we extend the primitive points to a $\mathbb{Q}$-basis $v_1, \ldots, v_{\dim B}$ of $N_\tau \otimes \mathbb{Q}$, then $\tau' = \tau$ and $N_{\tau'} \subset N_\tau$ is the sublattice of finite index with basis $n_{i_1}v_1, \ldots, n_{i_k}v_k, v_{k+1}, \ldots, v_{\dim B}$.

Proof. The proof is similar to the proof of Lemma 5.8 and we will only sketch it. Let x_i be local equations of n_iD_i, and let y_{ij} be local equations of H_{ij}. The cyclic cover B_{ij} is obtained from B by adjoining variables z_{ij} and relations

$$z_{ij}^{n_i} = x_i y_{ij}.$$

Since $\bigcap_j H_{ij} \cap D_i = \emptyset$ for all i, we may again consider the cases where either x_i is a unit, or y_{ij} is a unit for some j. In all cases we see that $\hat{\mathcal{O}}_{B',b'}$ is obtained from $\hat{\mathcal{O}}_{B,b}$

by adjoining variables $\tilde{x}_i, \tilde{y}_{ij}$ modulo the relations

$$\tilde{x}_i^{n_i} = x_i$$

$$\tilde{y}_{ij}^{n_i} = y_{ij}$$

If X_τ is a toric variety, and the cone τ is simplicial with primitive points u_i, then adjoining to the lattice M_τ the duals of u_i gives a new lattice M'_τ so that the primitive points u_i form a subset of a basis of the dual lattice $N'_\tau = (M'_\tau)^\vee$. $\square$

PROPOSITION 6.2. *Let $f : (U_X \subset X) \to (U_B \subset B)$ be a toroidal morphism, and $f_\Delta : \Delta_X \to \Delta_B$ the associated morphism of polyhedral complexes. Assume that B is quasi-projective Δ_B simplicial. Then there exists a finite morphism $p : B' \to B$ so that, if X' is the normalization of $X \times_B B'$ and $f' : X' \to B'$, $p' : X' \to X$ the two projections, we have*

$$\begin{array}{ccc} X' & \xrightarrow{p'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{p} & B \end{array}$$

(i) *$U_{B'} = p^{-1}(U_B) \subset B'$ and $U_{X'} = (p')^{-1}(U_X) \subset X'$ are toroidal embeddings with polyhedral complexes $\Delta_{X'}$ and $\Delta_{B'}$; moreover, f' is a toroidal morphism of these embeddings.*

(ii) *There exist morphisms*

$$p_\Delta \;:\; \Delta_{B'} \to \Delta_B$$

$$p'_\Delta \;:\; \Delta_{X'} \to \Delta_X,$$

and for all cones $\sigma' \in \Delta'_X$ and $\tau' \in \Delta'_B$ such that $f'_\Delta(\sigma) \subset \tau$, there exists a unique pair of cones $\sigma \in \Delta_X$ and $\tau \in \Delta_B$, with $f_\Delta(\sigma) \subset \tau$, satisfying

- *p_Δ maps τ' isomorphically to τ and induces an isomorphism between $N_{\tau'}$ and the sublattice of N_τ generated by the primitive points u_i of τ;*

- *p'_Δ maps σ' isomorphically to σ and induces an isomorphism between $N_{\sigma'}$ and the sublattice $N_\sigma \cap f_\Delta^{-1} p_\Delta(N_{\tau'})$ of N_σ;*
- *The following diagrams commute:*

$$\begin{array}{ccc} \sigma' & \xrightarrow[p'_\Delta]{\cong} & \sigma \\ {\scriptstyle f'_\Delta}\downarrow & & \downarrow{\scriptstyle f_\Delta} \\ \tau' & \xrightarrow[p_\Delta]{\cong} & \tau \end{array} \qquad\qquad \begin{array}{ccc} N_{\sigma'} & \xrightarrow[p'_\Delta]{\subset} & N_\sigma \\ {\scriptstyle f'_\Delta}\downarrow & & \downarrow{\scriptstyle f_\Delta} \\ N_{\tau'} & \xrightarrow[p_\Delta]{\subset} & N_\tau \end{array}$$

The proof of this proposition is the same as that of Proposition 5.9 with Lemma 6.1 substituted in place of Lemma 5.8. □

7. Combinatorial semistable reduction

We are going to restate the semistable reduction conjecture (Conjecture 0.5) in purely combinatorial terms. First let us define semistability combinatorially.

Definition 7.1. Let $f_\Delta : \Delta_X \to \Delta_B$ be a morphism of polyhedral complexes. We say that f_Δ is **weakly semistable** if the following conditions hold.

1. We have $f_\Delta^{-1}(0) = \{0\}$.
2. For any cone $\sigma \in \Delta_X$, we have $f_\Delta(\sigma) \in \Delta_B$.
3. For any cone $\sigma \in \Delta_X$, we have $f_\Delta(N_\sigma) = N_{f_\Delta(\sigma)}$.
4. Δ_B is nonsingular.

If also Δ_X is nonsingular, we say that f_Δ is **semistable**.

We now define the operations we are allowed to perform.

DEFINITION 7.2. Let $f_\Delta : \Delta_X \to \Delta_B$ be a morphism of polyhedral complexes. We refer to the following commutative diagram of polyhedral complexes and morphisms:

$$\begin{array}{ccc} \Delta_{X'} & \longrightarrow & \Delta_X \\ {\scriptstyle f'_\Delta}\downarrow & & \downarrow{\scriptstyle f_\Delta} \\ \Delta_{B'} & \longrightarrow & \Delta_B \end{array}$$

1. A **projective subdivision** f'_Δ of f_Δ is a commutative diagram as above, where $\Delta_{X'} \to \Delta_X$ is a projective subdivision of Δ_X and $\Delta_{B'} \to \Delta_B$ is a projective subdivision of Δ_B.
2. A **lattice alteration** f'_Δ of f_Δ is a commutative diagram as above, where $\Delta_{B'}$ is obtained from Δ_B by replacing the integral lattices N_τ with a consistent choice of sublattices of finite index $N'_\tau \subset N_\tau$, and $\Delta_{X'}$ is obtained from Δ_X by replacing the lattices N_σ with sublattices $N'_\sigma = N_\sigma \cap f_\Delta^{-1}(N'_\tau)$ where $f_\Delta(\sigma) \subset \tau$. The morphism f'_Δ is induced by f_Δ and the horizontal maps are the obvious ones.

Note that any sequence of projective subdivisions and lattice alterations can be decomposed as one subdivision followed by one lattice alteration.

CONJECTURE 7.3 (Combinatorial Semistable Reduction). Let $f_\Delta : \Delta_X \to \Delta_B$ be a morphism of polyhedral complexes, and assume that $f_\Delta^{-1}(0) = \{0\}$. Then there exists a projective subdivision f'_Δ of f_Δ and a lattice alteration f''_Δ of f'_Δ such that f''_Δ is semistable.

$$\begin{array}{ccccc} \Delta_{X''} & \to & \Delta_{X'} & \to & \Delta_X \\ \downarrow f''_\Delta & & \downarrow f'_\Delta & & \downarrow f_\Delta \\ \Delta_{B''} & \to & \Delta_{B'} & \to & \Delta_B \end{array}$$

The conjectures are tied together by the following proposition:

PROPOSITION 7.4. *Conjecture 7.3 implies Conjecture 0.5.*

Proof. Let $X \to B$ be a surjective morphism of projective varieties. By Theorem 2.1 we may assume that the morphism is toroidal, and by Proposition 3.1 we may also assume that X has no horizontal toroidal divisors. Let $f_\Delta : \Delta_X \to \Delta_B$ be the associated morphism of polyhedral complexes. By Conjecture 7.3 there exists a projective subdivision f'_Δ followed by a lattice alteration f''_Δ such that f''_Δ is semistable. We show that the two combinatorial operations have geometric analogues.

The projective subdivisions $\Delta_{X'}$ of Δ_X and $\Delta_{B'}$ of Δ_B give rise to birational projective morphisms $X' \to X$ and $B' \to B$ and the morphism f'_Δ represents the induced toroidal morphism $f' : X' \to B'$ (Lemma 1.13).

The lattice alteration f''_Δ of f'_Δ can be factored into two lattice alterations:

$$\begin{array}{ccccc} \Delta_{X''} & \to & \Delta_{X^o} & \to & \Delta_{X'} \\ \downarrow f''_\Delta & & \downarrow f^o_\Delta & & \downarrow f'_\Delta \\ \Delta_{B''} & \to & \Delta_{B^o} & \to & \Delta_{B'} \end{array}$$

where the lattices in Δ_{B^o} are sublattices $N^o_\tau \subset N'_\tau$ generated by the primitive points of τ (hence Δ_{B^o} is nonsingular), and the lattices in $\Delta_{B''}$ are sublattices $N''_\tau \subset N^o_\tau$ generated by multiples of primitive points $m_i u_i$. The first of these lattice alterations corresponds to the Cartier's cover (Proposition 6.2), and the second to the Kawamata's cover (Proposition 5.9) with ramification indices m_i. More precisely, there exists a finite morphism $p : B'' \to B'$ such that, letting X'' be the normalization of $B'' \times_{B'} X'$, the projection $f'' : X'' \to B''$ is a toroidal morphism with the associated morphism of polyhedral complexes $\bar{f}''_\Delta : \bar{\Delta}_{X''} \to \bar{\Delta}_{B''}$. Moreover, there is a natural commutative diagram

$$\begin{array}{ccc} \bar{\Delta}_{X''} & \xrightarrow{\pi_X} & \Delta_{X''} \\ {\scriptstyle \bar{f}''_\Delta}\downarrow & & \downarrow{\scriptstyle f''_\Delta} \\ \bar{\Delta}_{B''} & \xrightarrow{\pi_B} & \Delta_{B''} \end{array}$$

where π_X and π_B map cones and lattices isomorphically to cones and lattices:

$$\begin{aligned}\pi_X &: (\bar{\sigma}, N_{\bar{\sigma}}) \stackrel{\cong}{\to} (\sigma, N_\sigma)\\ \pi_B &: (\bar{\tau}, N_{\bar{\tau}}) \stackrel{\cong}{\to} (\tau, N_\tau)\end{aligned}$$

for all $\bar{\sigma} \in \bar{\Delta}_{X''}$ and $\bar{\tau} \in \bar{\Delta}_{B''}$. Since f''_Δ is semistable, so is $\bar{f}''_\Delta$, and also f'' by Lemmas 4.1 and 5.2. □

8. Properties of a weakly semistable morphism

In this section we show that weak semistability is preserved under certain base changes. Since the stronger condition of semistability is very far from being preserved under base changes, one may argue that weak semistability is a more natural condition than semistability. We also prove that a weakly semistable morphism satisfies a "universally rational Gorenstein" property:

Definition 8.1. A surjective morphism $X \to B$ is **mild** if for any dominant morphism $B_1 \to B$ where B_1 has at most rational Gorenstein singularities we have that $X \times_B B_1$ has at most rational Gorenstein singularities as well.

For a discussion of why mild morphisms are useful, see [ℵ].

Recall that a variety X is said to have Gorenstein singularities if it is Cohen-Macaulay and has an invertible dualizing sheaf ω_X. We say that a morphism $f : X \to B$ is Gorenstein if all its fibers are Gorenstein. By Proposition 9.6 in [**Har1**], if $f : X \to B$ is a flat morphism and B Gorenstein then X is Gorenstein if and only if f is Gorenstein. In this case we have $\omega_X \cong \omega_f \otimes f^*\omega_B$ where ω_f is the relative dualizing sheaf of f.

A normal variety X has rational singularities if for a resolution of singularities $r : X' \to X$ we have $r_*\mathcal{O}_{X'} \cong \mathcal{O}_X$ and $R^i r_* \mathcal{O}_{X'} = 0$ for all $i > 0$. If X is Gorenstein then these conditions are equivalent to $r_*\omega_{X'} = \omega_X$ ([**KKMS**], page 50).

Every toric variety is Cohen-Macaulay with rational singularities ([**KKMS**] Theorem 14, p. 52). The dualizing sheaf of an affine toric variety X_τ is the coherent sheaf $\omega_{X_\tau} = \mathcal{O}(X_\tau \setminus T_\tau)$. This sheaf is invertible if and only if there exists an element of M_τ (namely a linear function on τ taking integer values on N_τ) such that $\psi(v) = -1$ for all primitive points v of τ ([**F**], page 69).

LEMMA 8.2. *Let $f : X \to B$ be a weakly semistable morphism. Then X has rational Gorenstein singularities.*

Proof. Without loss of generality we may consider the morphism in local models $f : X_\sigma \to X_\tau$. Since X_τ is nonsingular, there exists an element $\psi \in M_\tau$ such that $\psi(u) = -1$ for all primitive points u of τ. Now $\psi \circ f_\Delta \in M_\sigma$ and $\psi \circ f_\Delta(v) = -1$ for all primitive points v of σ because f_Δ maps primitive points of σ to primitive points of τ. Hence X_σ has Gorenstein singularities. $\square$

Weak semistability of a toroidal morphism without horizontal divisors was defined by requiring that the fibers of the morphism be reduced and equidimensional. Since these properties are preserved under a base change, we see that the property of a morphism being weekly semistable is preserved under a base change, provided that we can give the new morphism a toroidal structure. The following lemma describes one situation when we can do this. The lemma is a special case of Kato's more general description of log-smooth morphisms (Theorem 3.5 in [**Kato**]).

LEMMA 8.3. *Let $f : X \to B$ be a weakly semistable morphism. Let $g : B' \to B$ be a morphism such that B' is nonsingular and $g^{-1}(B \setminus U_B)$ is a normal crossings* *divisor. Let* $X' = B' \times_B X$ *and* $g' : X' \to X$, $f' : X' \to B'$ *the two projections.*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{g} & B \end{array}$$

Then $(U_{B'} = g^{-1}(U_B) \subset B')$ *and* $(U_{X'} = (g')^{-1}(U_X) \subset X')$ *are toroidal embeddings, and* $f' : X' \to B'$ *is a weakly semistable toroidal morphism.*

Proof. Let f in local models be given by $f : X_\sigma \to X_\tau$. Clearly, $U_{B'} \subset B'$ is a nonsingular toroidal embedding, and we may choose a local model X_ρ for B'. The morphism of germs $g : X_\rho \to X_\tau$ need not be toric. We are going to construct a local model X_π for X' so that f' in the chosen local models is given by a toric morphism $f' : X_\pi \to X_\rho$.

$$\begin{array}{ccc} X_\pi & \xrightarrow{g'} & X_\sigma \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X_\rho & \xrightarrow{g} & X_\tau \end{array}$$

First assume that the closed orbit in X_τ consists of a point $\{x_\tau\}$. Then the primitive points of $\tau^\vee$ form a basis of M_τ. We let the images of these primitive points in the coordinate ring of X_τ be $x_1, \dots, x_m$. Since $f_\Delta^\vee$ embeds M_τ in M_σ as a saturated sublattice, we can extend the images of the primitive points of $\tau^\vee$ to a basis of M_σ, and we let the image of this basis in the coordinate ring of X_σ be $y_1, \dots, y_m, \dots, y_n$. Hence f is given by $x_i = y_i$ for $i = 1, \dots, m$. Also let w_l be a set of monomials that generate the coordinate ring of X_σ:

$$w_l = \prod_i y_i^{a_{li}}.$$

The morphism $g : X_\rho \to X_\tau$ is given by

$$x_i = u_i \prod_j z_j^{b_{ij}}$$

where z_j are the toric coordinates of X_ρ and u_i are units in $\hat{\mathcal{O}}_{X_\rho, x_\rho}$.

Assume further that $u_i = 1$ for all i. In this case g is a toric morphism. The normalization of $X_\rho \times_{X_\tau} X_\sigma$ is a union of toric varieties X_π by Lemma 5.4. But in fact $X_\rho \times_{X_\tau} X_\sigma$ is already normal: the group $M = M_\rho \times M_\sigma / g_\Delta^\vee \times (-f_\Delta^\vee)(M_\tau)$ has no torsion because $f_\Delta^\vee(M_\tau)$ is saturated in M_σ.

In case $u_i \neq 1$, the coordinate ring of $X_\rho \times_{X_\tau} X_\sigma$ (after suitably localized and completed) is

$$k[z_j, w_l, y_i]/(y_i - u_i \prod_j z_j^{b_{ij}}, w_l - \prod_i y_i^{b_{li}}).$$

If we now take new coordinates $\tilde{y}_i = u_i^{-1} y_i$ and $\tilde{w}_l = \prod_i \tilde{y}_i^{a_{li}}$, we get a toric model for X' as in the case $u_i = 1$.

In general, if $x_\tau \in X_\tau$ is not a closed orbit, write

$$f = f^o \times id : X_{\sigma'} \times \mathbb{G}_m^l \to X_{\tau'} \times \mathbb{G}_m^l$$

and consider the fiber squares

$$\begin{array}{ccccc}
X_\pi & \longrightarrow & X_{\sigma'} \times \mathbb{G}_m^l & \xrightarrow{\alpha} & X_{\sigma'} \\
{\scriptstyle f'}\Big\downarrow & & {\scriptstyle f}\Big\downarrow & & {\scriptstyle f^o}\Big\downarrow \\
X_\rho & \longrightarrow & X_{\tau'} \times \mathbb{G}_m^l & \xrightarrow{\beta} & X_{\tau'}
\end{array}$$

where α and β are projections. Clearly $X_\pi = X_\rho \times_{X_\tau} X_\sigma \cong X_\rho \times_{X_{\tau'}} X_{\sigma'}$, and since $\{x_{\tau'}\} \in X_{\tau'}$ is a closed orbit, we are reduced to the previous case. $\square$

Lemma 8.4. *(cf. [ℵ], Proposition 4.4) Consider the fiber square*

$$\begin{array}{ccc}
X' & \xrightarrow{h} & X \\
f' \downarrow & & \downarrow f \\
B' & \xrightarrow{r} & B.
\end{array}$$

where r is a modification (proper and birational), f is a flat Gorenstein morphism, and both B and B' have rational Gorenstein singularities (hence f', X and X' are Gorenstein as well). If X' has rational singularities, then X also has rational singularities.

Proof. By base change we have $h^*\omega_f = \omega_{f'}$. Since both B and B' have rational Gorenstein singularities, we have $\omega_B = r_*\omega_{B'}$. The flatness of f implies that $f^*r_*\omega_{B'} = h_*f'^*\omega_{B'}$. Therefore we have

$$\begin{aligned} h_*\omega_{X'} &= h_*(\omega_{f'} \otimes f'^*\omega_{B'}) \\ &= h_*(h^*\omega_f \otimes f'^*\omega_{B'}) \\ &= \omega_f \otimes h_*f'^*\omega_{B'} \\ &= \omega_f \otimes f^*r_*\omega_{B'} \\ &= \omega_f \otimes f^*\omega_B \\ &= \omega_X. \end{aligned}$$

Thus X has rational singularities. □

Proposition 8.5. *Let $f : X \to B$ be a weakly semistable morphism. Then f is mild.*

Proof. Let $B_1 \to B$ be a dominant morphism such that B_1 has rational Gorenstein singularities. We need to show that $X_1 = X \times_B B_1$ has rational Gorenstein singularities. By Lemma 8.2 f is a Gorenstein morphism, so X_1 has Gorenstein singularities.

Pick a resolution of singularities $B' \to B_1$ such that the inverse image of $B \setminus U_B$ in B' is a divisor with normal crossings. By lemma 8.3 we have that $X' = X \times_B B'$ is toroidal, therefore X' has rational singularities. By lemma 8.4 we have that X_1 has rational singularities as well, which is what we needed. □

9. Weak semistable reduction for families of pairs

So far we have considered the problem of making a family of varieties $f : X \to B$ as nice as possible. Now suppose that we are given a family of pairs $(X \to B, Z)$, where $X \to B$ is a morphism as before, and $Z \subset X$ is a divisor. We wish to perform weak semistable reduction so that the inverse image of Z becomes part of the toroidal divisor $X \setminus U_X$. In general, the inverse image of Z may contain horizontal components dominating B. So we need to extend the definition of a weakly semistable morphism to the case where X may have horizontal divisors.

The only time when we needed a semistable morphism $f : X \to B$ to have no horizontal divisors was when we proved in Lemma 8.2 that X has rational Gorenstein singularities. Now we define a kind of horizontal divisors so that everything we proved in Section 8 remains true in the presence of such divisors. In the following definitions we give both geometric and combinatorial versions.

Definition 9.1. (i) Let $f : X \to B$ be a toroidal morphism. We say that f has **good horizontal divisors** if at every point $x \in X$ we can find a local model of the form

$$X_\sigma = X_{\sigma'} \times \mathbb{A}^l$$

where the horizontal divisors through x are exactly the pullbacks of the coordinate hyperplanes in $\mathbb{A}^l$.

(ii) A morphism of polyhedral complexes $f_\Delta : \Delta_X \to \Delta_B$ has **good horizontal divisors** if for every $\sigma \in \Delta_X$ we can find splittings

$$\begin{aligned} N_\sigma &\cong N_{\sigma_1} \oplus N_{\sigma_2} \\ \sigma &\cong \sigma_1 \oplus \sigma_2 \end{aligned}$$

where $\sigma_1 = \sigma \cap Ker(f_\Delta)$ is a nonsingular cone with respect to the lattice N_{σ_1}.

The following lemma is obvious:

Lemma 9.2. *Let $f : X \to B$ be a toroidal morphism and $f_\Delta : \Delta_X \to \Delta_B$ the associated morphism of polyhedral complexes. Then f has good horizontal divisors if and only if f_Δ has good horizontal divisors.* $\square$

Notice that if X has good horizontal divisors, we can always remove them by taking new local models $X_{\sigma'} \times \mathbb{G}_m^l$. So, everything we proved about weakly semistable morphisms in Section 8 remains true if X is allowed to have good horizontal divisors. If X is nonsingular then all horizontal divisors are good.

Next we generalize the definition of (weak) semistability.

Definition 9.3.

(i) A toroidal morphism $f : X \to B$ with good horizontal divisors is called **weakly semistable** if
   1. the morphism f is equidimensional;
   2. all the fibers of f are reduced;
   3. B is nonsingular.

   If also X is nonsingular, we say that the morphism $f : X \to B$ is **semistable**.

(ii) Let $f_\Delta : \Delta_X \to \Delta_B$ be a morphism of polyhedral complexes with good horizontal divisors. We say that f_Δ is **weakly semistable** if
   1. For any cone $\tau \in \Delta_X$ we have $f_\Delta(\tau) \in \Delta_B$.
   2. For any cone $\tau \in \Delta_X$ we have $f_\Delta(N_\tau) = N_{f_\Delta(\tau)}$.
   3. Δ_B is nonsingular.

   If also Δ_X is nonsingular, we say that f_Δ is **semistable**.

The following lemma follows trivially from Lemma 9.2 and the results of the previous sections.

Lemma 9.4. *A toroidal morphism $f : X \to B$ is (weakly) semistable if and only if the morphism of polyhedral complexes $f_\Delta : \Delta_X \to \Delta_B$ is (weakly) semistable.* $\square$

Theorem 9.5 (Weak semistable reduction). (i) *Let $X \to B$ be a surjective morphism of projective varieties, and let $Z \subset X$ be a closed subset. There exist an alteration $B' \to B$ and a modification $X' \to X \times_B B'$, such that $X' \to B'$ is weakly semistable with good horizontal divisors and the inverse image of Z is part of the toroidal divisor $X' \setminus U_{X'}$.*

(ii) *Let $f_\Delta : \Delta_X \to \Delta_B$ be a morphism of polyhedral complexes with good horizontal divisors. Then there exists a projective subdivision f'_Δ of f_Δ and a lattice alteration f''_Δ of f'_Δ such that f''_Δ is weakly semistable with good horizontal divisors.*

Proof. First we show that (ii) implies (i). By Theorem 2.1, we may assume that X and B are nonsingular toroidal embeddings, the morphism f is toroidal and Z is part of the toroidal divisor. Let $f_\Delta : \Delta_X \to \Delta_B$ be the associated morphism of polyhedral complexes. Then f_Δ has good horizontal divisors by nonsingularity of X. Now we use part (ii) and the proof of Proposition 7.4, where it was shown that projective subdivisions and lattice alterations have geometric counterparts. Since the resulting morphism f''_Δ is weakly semistable with good horizontal divisors, so is the morphism $f'' : X'' \to B''$.

To prove part (ii), let Δ^o_X be the polyhedral complex that we get when we remove all horizontal divisors, i.e. replace (N_σ, σ) with (N_{σ_2}, σ_2) in the notation of Definition 9.1. Then Δ^o_X is a subcomplex of Δ_X. By Propositions 4.4 and 5.10 there exists a projective subdivision of $\Delta^o_X \to \Delta_B$ followed by a lattice alteration so that the resulting morphism is weakly semistable. It is easy to see that the two operations can be extended to the morphism $\Delta_X \to \Delta_B$. Indeed, we extend a projective subdivision of

Δ_X^o defined by a good function ψ by extending the function ψ as follows: set $\psi(v) = 0$ for v a primitive point corresponding to some horizontal divisor, and extend linearly. A lattice alteration of $\Delta_X^o \to \Delta_B$ can be extended to a lattice alteration of $\Delta_X \to \Delta_B$ by definition. Now it remains to note that after extending the two operations this way, the resulting morphism $\Delta_X'' \to \Delta_B''$ has good horizontal divisors and is weakly semistable. $\square$

Remark 9.6. Exactly the same way we can prove that semistable reduction for morphisms $X \to B$ implies semistable reduction for pairs $(X \to B, Z)$. In the next section we prove semistable reduction for morphisms of relative dimension at most 3, hence we also have semistable reduction for pairs in relative dimension at most 3.

CHAPTER 3

Semistable reduction for families of surfaces and three-folds

In this chapter we prove the semistable reduction conjecture for the case when the relative dimension of f is not bigger than three.

THEOREM 0.7. *Conjecture 7.3 of Chapter 2 is true if f_Δ has relative dimension ≤ 3. Hence, Conjecture 0.5 of Chapter 2 is true if f has relative dimension ≤ 3.*

To see that the second statement of the theorem follows from the first, consider a surjective morphism of affine toric varieties $f : X_\sigma \to X_\tau$ defined by a linear map of cones and lattices $f_\Delta : (\sigma, N_\sigma) \to (\tau, N_\tau)$. A general fiber of this morphism has dimension equal to the rank of the kernel of $f_\Delta : N_\sigma \to N_\tau$, and this is at least the relative dimension of $f_\Delta : \sigma \to \tau$. Therefore, if a toroidal morphism $f : X \to B$ has relative dimension $\leq d$, then the associated morphism of polyhedral complexes $f_\Delta : \Delta_X \to \Delta_B$ also has relative dimension $\leq d$.

The chapter is organized as follows. In Section 1 we use the construction of [**KKMS**] to make f_Δ semistable over the edges of Δ_B without increasing the multiplicity of Δ_X. In Section 2 we modify the barycentric subdivision of Δ_X so that we get a morphism to the barycentric subdivision of Δ_B. It is shown in Section 3 that in a certain situation we can choose a modified barycentric subdivision that decreases the multiplicity of Δ_X. The conditions when this happens are then verified for relative dimension ≤ 3 in Section 4. Section 5 gives an equivalent formulation of the combinatorial semistable reduction conjecture in terms of subdivisions only.

1. Notation and preliminaries

1.1. Notation. For a simplicial cone σ with primitive points $v_1, \dots, v_n$, the multiplicity of σ is

$$m(\sigma, N_\sigma) = [N_\sigma : \mathbb{Z}v_1 \oplus \ldots \oplus \mathbb{Z}v_n].$$

A polyhedral complex Δ is nonsingular if and only if $m(\sigma, N_\sigma) = 1$ for all $\sigma \in \Delta$. To compute the multiplicity of σ we can count the representatives $w \in N_\sigma$ of classes of $N_\sigma/\mathbb{Z}v_1 \oplus \ldots \oplus \mathbb{Z}v_n$ of the form

$$w = \sum_i \alpha_i v_i, \qquad 0 \leq \alpha_i < 1.$$

The set of all such points is denoted by $W(\sigma)$. For cones $\sigma_1, \sigma_2 \in \Delta$ we write $\sigma_1 \leq \sigma_2$ if σ_1 is a face of σ_2. Notice that if $\sigma_1 \leq \sigma_2$ then the multiplicity of σ_1 is at most the multiplicity of σ_2. Hence, to compute the multiplicity of a polyhedral complex Δ, it suffices to consider maximal cones only.

Let $f_\Delta : \Delta_X \to \Delta_B$ be a morphism of polyhedral complexes, and assume that Δ_B is simplicial. Let $u_1, \dots, u_m$ be the primitive points of Δ_B, and $M_1, \dots, M_m$ positive integers. By taking the $(M_1, \dots, M_m)$-sublattice at $u_1, \ldots, u_m$ we mean the lattice alteration $N'_\tau = \mathbb{Z}\{M_{i_1}u_{i_1}, \dots, M_{i_l}u_{i_l}\}$ for all cones $\tau \in \Delta_B$ with primitive points $u_{i_1}, \ldots, u_{i_l}$.

1.2. Applying the result of [KKMS]. Let $\sigma_1 \subset \mathbb{R}^{n_1}$ and $\sigma_2 \subset \mathbb{R}^{n_2}$ be two cones. We consider $\sigma_1 \times \sigma_2$ as a cone in $\mathbb{R}^{n_1+n_2}$. If $\{\sigma_{1,\alpha}\}_\alpha$ is a subdivision of σ_1, and $\{\sigma_{2,\beta}\}_\beta$ is a subdivision of σ_2, then $\{\sigma_{1,\alpha} \times \sigma_{2,\beta}\}_{\alpha,\beta}$ gives us a subdivision of $\sigma_1 \times \sigma_2$.

If Δ_X and Δ_B are simplicial, we say that $f_\Delta : \Delta_X \to \Delta_B$ is simplicial if $f_\Delta(\sigma) \in \Delta_B$ for all $\sigma \in \Delta_X$. Assume that $f_\Delta : \Delta_X \to \Delta_B$ is a simplicial map of simplicial complexes with $f_\Delta^{-1}(0) = \{0\}$. Let u_i, $i = 1, \dots, m$ be the primitive points of Δ_B, and let v_{ij}, $i = 1, \dots, m$ $j = 1, \dots, J_i$ be the primitive points of Δ_X such that v_{ij} is mapped to an integer multiple of u_i. For each $i = 1, \dots, m$ we denote by $\Delta_{X,i}$ the subcomplex of Δ_X lying over the cone $\langle u_i \rangle$ of Δ_B:

$$\Delta_{X,i} = f_\Delta^{-1}(\langle u_i \rangle).,$$

and for a cone $\sigma \in \Delta_X$, we let $\sigma_i = \sigma \cap f_\Delta^{-1}(\langle u_i \rangle)$. Then by the assumption that $f_\Delta^{-1}(0) \cap \sigma = \{0\}$, we have $\sigma \cong \sigma_1 \times \ldots \times \sigma_m$, although the lattice may not be a direct sum $N_\sigma \cap \operatorname{Span} \sigma_1 \otimes \ldots \otimes N_\sigma \cap \operatorname{Span} \sigma_m \subset N_\sigma$.

If $\Delta'_{X,i}$ are subdivisions of $\Delta_{X,i}$, we get a subdivision Δ'_X of Δ_X by taking the product subdivisions inside each cone $\sigma \in \Delta_X$.

Lemma 1.1. *If $\Delta'_{X,i}$ are projective subdivisions of $\Delta_{X,i}$ then Δ'_X is a projective subdivision of Δ_X.*

Proof. Let ψ_i be a convex piecewise linear function defining the subdivision $|\Delta'_{X,i}|$. Extend ψ_i linearly to the entire $|\Delta_X|$ by setting $\psi_i(|\Delta_{X,j}|) = 0$ for $j \neq i$. Then $\psi = \sum_i \psi_i$ is a convex piecewise linear function defining the subdivision Δ'_X. $\square$

Consider the restriction $f_\Delta|_{\Delta_{X,i}} : \Delta_{X,i} \to \langle u_i \rangle$. By the Main Theorem of Chapter 2 in [**KKMS**] there exist a subdivision $\Delta'_{X,i}$ of $\Delta_{X,i}$ and a positive integer M_i such that after taking the M_i-sublattice at u_i the induced morphism $f'_\Delta|_{\Delta'_{X,i}}$ is semistable. We let Δ'_X be the product of the subdivisions $\Delta'_{X,i}$, and we take the $(M_1, \dots, M_m)$-sublattice at $(u_1, \dots, u_n)$. Then $f'_\Delta : \Delta'_X \to \Delta'_B$ is a simplicial map and $f'_\Delta|_{\Delta'_{X,i}}$ is semistable for all i.

Lemma 1.2. *The multiplicity of Δ'_X is not greater than the multiplicity of Δ_X.*

Proof. Let $\sigma \in \Delta_X$ have primitive points v_{ij} and let $\sigma' \subset \sigma$ be a maximal cone in the subdivision with primitive points v'_{ij}. The multiplicity of σ' is the number of points in $W(\sigma')$. We show that $W(\sigma')$ can be mapped injectively to $W(\sigma)$, hence the multiplicity of σ' is not greater than the multiplicity of σ.

If $w' \in W(\sigma')$, we write

$$w' = \sum_{i,j} (\beta_{ij} + b_{ij}) v_{ij}, \qquad 0 \leq \beta_{ij} < 1, \qquad b_{ij} \in \mathbb{Z}_+.$$

Then $w = \sum_{ij} \beta_{ij} v_{ij} \in N_\sigma$ is in $W(\sigma)$. If two points $w'_1, w'_2 \in W(\sigma')$ give the same w, then their difference $w'_1 - w'_2$ is an integral linear combination of v_{ij}. But then $w'_1 - w'_2$ is also an integral linear combination of v'_{ij} because $\mathbb{Z}\{v'_{ij}\}_{i,j} = \mathbb{Z}\{v_{ij}\}_{i,j} \cap N_{\sigma'}$ (to see the inclusion $\mathbb{Z}\{v_{ij}\}_{i,j} \cap N_{\sigma'} \subset \mathbb{Z}\{v'_{ij}\}_{i,j}$ note that v'_{ij} form a basis of $N_\sigma \cap f^{-1}\langle u_i \rangle = \mathbb{Z}\{v'_{ij}\}_{i,j}$). Hence $w'_1 - w'_2 = 0$. □

2. Modified barycentric subdivisions

Let $f_\Delta : \Delta_X \to \Delta_B$ be a simplicial morphism. Consider the barycentric subdivision $BS(\Delta_B)$ of Δ_B. The 1-dimensional cones of $BS(\Delta_B)$ are of the form $\mathbb{R}_+ \hat{\tau}$ where $\hat{\tau} = \sum u_i$ is the barycenter of a cone $\tau \in \Delta_B$ with primitive points $u_1, \ldots, u_m$. A cone $\tau' \in BS(\Delta_B)$ is spanned by $\hat{\tau}_1, \ldots, \hat{\tau}_k$, where $\tau_1 \leq \tau_2 \leq \ldots \leq \tau_k$ is a chain of cones in Δ_B.

In general, f_Δ does not induce a morphism $BS(\Delta_X) \to BS(\Delta_B)$. For example, if $\sigma = \langle v_{11}, v_{12}, v_{21} \rangle$, $\tau = \langle u_1, u_2 \rangle$, and $f_\Delta : v_{ij} \mapsto u_i$, then f_Δ does not induce a morphism of barycentric subdivisions. To get a morphism we need to modify the barycenters $\hat{\sigma}$ of cones $\sigma \in \Delta_X$ so that they map to (multiples of) barycenters of Δ_B.

Definition 2.1. The data of **modified barycenters** consists of

1. A subset of cones $\tilde{\Delta}_X \subset \Delta_X$.
2. For each cone $\sigma \in \tilde{\Delta}_X$ a lattice point $b_\sigma \in \mathrm{int}(\sigma) \cap N_\sigma$ such that $f_\Delta(b_\sigma) \in \mathbb{R}_+ \hat{\tau}$ for some $\tau \in \Delta_B$.

Recall that for any total order $\prec$ on the set of cones in Δ_X refining the partial order $\leq$, the barycentric subdivision $BS(\Delta_X)$ can be realized as a sequence of star subdivisions at the barycenters $\hat{\sigma}$ for all cones $\sigma \in \Delta_X$ in the descending order $\prec$.

Definition 2.2. Given modified barycenters $(\tilde{\Delta}_X, \{b_\sigma\})$ and a total order $\prec$ on Δ_X refining the partial order $\leq$, we define the **modified barycentric subdivision** $MBS_{\tilde{\Delta}_X, \{b_\sigma\}, \prec}(\Delta_X)$ to be the sequence of star subdivisions at b_σ for all $\sigma \in \tilde{\Delta}_X$ in the descending order $\prec$.

Example 2.3. Let $f_\Delta : \langle v_{11}, v_{12}, v_{21} \rangle \to \langle u_1, u_2 \rangle$ be the morphism defined by $f_\Delta : v_{ij} \mapsto u_i$. Let $\tilde{\Delta}_X$ consist of the two cones $\tilde{\Delta}_X = \{\langle v_{11}, v_{21} \rangle, \langle v_{12}, v_{21} \rangle\}$, and let the modified barycenters be $\{b_\sigma\} = \{v_{11} + v_{21}, v_{12} + v_{21}\}$. Depending on whether $\langle v_{11}, v_{21} \rangle \prec \langle v_{12}, v_{21} \rangle$ or vice versa, we get two modified barycentric subdivisions as shown in Figure 1.

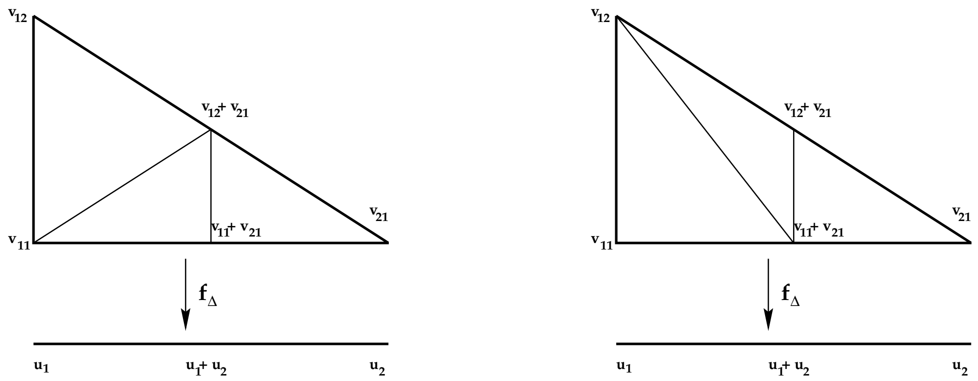

Figure 1. Two modified barycentric subdivisions from Example 2.3.

To simplify notation, we write $MBS_{\tilde{\Delta}_X}(\Delta_X)$ or simply $MBS(\Delta_X)$ instead of $MBS_{\tilde{\Delta}_X,\{b_\sigma\},\prec}(\Delta_X)$. By definition, $MBS(\Delta_X)$ is a projective simplicial subdivision of Δ_X. Next we show that, as in the case of the ordinary barycentric subdivision, the cones of $MBS(\Delta_X)$ can be characterized by chains of cones in Δ_X. We may assume that the zero- and one-dimensional cones of Δ_X are all in $\tilde{\Delta}_X$, and they precede all other cones in the order $\prec$. For a cone $\sigma \in \Delta_X$ let $\tilde{\sigma}$ be the maximal face of σ (w.r.t. $\prec$) in $\tilde{\Delta}_X$. Given a chain of cones $\sigma_1 \leq \ldots \leq \sigma_k$ in Δ_X, the cone $\langle b_{\tilde{\sigma}_1}, \ldots, b_{\tilde{\sigma}_k}\rangle$ is a subcone of σ_k. Let $C(\Delta_X)$ be the set of all such cones corresponding to chains $\sigma_1 \leq \ldots \leq \sigma_k$ in Δ_X.

PROPOSITION 2.4. *$C(\Delta_X) = MBS(\Delta_X)$.*

Proof. We do induction on the number of cones in $\tilde{\Delta}_X$ of dimension at least two. If $\tilde{\Delta}_X$ contains only zero- or one-dimensional cones then the statement is trivial. So, assume that $\tilde{\Delta}_X = \tilde{\Delta}_{X,0} \cup \{\sigma_0\}$, where $\sigma \prec \sigma_0$ for any $\sigma \in \tilde{\Delta}_{X,0}$, and assume that the proposition is proved for $\tilde{\Delta}_{X,0}$.

Without loss of generality we may assume that Δ_X consists of cones containing σ_0 and their faces only. We get $MBS_{\tilde{\Delta}_X}(\Delta_X)$ from Δ_X if we first subdivide at b_{σ_0} and then at b_σ for $\sigma \in \tilde{\Delta}_{X,0}$ in the descending order $\prec$. If $\Delta_{X,0}$ is the subcomplex of Δ_X consisting of cones *not* containing σ_0, then the star subdivision of Δ_X at b_{σ_0} is $\Delta_{X,0} \times \langle b_{\sigma_0}\rangle$. Since σ_0 is greater than any $\sigma \in \tilde{\Delta}_{X,0}$ with respect to $\prec$, all $b_\sigma \in \Delta_{X,0}$, and we see that

$$MBS_{\tilde{\Delta}_X}(\Delta_X) = MBS_{\tilde{\Delta}_{X,0}}(\Delta_{X,0}) \times \langle b_{\sigma_0}\rangle.$$

A cone in $MBS_{\tilde{\Delta}_{X,0}}(\Delta_{X,0}) \times \langle b_{\sigma_0}\rangle$ is of the form $\sigma \times \rho$, where ρ is a face of $\langle b_{\sigma_0}\rangle$, i.e. either $\{0\}$ or $\langle b_{\sigma_0}\rangle$ itself, and where σ is a cone in $MBS_{\tilde{\Delta}_{X,0}}(\Delta_{X,0})$. Applying induction hypothesis to $MBS_{\tilde{\Delta}_{X,0}}(\Delta_{X,0})$, we get that $\sigma = \langle b_{\tilde{\sigma}_1}, \ldots, b_{\tilde{\sigma}_l}\rangle$ for a chain of cones $\sigma_1 \leq \ldots \leq \sigma_l$ in $\Delta_{X,0}$. Now if $\rho = \{0\}$ then $\sigma \times \rho = \langle b_{\tilde{\sigma}_1}, \ldots, b_{\tilde{\sigma}_l}\rangle \in C(\Delta_X)$. If $\rho = \langle b_{\sigma_0}\rangle$, we let σ_{l+1} be a cone in Δ_X that contains both σ_l and σ_0. Then $\tilde{\sigma}_{l+1} = \sigma_0$, and $\sigma \times \rho = \langle b_{\tilde{\sigma}_1}, \ldots, b_{\tilde{\sigma}_l}, b_{\tilde{\sigma}_{l+1}}\rangle \in C(\Delta_X)$.

Conversely, let $\langle b_{\tilde{\sigma}_1}, \ldots, b_{\tilde{\sigma}_l}\rangle$ be a cone in $C(\Delta_X)$ for some chain $\sigma_1 \leq \ldots \leq \sigma_l$ in Δ_X. Then for some $k \leq l$ we have that $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_k \in \tilde{\Delta}_{X,0}$, and $\tilde{\sigma}_{k+1} = \ldots = \tilde{\sigma}_l = \sigma_0$. By induction hypothesis, the cone $\langle b_{\tilde{\sigma}_1}, \ldots, b_{\tilde{\sigma}_k}\rangle$ coming from the chain $\sigma_1 \leq \ldots \leq \sigma_k$ in $\Delta_{X,0}$ is in $MBS_{\tilde{\Delta}_{X,0}}(\Delta_{X,0})$. Hence the cone $\langle b_{\tilde{\sigma}_1}, \ldots, b_{\tilde{\sigma}_l}\rangle$ is of the form $\sigma \times \rho$, where $\sigma = \langle b_{\tilde{\sigma}_1}, \ldots, b_{\tilde{\sigma}_k}\rangle \in MBS_{\tilde{\Delta}_{X,0}}(\Delta_{X,0})$, and $\rho = \langle b_{\sigma_0}\rangle$ if $k < l$, and $\rho = \{0\}$ if $k = l$. $\square$

Corollary 2.5. *Assume that $f_\Delta : \Delta_X \to \Delta_B$ is a simplicial morphism. If $f_\Delta(\tilde{\sigma}) = f_\Delta(\sigma)$ for all $\sigma \in \Delta_X$ then f_Δ induces a simplicial morphism*

$$f'_\Delta : MBS(\Delta_X) \to BS(\Delta_B).$$

Proof. Let $\sigma' \in MBS(\Delta_X)$ correspond to a chain $\sigma_1 \leq \ldots \leq \sigma_k$ in Δ_X. Since f_Δ is simplicial, we have a chain of cones $f_\Delta(\sigma_1) \leq \ldots \leq f_\Delta(\sigma_k)$ in Δ_B. Recall that $b_{\tilde{\sigma}_i}$ is mapped to a multiple of a barycenter: $f_\Delta(b_{\tilde{\sigma}_i}) = \mathbb{R}_+\hat{\tau}$ for some $\tau \in \Delta_B$. The assumption that $f_\Delta(\tilde{\sigma}_i) = f_\Delta(\sigma_i)$ implies that $f_\Delta(b_{\tilde{\sigma}_i}) \in \mathbb{R}_+\widehat{f_\Delta(\sigma_i)}$, hence the cone $\langle b_{\tilde{\sigma}_1}, \ldots, b_{\tilde{\sigma}_k}\rangle$ is mapped onto the cone $\langle \widehat{f_\Delta(\sigma_1)}, \ldots, \widehat{f_\Delta(\sigma_k)}\rangle \in BS(\Delta_B)$. $\square$

Example 2.6. Assume that $f_\Delta : \Delta_X \to \Delta_B$ is a simplicial morphism taking primitive points of Δ_X to primitive points of Δ_B. Then for a cone $\sigma \in \Delta_X$ such that $f_\Delta : \sigma \overset{\simeq}{\to} \tau$ for some $\tau \in \Delta_B$, we have $f_\Delta(\hat{\sigma}) = \hat{\tau}$.

Let $\overline{\Delta}_X = \{\sigma \in \Delta_X : f_\Delta|_\sigma \text{ is injective}\}$, $b_\sigma = \hat{\sigma}$. Clearly, the hypothesis of the lemma is satisfied, and we have a simplicial morphism $f'_\Delta : MBS_{\overline{\Delta}_X}(\Delta_X) \to BS(\Delta_B)$.

Conversely, if $(\tilde{\Delta}_X, \{b_\sigma\})$ is the data of modified barycenters such that f_Δ induces a morphism $f'_\Delta : MBS_{\tilde{\Delta}_X}(\Delta_X) \to BS(\Delta_B)$, then $\overline{\Delta}_X \subset \tilde{\Delta}_X$. Thus, we may always assume that $\overline{\Delta}_X \subset \tilde{\Delta}_X$.

3. Reducing the multiplicity of Δ_X

PROPOSITION 3.1. *Let $f_\Delta : \Delta_X \to \Delta_B$ be a simplicial morphism taking primitive points to primitive points. Assume that Δ_B is nonsingular, Δ_X is singular, and every singular cone $\sigma \in \Delta_X$ contains a point $w \in W(\sigma) \setminus \{0\}$ mapping to a barycenter in Δ_B. Then there exists a modified barycentric subdivision $MBS(\Delta_X)$ of Δ_X having multiplicity strictly less than the multiplicity of Δ_X such that f_Δ induces a simplicial morphism $f'_\Delta : MBS(\Delta_X) \to BS(\Delta_B)$.*

Proof. For every singular cone $\sigma \in \Delta_X$ we choose a point w_σ as follows. By assumption, there exists a point $w \in W(\sigma) \setminus \{0\}$ mapping to a barycenter of Δ_B: $f_\Delta(w) = \hat{\tau}$. Then for a unique cone $\tau_0 \in \Delta_B$ we have $f_\Delta(\sigma) = \tau \times \tau_0$. We choose a face $\sigma_0 \leq \sigma$ such that $f_\Delta : \sigma_0 \stackrel{\simeq}{\to} \tau_0$. Set $w_\sigma = w + \hat{\sigma}_0$; then

$$f_\Delta(w_\sigma) = f_\Delta(w) + f_\Delta(\hat{\sigma}_0) = \hat{\tau} + \hat{\tau}_0 = \widehat{f_\Delta(\sigma)}.$$

Having chosen the set $\{w_\sigma\}$, we may remove or rename some of the points w_σ if necessary so that every simplex $\rho \in \Delta_X$ contains at most one w_σ in its interior, and the face of ρ containing w_ρ in its interior is maximal with respect to $\leq$ among all faces of ρ containing some w_σ in their interior. With $\overline{\Delta}_X$ as in Example 2.6, let $\tilde{\Delta}_X = \overline{\Delta}_X \cup \{\rho \in \Delta_X | w_\sigma \in \text{int}(\rho) \text{ for some singular } \sigma\}$, $b_\rho = \hat{\rho}$ if $\rho \in \overline{\Delta}_X$, and $b_\rho = w_\sigma$ if $w_\sigma \in \text{int}(\rho)$.

Next we specify the order $\prec$. We refine the partial order $\leq$ as follows: for two faces σ_1 and σ_2 of a cone $\sigma \in \Delta_X$ we set $\sigma_1 \prec_0 \sigma_2$ if $\dim f_\Delta(\sigma_1) < \dim f_\Delta(\sigma_2)$. Since

$\overline{\Delta}_X \subset \tilde{\Delta}_X$, this ensures that the condition $f_\Delta(\tilde{\sigma}) = f_\Delta(\sigma)$ of Corollary 2.5 is satisfied for any refinement of $\prec_0$. Now if σ is singular, then the point w_σ constructed above lies in the interior of a face ρ_σ such that $f_\Delta(\rho_\sigma) = f_\Delta(\sigma)$. We further refine the order $\prec_0$ by setting $\sigma_1 \prec_0 \rho_\sigma$ for any face σ_1 of the singular cone σ which does not contain ρ_σ. Then $\tilde{\sigma} = \rho_\sigma$ whenever σ is singular. Finally we let $\prec$ be any refinement of $\prec_0$ to a total order.

Let $\sigma \in \Delta_X$ be a cone, and let a maximal cone $\sigma' \in MBS(\Delta_X)$ be given by a maximal chain of faces of σ: $\sigma_1 \leq \ldots \leq \sigma_n$. We have to show that $m(\sigma', N_\sigma) \leq m(\sigma, N_\sigma)$, and if σ is singular then the inequality is strict.

We can order the primitive points $v_1, \ldots, v_n$ of σ so that $\sigma_1 = \langle v_1 \rangle$, $\sigma_2 = \langle v_1, v_2 \rangle$, $\ldots$, $\sigma_n = \langle v_1, \ldots, v_n \rangle$. Since $b_{\tilde{\sigma}_i} \in \sigma_i$, the primitive points of $\sigma' = \langle b_{\tilde{\sigma}_1}, \ldots, b_{\tilde{\sigma}_n} \rangle$ can be written as

$$
\begin{array}{ccccccccc}
v'_1 & = & \frac{1}{\mu_1} b_{\tilde{\sigma}_1} & = & a_{11} v_1 & & & & \\
v'_2 & = & \frac{1}{\mu_2} b_{\tilde{\sigma}_2} & = & a_{21} v_1 & + & a_{22} v_2 & & \\
 & \cdots & & & & & & & \\
v'_n & = & \frac{1}{\mu_n} b_{\tilde{\sigma}_n} & = & a_{n1} v_1 & + & \ldots & + & a_{nn} v_n
\end{array}
$$

for some $a_{ij} \geq 0$ and integers $\mu_i \geq 1$. The multiplicity of σ' is $a_{11} \cdot a_{22} \cdots a_{nn}$ times the multiplicity of σ. By the choice of b_ρ above, all $a_{ii} \leq 1$, hence $m(\sigma', N_{\sigma'}) \leq m(\sigma, N_\sigma)$. If σ is singular, let i be the smallest index such that the face σ_i is singular. Then, with notation as above, $b_{\tilde{\sigma}_i} = w + \hat{\sigma}_0$ for some $w \in W(\sigma_i) \setminus \{0\}$, and $\sigma_0 \leq \sigma_i$. Now if $a_{ii} = 1$, then $w \in \langle v_1, \ldots, v_{i-1} \rangle$, and this gives a contradiction with the choice of i. Hence $a_{ii} < 1$ and $m(\sigma', N_{\sigma'}) < m(\sigma, N_\sigma)$. $\square$

4. Families of surfaces and three-folds

Proof of Theorem 0.7. Let $f_\Delta : \Delta_X \to \Delta_B$ be a surjective morphism of polyhedral complexes of relative dimension ≤ 3 such that $f_\Delta^{-1}(0) = \{0\}$. By Proposition 4.4 and Remark 4.5 in Chapter 2, we may assume that Δ_X is simplicial, Δ_B is nonsingular, and $f_\Delta : \Delta_X \to \Delta_B$ is a simplicial morphism.

Applying the construction of [**KKMS**] over the edges of Δ_B (Subsection 1.2), we can make $f_\Delta|_{\Delta_{X,i}}$ semistable without increasing the multiplicity of Δ_X. We show below that every singular simplex $\sigma \in \Delta_X$ contains a point $w \in W(\sigma) \setminus \{0\}$ mapping to a barycenter of Δ_B. By Proposition 3.1, there exists a modified barycentric subdivision such that f_Δ induces a simplicial morphism $f'_\Delta : MBS(\Delta_X) \to BS(\Delta_B)$, with multiplicity of $MBS(\Delta_X)$ strictly less than the multiplicity of Δ_X. Since f'_Δ is simplicial and $BS(\Delta_B)$ nonsingular, the proof is completed by induction on the multiplicity of Δ_X.

Consider the restriction of f_Δ to a singular simplex $f_\Delta : \sigma \to \tau$, where τ has primitive points $u_1, \ldots, u_m$, σ has primitive points $v_{ij}, i = 1, \ldots, m, j = 1, \ldots, J_i$, and $f_\Delta(v_{ij}) = u_i$. Since σ is singular, it contains a point $w \in W(\sigma) \setminus \{0\}$:

$$w = \sum_{i,j} \alpha_{ij} v_{ij}, \qquad 0 \leq \alpha_{ij} < 1, \qquad \sum \alpha_{ij} > 0.$$

Considering a face of σ if necessary, we may assume that w lies in the interior of σ, hence $0 < \alpha_{ij}$. Since $f_\Delta(w) \in N_\tau$, it follows that $\sum_j \alpha_{ij} \in \mathbb{Z}$ for all i. In particular, if $J_{i_0} = 1$ for some i_0 then $\alpha_{i_0 1} = 0$, and w lies in a face of σ. So we may assume that $J_i > 1$ for all i. Since the relative dimension of f_Δ is $\sum_i (J_i - 1)$, we have to consider all possible decompositions $\sum_i (J_i - 1) \leq 3$, where $J_i > 1$ for all i.

The cases when the relative dimension of f_Δ is 0 or 1 are trivial and left to the reader.

If the relative dimension of f_Δ is 2, then either $J_1 = 3$, or $J_1 = J_2 = 2$. In the first case we have that $\langle v_{11}, v_{12}, v_{13}\rangle$ is singular, contradicting the semistability of $f_\Delta|_{\Delta_{X,1}}$. In the second case, $\alpha_{11} + \alpha_{12}, \alpha_{21} + \alpha_{22} \in \mathbb{Z}$ and $0 < \alpha_{ij} < 1$ imply that $\alpha_{11} + \alpha_{12} = \alpha_{21} + \alpha_{22} = 1$. Hence $f_\Delta(w) = u_1 + u_2$ is a barycenter.

In relative dimension 3, either $J_1 = 4$, or $J_1 = 3, J_2 = 2$, or $J_1 = J_2 = J_3 = 2$. In the first case we get a contradiction with the semistability of $f_\Delta|_{\Delta_{X,1}}$; the third case gives $\alpha_{11} + \alpha_{12} = \alpha_{21} + \alpha_{22} = \alpha_{31} + \alpha_{32} = 1$ as for relative dimension 2. In the second case, either $\alpha_{11} + \alpha_{12} + \alpha_{13} = \alpha_{21} + \alpha_{22} = 1$ and w maps to a barycenter, or $\alpha_{11} + \alpha_{12} + \alpha_{13} = 2, \alpha_{21} + \alpha_{22} = 1$ and $(\sum v_{ij}) - w$ maps to a barycenter. $\square$

Example 4.1. We show by an example that the previous construction of modified barycentric subdivisions does not work in relative dimension ≥ 4. Let $\tau = \langle u_1, u_2\rangle$ and $\sigma = \langle v_{11}, v_{12}, v_{13}, v_{14}, v_{21}, v_{22}\rangle$, with lattices $N_\tau = \mathbb{Z}\{u_1, u_2\}$ and $N_\sigma = \mathbb{Z}\{v_{11}, \dots, v_{22}, \frac{1}{2}(v_{11} + \dots + v_{22})\}$. Then $W(\sigma) \setminus \{0\}$ consists of a single point $w = \frac{1}{2}(v_{11} + \dots + v_{22})$, and if f_Δ maps v_{ij} to u_i then w is mapped to $2u_1 + u_2$, which is not a barycenter.

5. Another formulations of the semistable reduction conjecture

Let $f_\Delta : (N_\sigma, \sigma) \to (N_\tau, \tau)$ be a simplicial morphism of lattices and simplicial cones (i.e., taking faces onto faces), let $v_1, \dots, v_n$ be the primitive points of σ and $u_1, \dots, u_m$ the primitive points of τ. Let $\sigma_i = f_\Delta^{-1}(\mathbb{R}u_i)$, and assume that $f_\Delta^{-1}(0) \cap \sigma = \{0\}$, so that $\sigma = \sigma_1 \times \dots \times \sigma_m$.

Denote $N_\tau^0 = f_\Delta(\mathbb{Z}v_1 \oplus \ldots \oplus \mathbb{Z}v_n)$, $N_\sigma^0 = f_\Delta^{-1}(N_\tau^0) \cap N_\sigma$, and consider the tower of lattices:

$$\mathbb{Z}v_1 \oplus \ldots \oplus \mathbb{Z}v_n \subset N_\sigma^0 \cap \text{Span } \sigma_1 \oplus \ldots \oplus N_\sigma^0 \cap \text{Span } \sigma_m \subset N_\sigma^0 \subset N_\sigma.$$

The multiplicity of σ is by definition the index between the first and the fourth lattices:

$$m(\sigma, N_\sigma) = [N_\sigma : \mathbb{Z}v_1 \oplus \ldots \oplus \mathbb{Z}v_n].$$

We define the **horizontal multiplicity** of σ to be the index between the second and the third lattices:

$$m_{hor}(\sigma, N_\sigma, f_\Delta) = [N_\sigma^0 : N_\sigma^0 \cap \text{Span } \sigma_1 \oplus \ldots \oplus N_\sigma^0 \cap \text{Span } \sigma_m].$$

For simplicity, we omit some of the arguments of m_{hor} if no confusion can occur. We also say that a morphism $f_\Delta : \Delta_X \to \Delta_B$ has horizontal multiplicity equal to the maximum of $m_{hor}(\sigma, N_\sigma, f_\Delta)$ over all cones $\sigma \in \Delta_X$.

Let $v_{i_1}, \ldots, v_{i_{n_i}}$ be the primitive points of σ_i (with respect to either N_σ or N_σ^0). Then

$$[N_\sigma^0 \cap \text{Span } \sigma_i : \mathbb{Z}v_{i_1} \oplus \ldots \oplus \mathbb{Z}v_{i_{n_i}}] = m(\sigma_i, N_\sigma^0),$$

and we can write

$$m_{hor}(\sigma, N_\sigma, f_\Delta) = \frac{m(\sigma, N_\sigma)}{[N_\sigma : N_\sigma^0] \cdot \prod_i m(\sigma_i, N_\sigma^0)} = \frac{m(\sigma, N_\sigma^0)}{\prod_i m(\sigma_i, N_\sigma^0)}.$$

In particular, if $N_\sigma = N_\sigma^0$ then the horizontal multiplicity is obtained by dividing the usual multiplicity by the "vertical multiplicity" $\prod_i m(\sigma_i, N_\sigma)$.

LEMMA 5.1. *Let $f_\Delta : (N_\sigma, \sigma) \to (N_\tau, \tau)$ be a simplicial morphism, and consider a lattice alteration given by $N'_\tau \subset N_\tau$ a sublattice of finite index and $N'_\sigma = f_\Delta^{-1}(N_\tau) \cap N_\sigma$. Then*

$$m_{hor}(\sigma, N_\sigma) = m_{hor}(\sigma, N'_\sigma).$$

Proof. Let $u_1, \ldots, u_m$ be the primitive points of τ with respect to N_τ^0. First note that $m(\tau, N_\tau^0) = 1$ because N_τ^0 is generated by elements $f_\Delta(v_i) \in \mathbb{Z}u_i$, hence $N_\tau^0 = \mathbb{Z}u_1 \oplus \ldots \oplus \mathbb{Z}u_m$. Similarly, $m(\tau, {N'_\tau}^0) = 1$ and since ${N'_\tau}^0 \subset N_\tau^0$, we see that ${N'_\tau}^0$ must be of the form ${N'_\tau}^0 = \mathbb{Z}M_1u_1 \oplus \ldots \oplus \mathbb{Z}M_mu_m$ for some integers $M_1, \ldots, M_m > 0$.

Consider the diagram

$$\begin{array}{ccccccc} N_\sigma^0 \cap \text{Span } \sigma_1 \oplus \ldots \oplus N_\sigma^0 \cap \text{Span } \sigma_m & \subset & N_\sigma^0 & \xrightarrow{f} & N_\tau^0 & = & \mathbb{Z}u_1 \oplus \ldots \oplus \mathbb{Z}u_m \\ \cup & & \cup & & \cup & & \cup \\ {N'_\sigma}^0 \cap \text{Span } \sigma_1 \oplus \ldots \oplus {N'_\sigma}^0 \cap \text{Span } \sigma_m & \subset & {N'_\sigma}^0 & \xrightarrow{f} & {N'_\tau}^0 & = & \mathbb{Z}M_1u_1 \oplus \ldots \oplus \mathbb{Z}M_mu_m \end{array}$$

The indices of the horizontal inclusion in the left square are $m_{hor}(\sigma, N_\sigma)$ and $m_{hor}(\sigma, N'_\sigma)$ respectively. To show that they are equal is equivalent to showing that the indices of the vertical inclusions are the same. Since the middle square is a fiber square with both horizontal maps surjective, we have

$$[N_\sigma^0 : {N'_\sigma}^0] = [N_\tau^0 : {N'_\tau}^0],$$

and we may split the diagram into m fiber squares

$$\begin{array}{ccc} N_\sigma^0 \cap \text{Span } \sigma_i & \xrightarrow{f} & \mathbb{Z}u_i \\ \cup & & \cup \\ {N'_\sigma}^0 \cap \text{Span } \sigma_i & \xrightarrow{f} & \mathbb{Z}M_iu_i \end{array}$$

where the horizontal maps are surjective, hence the vertical inclusions have the same index M_i. $\square$

LEMMA 5.2. *Let $f_\Delta : (N_\sigma, \sigma) \to (N_\tau, \tau)$ be a simplicial morphism, and assume that $N_\sigma = N_\sigma^0$ and $m_{hor}(\sigma, N_\sigma) = 1$. If Δ_i is a nonsingular subdivision of σ_i for $i = 1, \ldots, m$, then the product $\Delta_1 \times \ldots \times \Delta_m$ is a nonsingular subdivision of σ.*

Proof. By assumption, $m(\sigma, N_\sigma) = \prod_i m(\sigma_i, N_\sigma)$. If σ' is a maximal cone in the subdivision then σ' has the same horizontal multiplicity as σ, and since σ'_i have multiplicity 1, we get that σ' also has multiplicity 1. $\square$

Proposition 5.3. *Let $f_\Delta : \Delta_X \to \Delta_B$ be a morphism of polyhedral complexes. Then the following are equivalent:*

(i) *There exists a subdivision $f'_\Delta : \Delta'_X \to \Delta'_B$ of $f_\Delta : \Delta_X \to \Delta_B$ followed by a lattice alteration $f''_\Delta : \Delta''_X \to \Delta''_B$ such that f''_Δ is semistable.*

(ii) *There exists a subdivision $f'_\Delta : \Delta'_X \to \Delta'_B$ of $f_\Delta : \Delta_X \to \Delta_B$ such that f'_Δ is simplicial and has horizontal multiplicity* 1.

Proof.

$(i) \Rightarrow (ii)$. Consider $f'_\Delta : (\sigma, N_\sigma) \to (\tau, N_\tau)$ in $f'_\Delta : \Delta'_X \to \Delta'_B$ such that after taking a suitable sublattice $N'_\tau \subset N_\tau$, the cone σ becomes nonsingular with respect to the lattice $N'_\sigma = N_\sigma \cap {f'_\Delta}^{-1}(N'_\tau)$. Since $m_{hor}(\sigma, N'_\sigma) \leq m(\sigma, N'_\sigma) = 1$, by Lemma 5.1 we have $m_{hor}(\sigma, N_\sigma) = 1$.

$(ii) \Rightarrow (i)$. Consider $f'_\Delta : \Delta'_X \to \Delta'_B$ simplicial with horizontal multiplicity 1. We use the construction of [**KKMS**] to find a lattice alteration $f''_\Delta : \Delta''_X \to \Delta''_B$ followed by a subdivision $f'''_\Delta : \Delta'''_X \to \Delta'''_B$ so that f'''_Δ is semistable over the 1-dimensional cones of Δ_B. By Lemma 5.1, f''_Δ has horizontal multiplicity 1, and by Lemma 5.2, Δ'''_X is nonsingular. Since f'''_Δ maps primitive points to primitive points, it is semistable. $\square$

CHAPTER 4

Minimal models and boundedness of stable varieties

It is known that there exists a quasi-projective coarse moduli space for nonsingular n-dimensional varieties X with ample canonical divisor K_X and a given Hilbert polynomial $H(l) = h^0(X, \mathcal{O}(lK_X))$ for large l (e.g. see [**Vie**]). The starting point in the construction of this moduli space is Matsusaka's big theorem [**Mat1**] which states that the class of these varieties is bounded: there exists an integer $\nu_0 > 0$, independent of X, such that $\nu_0 K_X$ is very ample and has no higher cohomology. This condition is equivalent to the existence of a family $\mathcal{X} \to \mathcal{B}$ over a scheme of finite type whose fibers contain all such varieties.

In order to compactify the moduli space of smooth n-folds, one has to add points at the boundary, corresponding to limits of smooth varieties. The natural way to construct these limits is to take a one-parameter semistable family of n-folds, and find the relative canonical model of this family in the sense of the minimal model program in dimension $n+1$ ($MMP(n+1)$). All such limits have semi-log-canonical singularities (see the discussion after Definition 1.8).

We consider the class of stable (almost) smoothable n-folds X with semi-log-canonical singularities and ample canonical divisor K_X. Smoothability here means that X admits a deformation to a variety with rational Gorenstein singularities. The moduli functor $\mathcal{M}_H^{sm}$ assigns to a scheme S the set of isomorphism classes of families of stable smoothable n-folds over S with a given Hilbert polynomial H. The main result of this paper is that the minimal model program in dimension $n+1$ implies boundedness of the functor $\mathcal{M}_H^{sm}$.

THEOREM 0.4. *Assuming MMP(n+1), there exists a family* $\tilde{X} \to B$ *over a projective scheme* B *in* $\mathcal{M}_H^{sm}(B)$ *whose geometric fibers include all stable smoothable n-folds with Hilbert polynomial* H.

If, in addition to boundedness, $\mathcal{M}_H^{sm}$ is locally closed, separated, complete, with tame automorphisms, and if the canonical polarization is semi-positive, then Kollár's theorem [**Kol2**] states that there exists a projective moduli space M_H^{sm} coarsely representing the functor $\mathcal{M}_H^{sm}$. Semi-positivity of the canonical polarization has been proved by Kollár (Theorem 4.12, [**Kol2**]). Since K_X is ample, X has a finite automorphism group by a result of Iitaka (Section 11.7, [**Iit**]). Separatedness of $\mathcal{M}_H^{sm}$ follows from the uniqueness of log-canonical models (Lemma 1.9). In Lemma 1.6 we prove that semi-log-canonical varieties deform to semi-log-canonical varieties; since the other conditions we require from stable smoothable n-folds are either open or closed, the functor $\mathcal{M}_H^{sm}$ is locally closed. Completeness then follows because M_H^{sm} is the image of the projective scheme B. This proves the existence of coarse moduli spaces:

COROLLARY 0.5. *Assuming MMP(n+1), the moduli functor* $\mathcal{M}_H^{sm}$ *is coarsely represented by a projective scheme* M_H^{sm}.

We remark that we do not need to assume the minimal model program in its full generality; only the semistable minimal model program is required (see Assumption 1.2). Nevertheless, the minimal model program MMP(n+1) is known only in dimensions $n+1 \leq 3$. Boundedness for semi-log-canonical surfaces has been proved by Alexeev [**Ale1**]. With this proof he finished the construction of projective coarse moduli spaces for semi-log-canonical surfaces that was started in [**KSB**]. The proof of Theorem 0.4 simplifies Alexeev's result of boundedness for semi-log-canonical surfaces, and generalizes it to n-folds, subject to the minimal model program assumption.

In [**Ale2**] Alexeev considered the problem of constructing projective coarse moduli spaces for stable pairs (X, B) consisting of an n-fold X and a divisor B in X. He showed that the log-minimal model program in dimension $n+1$ together with the boundedness assumption imply the existence of coarse moduli spaces for stable pairs. At the end of the paper we indicate what are the difficulties in extending the proof of Theorem 0.4 directly to the case of stable pairs.

0.1. Sketch of the proof. We imitate the construction of a stable n-fold X as a limit of a one-parameter family of smooth varieties, and try to find all these limits at the same time. By Matsusaka's theorem [**Mat2**], the class of normal varieties with rational Gorenstein singularities, ample canonical bundle and a given Hilbert polynomial is bounded. We start with a compactification of the Hilbert scheme parameterizing these varieties, and the universal family over it. To this morphism we apply weak semistable reduction and then take the relative canonical model of the weakly semistable family. The crucial step in showing that this canonical model exists is a theorem of Siu and Kawamata on the invariance of plurigenera in families of canonical varieties [**Kaw3**]. If $f : X \to B$ is a weakly semistable morphism, then for a general nonsingular curve $C \subset B$ the restriction $X_C = f^{-1}(C)$ has canonical singularities, so we can find its relative canonical model by the $MMP(n+1)$ assumption. Now invariance of plurigenera implies that sections of $f_*\mathcal{O}_{X_C}(K_{X_C})$ can be lifted to sections of $f_*\mathcal{O}_X(K_X)$, and it follows from this that the relative canonical ring of $f : X \to B$ is a finitely generated $\mathcal{O}_B$-algebra.

Most of the results we prove about semi-log-canonical varieties and the moduli functor $\mathcal{M}_H^{sm}$ are direct generalizations of the corresponding results for surfaces given in [**KSB, Ale3**], or simplifications of the results for stable pairs [**Ale2**].

1. Preliminaries

By a variety X we mean a reduced (not necessarily irreducible) separated scheme of finite type over the field k. A birational morphism $f : X \to Y$ is a morphism that maps every component of X birationally to a component of Y, and has a birational inverse. Whenever we talk about a Weil divisor D in X we assume that the generic point of every component of D lies in the smooth locus of X. A $\mathbb{Q}$-linear combination of Weil divisors $D = \sum a_i D_i$ in X is $\mathbb{Q}$-Cartier if a multiple lD is Cartier for some integer $l > 0$. If $f : Y \to X$ is a morphism, we let $f^*(D)$ be the $\mathbb{Q}$-Cartier divisor $\frac{1}{l} f^*(lD)$; and if f is birational then $f_*^{-1}(D) = \frac{1}{l} f_*^{-1}(lD)$ is the strict transform of D.

Suppose that X contains an open subvariety $U \subset X$ with complement of codimension at least two such that U is Gorenstein, i.e. the dualizing sheaf ω_U is invertible. We choose a canonical divisor K_U, where $\mathcal{O}_U(K_U) \cong \omega_U$, and let K_X be its closure in X. The variety X is called $\mathbb{Q}$-Gorenstein if K_X is $\mathbb{Q}$-Cartier. For a morphism $f : Y \to X$, we define similarly the divisor $K_{Y/X}$ with the help of the relative dualizing sheaf $\omega_{Y/X}$. We say that the morphism f is $\mathbb{Q}$-Gorenstein if $K_{Y/X}$ is $\mathbb{Q}$-Gorenstein. If both X and $f : Y \to X$ are $\mathbb{Q}$-Gorenstein, then so is Y, and $\mathcal{O}_Y(K_Y) \cong \mathcal{O}_Y(K_{Y/X}) \otimes f^*\mathcal{O}_X(K_X)$.

Definition 1.1. Let X be a normal $\mathbb{Q}$-Gorenstein variety, and let D be a $\mathbb{Q}$-Cartier divisor in X. We say that the pair (X, D) has canonical (resp. log canonical) singularities if for any birational morphism $f : Y \to X$ from a normal $\mathbb{Q}$-Gorenstein variety Y we have

$$K_Y = f^*(K_X + D) - f_*^{-1}(D) + \sum_i a_i E_i$$

where all $a_i \geq 0$ (resp. $a_i \geq -1$).

To show that (X, D) has log-canonical singularities, or that $(X, \emptyset)$ has canonical singularities, it suffices to check the numerical condition for a resolution $f : Y \to X$ of the pair (X, D), i.e. such that Y is nonsingular and the inverse image $f^*(D)$ is a divisor with nonsingular components crossing normally.

A flat projective morphism $f : X \to C$ from a nonsingular variety X to a nonsingular curve C is semistable if its fibers are reduced divisors of simple normal crossing. If $f : X \to C$ is any flat projective morphism to a nonsingular curve, we say that X admits a semistable resolution if there exists a birational morphism $g : \tilde{X} \to X$ such that $f \circ g : \tilde{X} \to C$ is semistable.

To prove Theorem 0.4 we need to assume the existence of relative canonical models (cf. [**KMM**]) in the following two cases:

Assumption 1.2. (MMP(n+1)). Let X be a normal irreducible $\mathbb{Q}$-Gorenstein $(n+1)$-fold with at most canonical singularities (hence mK_X is a Cartier divisor for some $m > 0$), and let $f : X \to Y$ be a morphism of one of the two types:

1. f is birational and for some morphism $g : Y \to C$ to a nonsingular curve C, the composition $g \circ f : X \to C$ is semistable; or
2. f is a flat projective morphism to a nonsingular curve Y with fibers of general type.

Then the relative canonical ring

$$R_{X/Y} = \bigoplus_{l \geq 0} f_* \mathcal{O}_X(lmK_X)$$

is a finitely generated $\mathcal{O}_Y$-algebra. The scheme $X_{can} = \text{Proj } R_{X/Y}$ (or the morphism $f_{can} : X_{can} \to Y$) is called a relative canonical model.

Relative canonical models are unique: if $X' \to Y$ is another morphism satisfying conditions 1. or 2. above, and if X' is birational to X over Y, then X'_{can} is isomorphic to X_{can} over Y.

1.1. Semi-log-canonical singularities.

DEFINITION 1.3. (cf. [**Ale3**] Def. 2.8, [**Kol2**] Def. 4.10) We say that a $\mathbb{Q}$-Gorenstein variety X has semi-log-canonical singularities if

(i) X satisfies Serre's condition S_2;
(ii) X has normal crossing singularities in codimension 1;
(iii) for any birational morphism $f : Y \to X$ from a normal $\mathbb{Q}$-Gorenstein variety Y we have

$$K_Y = f^*(K_X) + \sum_i a_i E_i$$

where all $a_i \geq -1$.

REMARK 1.4. Compared to log-canonical singularities, in the definition of semi-log-canonical singularities we relax the condition of the variety being normal: X has to satisfy Serre's condition S_2, but it may have the simplest kind of singularities in codimension 1.

The only semi-log-canonical varieties we are going to consider are fibers of morphisms $Y \to C$ from a variety with canonical singularities to a nonsingular curve.

These varieties are Cohen-Macaulay, so condition (i) is automatically satisfied. If, moreover, the family admits a semistable resolution, then (ii) is also satisfied: localizing at a codimension two point of Y (codimension one in $X = Y_0$) we get a germ of a surface with canonical singularities, admitting a semistable resolution; the fibers of such a surface are curves with normal crossing singularities.

Condition (ii) implies that X is Gorenstein in codimension one, so K_X is well defined. We call the closure of the singular locus in codimension one the double divisor of X. Let $\nu : X^\nu \to X$ be the normalization and $cond(\nu)$ the inverse image of the double divisor (with coefficients equal to one). Then

$$K_{X^\nu} = \nu^*(K_X) - cond(\nu).$$

and condition (iii) is equivalent to the pair $(X, cond(\nu))$ having log-canonical singularities.

THEOREM 1.5. *(cf.* [**KSB**] *Thm. 5.1.) Assume MMP(n+1). Let $\pi : X \to C$ be a flat projective morphism from a normal $\mathbb{Q}$-Gorenstein $(n+1)$-fold X to a germ of a nonsingular curve $(C, 0)$.*

(i) *Suppose that X admits a semistable resolution. Then X has canonical singularities if and only if the special fiber X_0 has semi-log-canonical singularities and the general fiber X_t has canonical singularities.*
(ii) *The product $X \times_C C'$ has canonical singularities for all finite base changes $(C', 0') \to (C, 0)$ if and only if the special fiber X_0 has semi-log-canonical singularities and the general fiber X_t has canonical singularities.*

Proof. $(i) \Rightarrow$. Let $g : \tilde{X} \to X$ be a semistable resolution, and assume that X has canonical singularities:

$$K_{\tilde{X}} = g^* K_X + \sum_i a_i E_i + \sum_j b_j F_j,$$

where E_i are the exceptional divisors mapping to $0 \in C$, F_j are flat over C, and $a_i, b_j \geq 0$ for all i, j. If $X_0' = g_*^{-1} X_0$ is the strict transform of X_0, then by adjunction

$$K_{X_0'} = (K_{\tilde{X}} + X_0')|_{X_0'} = (K_{\tilde{X}} - \sum E_i)|_{X_0'} = (g^* K_X + \sum_i (a_i - 1) E_i + \sum_j b_j F_j)|_{X_0'}.$$

Let $\nu : Y \to X_0'$ be the normalization map. Then $K_Y = \nu^* K_{X_0'} - cond(\nu)$ where $cond(\nu)$ is a reduced effective normal crossing divisor, not containing any components supported on E_i. This proves that the special fiber X_0 is semi-log-canonical because $Y \to X_0$ is a resolution. Since X is canonical, the general fiber X_t is also canonical.

$\Leftarrow$. Let $g : \tilde{X} \to X$ be the relative canonical model over X obtained by applying the MMP(n+1) assumption to a semistable resolution of X. From the assumption that the general fiber of π is canonical it follows that g is an isomorphism away from the special fibers; in particular, there are no exceptional divisors F_j flat over C (with notation as above).

Suppose that some $a_i \geq 0$, say $a_0 \geq 0$ and a_0 is maximal among the a_i. Let $Y \subset E_0$ be a curve mapping to a point in X, not lying in any other component of $\tilde{X}_0$. Then

$$K_{\tilde{X}}.Y = \sum a_i E_i.Y = \sum_{i \neq 0} a_i E_i.Y - \sum_{i \neq 0} a_0 E_i.Y \leq 0$$

and this contradicts the g-ampleness of $K_{\tilde{X}}$. So, all $a_i < 0$, and since X_0 is semi-log-canonical, the adjunction formula above shows that there are no exceptional divisors E_i. Thus, X has canonical singularities.

$(ii) \Rightarrow$. We know that after a finite base change $(C', 0') \to (C, 0)$ the fiber product $X' = X \times_C C'$ admits a semistable resolution. By part (i) the special fiber $X'_{0'} \cong X_0$ is semi-log-canonical and the general fiber $X'_{t'} \cong X_t$ is canonical.

$\Leftarrow$. It suffices to prove that X has canonical singularities. For some finite base change $(C', 0') \to (C, 0)$, the fiber product $X' = X \times_C C'$ admits a semistable resolution and has canonical singularities by (i). Let $h : \tilde{X} \to X$ be a resolution of (X, X_0), and let $\tilde{X}' = \tilde{X} \times_X X'$:

$$\begin{array}{ccc} \tilde{X}' & \xrightarrow{\tilde{f}} & \tilde{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{f} & X \end{array}$$

First assume that $\tilde{X}'$ is normal. Tracing the diagram in two ways we get

$$K_{\tilde{X}'} = g^* K_{X'} + \sum a_i E_i = g^* f^* K_X + m\tilde{X}'_0 + \sum a_i E_i,$$
$$K_{\tilde{X}'} = \tilde{f}^* K_{\tilde{X}} + m\tilde{X}'_0 = \tilde{f}^* h^* K_X + \tilde{f}^* \sum b_i F_i + m\tilde{X}'_0.$$

Since X' has canonical singularities, all $a_i \geq 0$, hence all $b_i \geq 0$ and X has canonical singularities.

If $\tilde{X}'$ is not normal, let $\nu : \tilde{X}^\nu \to \tilde{X}'$ be the normalization, and $cond(\nu)$ the conductor:

$$K_{\tilde{X}^\nu} = \nu^* K_{\tilde{X}'} - cond(\nu).$$

Computing $K_{\tilde{X}^\nu}$ in two ways as above, we get

$$\nu^* \tilde{f}^* \sum_i b_i F_i = \sum a_i E_i + cond(\nu),$$

where $a_i \geq 0$. Since $cond(\nu)$ is effective, we see that $b_i \geq 0$ and X has canonical singularities. □

The proof of the previous theorem can be easily modified to show that semi-log-canonical singularities deform to semi-log-canonical singularities.

LEMMA 1.6. *(cf. [**KSB**] Cor. 5.5.) Assume $MMP(n+1)$. Let $\pi : X \to C$ be a flat projective morphism from a $\mathbb{Q}$-Gorenstein $(n+1)$-fold X to a germ of a nonsingular curve $(C, 0)$. If the special fiber X_0 has semi-log-canonical singularities, then the general fiber X_t also has semi-log-canonical singularities.*

Proof. First assume that X is normal. Taking a finite base change $C' \to C$ if necessary, we may also assume that $X \to C$ admits a semistable resolution. Let $g : \tilde{X} \to X$ be the relative canonical model over X of this resolution. As in the proof of part (i) of the previous theorem, write

$$K_{\tilde{X}} = g^* K_X + \sum a_i E_i + \sum b_j F_j$$

where E_i are the exceptional divisors mapping to the special fiber X_0, and F_j are the exceptional divisors flat over C.

As before, we get from the adjunction formula that there are no exceptional divisors E_i mapping to X_0. When we restrict to the general fiber X_t, it follows from the g-ampleness of $K_{\tilde{X}}$ that all $b_j < 0$ ([**Kol3**], Lemma 2.19). Since the special fiber X_0 is semi-log-canonical, the coefficients $b_j \geq -1$, and we get that the general fiber X_t is semi-log-canonical because $\tilde{X}_t \to X_t$ is a resolution.

If X is not normal, we let $\nu : X^\nu \to X$ be the normalization. We may assume that the components of X^ν admit a semistable resolution (with the same base change) $g : \tilde{X} \to X^\nu \to X$. The conductor $cond(\nu)$ maps flat onto C, and as in the normal case, the discrepancies b_j of the general fiber are not bigger than the discrepancies of the special fiber. □

1.2. The moduli functor $\mathcal{M}_H^{sm}$.

DEFINITION 1.7. (cf. [**Kol2**], Def. 5.2)

1. A stable n-fold X with Hilbert polynomial H is a connected projective n-dimensional variety (not necessarily irreducible) over an algebraically closed field of characteristic zero with semi-log-canonical singularities and ample canonical divisor K_X such that $h^0(X, lK_X) = H(l)$ for all large and divisible l.
2. A stable n-fold with Hilbert polynomial H over a scheme S is a flat projective $\mathbb{Q}$-Gorenstein morphism $\pi : Y \to S$ such that
   (a) every geometric fiber $Y_s = \pi^{-1}(s)$ is a stable n-fold with Hilbert polynomial H; and
   (b) we have a natural isomorphism
   $$\mathcal{O}_{Y_s}(lK_{Y_s}) \cong \mathcal{O}_Y(lK_{Y/S})|_{Y_s}$$
   for all large and divisible l and all point $s \in S$.
3. The moduli functor $\mathcal{M}_H$: (Schemes) $\to$ (Sets) is defined by $\mathcal{M}_H(S) = \{$isomorphism classes of stable n-folds with Hilbert polynomial H over $S\}$.

Note that condition 2.(a) in Definition 1.7 is open by Lemma 1.6 and Remark 2.3 below. Condition 2.(b) on the other hand is closed by a theorem of Kollár [**Kol2**]. Hence the functor $\mathcal{M}_H$ is locally closed. One can define an open subfunctor of $\mathcal{M}_H$ by requiring that all fibers of $Y \to S$ be normal varieties with at most rational Gorenstein singularities. As we mentioned in the introduction, this subfunctor is coarsely represented by a quasi-projective scheme. Since our goal is to compactify this scheme, we define a closed subfunctor $\mathcal{M}_H^{sm}$ of $\mathcal{M}_H$ corresponding to "smoothable" n-folds.

DEFINITION 1.8. (cf. [**Kol2**], Def. 5.4)

1. A stable n-fold Y_0 is smoothable if there exists a flat projective $\mathbb{Q}$-Gorenstein one-parameter family $\pi : Y \to C$ of stable n-folds such that the special fiber

of π is Y_0 and the general fiber Y_t is a normal n-fold with at most rational Gorenstein singularities.

2. A stable n-fold $\pi : Y \to S$ over a scheme S is smoothable if the following is satisfied for all points $s \in S$. Let $p : U \to D$ be a versal deformation of the fiber $Y_s \cong p^{-1}(0)$, so we get a morphism q from the formal germ of S at s to D. Let D^{st} be a locally closed subscheme of D such that p is a stable family over D^{st} (hence the morphism q factors through D^{st}), and let D^{sm} be the open subscheme of D^{st} such that p is a family of normal n-folds with at most rational Gorenstein singularities over D^{sm}. Then the morphism q has to factor through the closure of D^{sm}.
3. The moduli functor $\mathcal{M}_H^{sm}$: (Schemes) $\to$ (Sets) is defined by $\mathcal{M}_H^{sm}(S) =$ {isomorphism classes of stable smoothable n-folds with Hilbert polynomial H over S}.

Given a smoothing $\pi : Y \to C$ of a stable n-fold Y_0 as in Definition 1.8, the total space Y has canonical singularities by Theorem 1.5 (ii), and K_Y is π-ample. So, $\pi : Y \to C$ is a relative canonical model. Conversely, if $\pi : Y \to C$ is a relative canonical model which admits a semistable resolution, the special fiber Y_0 is a stable n-fold by Theorem 1.5 (i); and if the generic fiber of π is Gorenstein, then $\pi : Y \to C$ is in fact a smoothing of Y_0. This shows that the moduli functor $\mathcal{M}_H^{sm}$ can be defined purely in terms of special fibers of canonical models.

Lemma 1.9. *The functor $\mathcal{M}_H^{sm}$ is separated; i.e. any family $X \to C \setminus \{0\}$ of stable smoothable n-folds over a punctured curve can be extended to a family over C in no more than one way.*

Proof. (Theorem 2.9, [**Ale2**]). Suppose that we have two extensions $f_1 : X_1 \to C$ and $f_2 : X_2 \to C$. Let $\tilde{X}$ be a common resolution of X_1 and X_2, $g_1 : \tilde{X} \to X_1$, $g_2 : \tilde{X} \to X_2$. Write

$$K_{\tilde{X}} = g_1^* K_{X_1} + \sum a_i E_i + \sum b_j F_j$$

where E_i are the exceptional divisors mapping to $0 \in C$, and F_j are flat over C. Taking a finite base change $C' \to C$ if necessary, we may assume that $\tilde{X}$ admits a semistable resolution.

First assume that X_1 is normal. As in the proof of Theorem 1.5, we get that $a_i \geq 0$ and $b_j \geq -1$. Since K_{X_1} is f_1-ample, X_1 is the log-canonical model of the pair $(\tilde{X}, K_{\tilde{X}} + \sum F_j$:

$$X_1 \cong \text{Proj} \bigoplus_{l \geq 0} (f_1 \circ g_1)_* \mathcal{O}_{\tilde{X}}(l(K_{\tilde{X}} + \sum F_j)).$$

The same is true for X_2, and since X_1 and X_2 are isomorphic away from the special fibers, the divisors F_j are the same in both cases. Hence X_1 and X_2 are isomorphic.

If X_1 is not normal, we can recover its normalization X_1^ν as the log-canonical model of the divisor $K_{\tilde{X}} + \sum F_j$, where now $\sum F_j$ also includes the components of the inverse image of *cond*(ν) flat over C. This proves that X_1 and X_2 are isomorphic in codimension one and have the same normalization. Let $\overline{\Gamma} \subset X_1 \times X_2$ be the closure of the graph of the rational map $X_1 \to X_2$. The projection $\pi_i : \overline{\Gamma} \to X_i$ is birational, finite, and isomorphism in codimension 1 for $i = 1, 2$. Since X_i satisfies Serre's condition S_2, the projection $\pi_i : \overline{\Gamma} \to X_i$ is an isomorphism. □

1.3. Weak semistable reduction. We recall some properties of a weakly semistable morphism proved in Chapter 2.

LEMMA 1.10. *Let $f : X \to B$ be weakly semistable. Then*

1. *X has rational Gorenstein (hence canonical) singularities (Lemma 8.2 in Chapter 2).*
2. *If $g : C \to B$ is a morphism from a nonsingular curve where $g(C) \not\subset B \setminus U_B$ then C and $X_C = X \times_B C$ can be given toroidal structures such that $X_C \to C$ is again toroidal and weakly semistable; in particular, X_C has canonical singularities (Lemma 8.3 in Chapter 2).* □

LEMMA 1.11. *Let $f : X \to C$ be a weakly semistable morphism from a variety X with $\dim X = n+1$ to a curve C. Assume that the fibers of f are of general type, and let $f_{can} : X_{can} \to C$ be the relative canonical model. Then the special fiber $X_{can,0} = f_{can}^{-1}(0)$ is a stable n-fold.*

Proof. By Lemma 1.10 part 2, if $g : (C', 0') \to (C, 0)$ is a finite base change, then $X' = X \times_C C'$ is again canonical. Since $K_{X'} = g^* K_X + m X'_{0'}$, we see that finding the canonical model commutes with taking a finite base change:

$$X_{can} \times_C C' \cong (X \times_C C')_{can}$$

and we can apply Theorem 1.5 (ii) to conclude that $X_{can,0}$ has semi-log-canonical singularities. □

It can be shown directly that the fibers of a weakly semistable morphism have semi-log-canonical singularities. By Lemma 8.3 in Chapter 2, we may assume that the base is a curve. Then in a local model the special fiber is the complement of the big torus in a toric variety. Alexeev has shown [**Ale2**] that this complement is semi-log-canonical.

LEMMA 1.12. *Let $f : X \to C$ be as in the previous lemma, and assume that the genus $g(C) \geq 3$. Then the sheaf $f_* \mathcal{O}_X(lK_X)$ is generated by global sections for all $l \geq 2$.*

Proof. (Kollár [**Kol2**]) By Theorem 1 in [**Kaw2**], the vector bundle $E_l = f_*\mathcal{O}_X(lK_{X/C})$ is semipositive for all $l \geq 0$; that means, any quotient line bundle of E_l has non-negative degree on C. If L is a line bundle with $\deg(L) > 2g(C) - 2$ then $\mathcal{O}_C(K_C)$ cannot be a quotient of $E_l \otimes L$. Hence $H^1(C, E_l \otimes L) = 0$. Now if $\deg(L) > 2g(C)$ then $E_l \otimes L$ is generated by global sections. In particular, $f_*\mathcal{O}_X(lK_X) = E_l \otimes \mathcal{O}_C(lK_C)$ is generated by global sections for all $l \geq 2$. ☐

1.4. Deformations of canonical singularities and invariance of plurigenera. The following two theorems were proved by Kawamata [**Kaw3**], generalizing results of Siu [**Siu**].

THEOREM 1.13. *Let $\pi : X \to S$ be a flat morphism from a germ of a variety (X, x_0) to a germ of a smooth curve (S, s_0) whose special fiber $X_0 = \pi^{-1}(s_0)$ has only canonical singularities. Then X has only canonical singularities. In particular, the fibers $X_s = \pi^{-1}(s)$ have only canonical singularities.* ☐

THEOREM 1.14. *Let $\pi : X \to S$ be a projective flat morphism from a normal variety to a germ of a smooth curve (S, s_0). Assume that the fibers $X_s = \pi^{-1}(s)$ have only canonical singularities and are of general type for all $s \in S$. Then the plurigenus $P_m(X_s) = \dim H^0(X_s, mK_{X_s})$ is constant as a function of $s \in S$ for any $m > 0$.* ☐

The two theorems for families of n-folds follow from the minimal model program assumption in dimension $n + 1$ [**Nak**]. However, we are going to apply them for families of $(n + 1)$-dimensional varieties, and we do not want to assume minimal model program in dimension $n + 2$.

2. Proof of boundedness

By Matsusaka's theorem (Theorem 2.4, [**Mat2**]) there exists an integer $\nu_0 > 0$ such that if X is a normal variety with rational Gorenstein singularities, with ample canonical divisor K_X, and with a given Hilbert polynomial $H(l) = h^0(X, lK_X)$ for large l, then $\nu_0 K_X$ is very ample and has no higher cohomology. Thus, the Hilbert scheme parameterizing embeddings $X \in \mathbb{P}^{H(\nu_0)-1}$ with Hilbert polynomial $H(\nu_0 t)$ has a locally closed subscheme parameterizing ν_0-canonical embeddings of varieties X with rational Gorenstein singularities. We give this subscheme the induced reduced structure and call its closure B_0. Let $f_0 : X_0 \to B_0$ be the universal family over B_0. We apply weak semistable reduction (Theorem 0.6 in Chapter 2) to the morphism f_0 to get a weakly semistable morphism

$$f : X \to B.$$

By Lemma 1.10 we know that for a general nonsingular curve $C \subset B$ the inverse image $X_C = f^{-1}(C)$ has canonical singularities. We can then apply the MMP(n+1) assumption and find a canonical model $X_{C,can}$ for X_C. Since the general fiber of $X_{C,can} \to C$ has rational Gorenstein singularities, we see that the special fiber is a stable smoothable n-fold with Hilbert polynomial H.

The idea of the proof is to construct a relative canonical model for the whole family $f : X \to B$ and to show that the restriction of this canonical model to a curve C is the canonical model $X_{C,can}$.

LEMMA 2.1. *The relative canonical ring*

$$R_{X/B} = \bigoplus_{l \geq 0} f_* \mathcal{O}_X(lK_X)$$

is a finitely generated $\mathcal{O}_B$*-algebra.*

To prove the lemma we sweep B locally with nonsingular curves. By Bertini's theorem a general hyperplane section through $b \in B$ is nonsingular, and the same is true for nearby hyperplane sections. Applying this dim $B - 1$ times we get the following diagram:

$$\begin{tikzcd} X_1 = B_1 \times_B X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ B_1 \arrow[r, "\phi"] \arrow[d, "g"'] & B \\ S & \end{tikzcd}$$

where

1. $(g : B_1 \to S, \phi)$ is a family of nonsingular curves in B parameterized by S, no curve lying in $B \setminus U_B$;
2. there exist $b_1 \in B_1$, $\phi(b_1) = b$, and open neighborhoods $b_1 \in U_1$, $b \in U$ such that $\phi : U_1 \to U$ is an isomorphism;
3. we may assume that the hyperplane sections have high degree, so the fibers of g have high genera (≥ 3).

Since Lemma 2.1 is local in B we may replace B by B_1 and X by X_1.

LEMMA 2.2. *Consider the Cartesian diagram*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] & X \arrow[d, "f"] \\ C \arrow[r] \arrow[d, "g"'] & B \arrow[d, "g"] \\ \{s\} \arrow[r] & S \end{tikzcd}$$

where $s \in S$ is a closed point, $C = g^{-1}(s)$, and $Y = f^{-1}(C)$. Then the natural morphism

$$f_* \mathcal{O}_X(lK_X) \otimes_{\mathcal{O}_B} \mathcal{O}_C \to f_* \mathcal{O}_Y(lK_Y)$$

is an isomorphism for every $s \in S$ and every $l \geq 2$.

Proof. By "cohomology and base change" [**Har2**] it suffices to prove the surjectivity of this morphism. Lemma 1.12 shows that the sheaf $f_*\mathcal{O}_Y(lK_Y)$ is generated by global sections $H^0(Y, \mathcal{O}_Y(lK_Y))$ for every $l \geq 2$. The family $g \circ f : X \to S$ has fibers of general type (by sub-additivity of Kodaira dimension [**Kaw2, Vie1**], since the fibers of both f and g are of general type) and with canonical singularities (by Lemma 1.10). Theorem 1.14 now implies that the plurigenera of the fibers of $g \circ f$ are locally constant functions on S. It follows from semicontinuity that the sheaf $(g \circ f)_*\mathcal{O}_X(lK_X)$ is locally free and the natural morphism

$$(g \circ f)_*\mathcal{O}_X(lK_X)|_{\{s\}} \to H^0(Y, \mathcal{O}_Y(lK_Y))$$

is surjective. So, a section in $H^0(Y, \mathcal{O}_Y(lK_Y))$ can be lifted to a local section of $(g \circ f)_*\mathcal{O}_X(lK_X)$, which gives a section of $f_*\mathcal{O}_X(lK_X)$. $\square$

Remark 2.3. For the next proof we note that ampleness is an open condition: given a flat projective morphism $f : X \to B$ and a $\mathbb{Q}$-Cartier divisor D on X such that $D|_{X_b}$ is ample for some $b \in B$, then D is f-ample over a neighborhood of b in B. This follows from Kleiman's numerical criterion for ampleness [**Kle**]. If $Z_1(X/B)$ is the free group generated by reduced irreducible curves in X mapping to a point in B, and $N_1(X/B)$ is $Z_1(X/B) \otimes \mathbb{R}$ modulo numerical equivalence (see [**KMM**] for notations), then we have natural linear maps $Z_1(X_b) \to N_1(X/B)$ and $Z_1(X_\eta) \to N_1(X/B)$ where η is the generic point of B. Now the second map factors through the first: for a curve over the generic point of B we take its closure and restrict to the special fiber. Also, the maps restricted to effective cones factor. If D is ample on X_b then D is positive on the effective cone $\overline{NE}(X_b) \setminus \{0\}$, hence it is positive on $\overline{NE}(X_\eta) \setminus \{0\}$, and so D is ample on X_η.

Proof of Lemma 2.1. The statement is local in B , so we may replace B by a smaller open neighborhood of $b \in B$ if necessary. We use the same notation as in Lemma 2.2.

By the minimal model program assumption MMP(n+1) applied to $f : Y \to C$, the sheaf $R_{Y/C}$ is a finitely generated $\mathcal{O}_C$-algebra, and by Lemma 2.2, $R_{X/B} \otimes \mathcal{O}_C \cong R_{Y/C}$, at least in high degrees. Since the graded pieces $(R_{X/B})_l = f_* \mathcal{O}_X(lK_X)$ are finite $\mathcal{O}_B$-modules we can apply Nakayama's lemma and conclude that $R_{X/B} \otimes \mathcal{O}_{B,b}$ is a finitely generated $\mathcal{O}_{B,b}$-algebra, which is a localization of a finitely generated $\mathcal{O}_U$-algebra $\tilde{R}$ for some open neighborhood $b \in U \subset B$. We may replace B by U and assume that $\tilde{R}$ is a finitely generated $\mathcal{O}_B$-algebra. Let $\tilde{X} = \text{Proj } \tilde{R}$, and $\tilde{f} : \tilde{X} \to B$.

By construction, $\tilde{Y} = \tilde{f}^{-1}(C)$ is a relative canonical model of $f : Y \to C$; in particular, it has canonical singularities. By Theorem 1.13 canonical singularities deform to canonical singularities, hence $\tilde{X}$ has canonical singularities.

The canonical divisor $K_{\tilde{X}}$ is $\tilde{f}$-ample when restricted to $\tilde{Y}$. Using the numerical criterion for ampleness, $K_{\tilde{X}}$ is $\tilde{f}$-ample over a neighborhood of b. Again, we may replace B by a smaller open set and assume that $K_{\tilde{X}}$ is $\tilde{f}$-ample.

Now $\tilde{f} : \tilde{X} \to B$ is a relative canonical model, birational to $f : X \to B$ over B, hence the relative canonical rings are the same

$$R_{X/B} \cong R_{\tilde{X}/B}.$$

Since the latter is finitely generated as an $\mathcal{O}_B$-algebra, so is the former. □

Proof of Theorem 0.4 We show that $\tilde{f} : \tilde{X} = \text{Proj } R_{X/B} \to B$ is the required family.

It is clear from the construction that every closed fiber $\tilde{f}^{-1}(b)$ is a stable smoothable n-fold with Hilbert polynomial H. Indeed, we have constructed through every closed point $b \in B$ a nonsingular curve C such that $\tilde{f}^{-1}(C)$ is the relative canonical model of a weakly semistable family over C with Gorenstein generic fiber, so the special fiber is stable and smoothable by Lemma 1.11.

To prove the converse, namely that the fibers of $\tilde{f}$ contain all stable smoothable n-folds with Hilbert polynomial H, we go back to the construction of $\tilde{f} : \tilde{X} \to B$ from the universal family over the Hilbert scheme. We have the commutative diagram

$$\begin{array}{ccccccc}
\tilde{X} & \longleftarrow & X & \longrightarrow & X_1 = X_0 \times_B B_0 & \longrightarrow & X_0 \\
{\scriptstyle \tilde{f}}\big\downarrow & & {\scriptstyle f}\big\downarrow & & {\scriptstyle f_1}\big\downarrow & & {\scriptstyle f_0}\big\downarrow \\
B & \xleftarrow{=} & B & \xrightarrow{=} & B & \longrightarrow & B_0
\end{array}$$

where $f_0 : X_0 \to B_0$ is the universal family, the morphisms $B \to B_0$ and $X \to X_1$ come from weak semistable reduction, and the map $X \to \tilde{X}$ is only rational.

Let $V_{B_0} \subset B_0$ be the open set parameterizing ν_0-canonical embeddings of normal varieties with rational Gorenstein singularities; and let $V_B \subset B, V_{X_1} \subset X_1$, and $V_{\tilde{X}} \subset \tilde{X}$ be the inverse images of V_{B_0}. Now $f_1 : V_{X_1} \to V_B$ is a family of normal varieties with rational Gorenstein singularities and ample canonical bundle over a nonsingular base. It follows that V_{X_1} is normal with canonical singularities and $f_1 : V_{X_1} \to V_B$ is a relative canonical model, birational to $\tilde{f} : V_{\tilde{X}} \to V_B$ over V_B. By uniqueness of the relative canonical model, the two are isomorphic $V_{\tilde{X}} \cong V_{X_1}$ over V_B.

Let $\pi : Y \to C$ be a smoothing of a stable n-fold Y_0, and let $Y_\eta = \pi^{-1}(\eta)$ be the fiber over the generic point η of C. Since Y_η is a projective rational Gorenstein scheme with ample canonical bundle and Hilbert polynomial H, we can find a morphism $\eta \to V_{B_0}$ such that the universal family V_{X_0} pulls back to Y_η. There exists a finite morphism $(C', \eta') \to (C, \eta)$ such that we can lift $\eta \to V_{B_0}$ to $\eta' \to V_B$. By completeness of B we get a morphism $C' \to B$. Let $\tilde{Y} = C' \times_B \tilde{X}$, and let $Y' = C' \times_C Y$. The two families $\tilde{Y} \to C'$ and $Y' \to C'$ are isomorphic over the generic point, so their special fibers must be isomorphic by separatedness of the functor $\mathcal{M}_H^{sm}$ (Lemma 1.9). $\square$

2.1. Boundedness for stable smoothable pairs. With the notation of [**Ale2**], we consider stable pairs (Y_0, D_0) where Y_0 is a connected projective variety and D_0 is a reduced effective Weil divisor in Y_0 such that the pair (Y_0, D_0) has semi-log-canonical singularities and $K_{Y_0} + D_0$ is an ample $\mathbb{Q}$-Cartier divisor. We say that (Y_0, D_0) is smoothable if it admits a $\mathbb{Q}$-Gorenstein 1-parameter deformation (Y, D) such that the general fiber Y_t has rational singularities and $K_{Y_t} + D_t$ is Cartier. Further, we fix the Hilbert polynomials $\dim H^0(Y_0, l(K_{Y_0} + D_0))$ and $\dim H^0(D_0, l(K_{Y_0} + D_0)|_{D_0})$ for large l.

By Matsusaka's theorem, for some $\nu_0 > 0$ the divisor $\nu_0(K_{Y_t} + D_t)$ is very ample and without higher cohomology for all pairs (Y_t, D_t) as above. There exists a Hilbert scheme parameterizing embeddings $Y_t \subset \mathbb{P}^N$ with the given Hilbert polynomial. Since $D_t \subset Y_t \subset \mathbb{P}^N$ also has a fixed Hilbert polynomial, there is a Hilbert scheme parameterizing such embeddings. In the product of the two Hilbert schemes we can find a locally closed subscheme parameterizing the required embeddings $(Y_t, D_t) \subset \mathbb{P}^N$ via $\nu_0(K_{Y_t} + D_t)$; there is also a universal family $(\mathcal{Y}, \mathcal{D})$ over this subscheme.

As before, we complete the universal family and apply weak semistable reduction to it. We can make sure that the proper transform D of the universal divisor $\mathcal{D}$ in the weakly semistable family $f : X \to B$ is the union of horizontal toroidal divisors. The proof that $\bigoplus_l \mathcal{O}_X(l(K_X + D))$ is a finitely generated $\mathcal{O}_B$-algebra goes as before, replacing the $MMP(n{+}1)$ with $logMMP(n{+}1)$ assumption, if Kollár's semipositivity result and Siu-Kawamata theorems on invariance of plurigenera and deformations of canonical singularities also hold for pairs. I do not know if these are true. Kollár's semipositivity theorem can be avoided in the proof of boundedness by considering ramified cyclic covers of the base variety S; but to construct the moduli space we would still need it. As remarked above, the theorems on invariance of plurigenera and deformations of canonical singularities can be replaced by the $logMMP(n+2)$ assumption; this is, however, a very strong assumption, considering for example the case of surfaces $n = 2$.

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Curriculum Vitae

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