CUBICAL SMALL-CANCELLATION QUOTIENTS OF NON-PRODUCTS

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Abstract. We show that if X is a compact nonpositively curved cube complex which is a non-product, then $\pi_1 X$ has a proper quotient that is π_1 of a cube complex with these same properties.

Contents

1. Introduction

Burger and Mozes [BM97] constructed nonpositively curved cube complexes X with infinite simple $\pi_1 X$. In that case $\widetilde{X}$ is a product of two trees. A nonpositively curved cube complex X is a *non-product* if $\pi_1 X$ is nontrivial, non-cyclic, and the universal cover $\widetilde{X}$ does not contain a $\pi_1 X$-invariant convex subcomplex isomorphic to a nontrivial product of unbounded cube complexes.

We obtain the following attractive counterpoint to the Burger-Mozes result:

Theorem 1.1. [Cubically non-simple] Let $G = \pi_1 X$ where X is a finite dimensional non-product nonpositively curved cube complex. Then G has a nontrivial proper quotient $\bar{G} = \pi_1 \bar{X}$ with $\bar{X}$ a nonproduct nonpositively curved cube complex. Moreover, if X is compact, then $\bar{X}$ is compact.

Remark 1.2. In Theorem 1.1, for any finite set $S \subset G$, the quotient $\bar{G}$ can be chosen so that the image of every nontrivial element in S is nontrivial in $\bar{G}$.

This result raises the following possibility, which is out of reach with current technology in general. Our work resolves it in the cubical setting.

Conjecture 1.3. Every CAT(0) group with a rank one element has a proper quotient with the same properties.

To prove Theorem 1.1, we work in the context of cubical small-cancellation theory. A cubical presentation $\langle X \mid \{Y_i \to X\}\rangle$ consists of a nonpositively curved cube complex X and a collection of local isometries $Y_i \to X$. This data determines a quotient $\pi_1 X/\langle\langle \pi_1 Y_i\rangle\rangle$ that arises as π_1 of the space X^* obtained by coning-off each Y_i in X. Cubical presentations are natural generalizations of group presentations, and are used to study quotients of cubulated groups, similar to how ordinary presentations are used to study quotients of free groups. *Classical* small-cancellation-theory yields tractable quotients of free groups, and *cubical* small-cancellation theory is the corresponding framework in the cubical setting. See Section 4 for the relevant definitions. Cubical presentations and cubical small-cancellation theory were introduced in [Wis21]. They played an essential role in the proof of the Malnormal Special Quotient Theorem, and thus in the proofs of the Virtual Haken and Virtual Fibering conjectures [Ago13]. Subsequently, cubical small-cancellation theory has been further studied and utilized [Jan17, AH22, JW22, Are24a, FW24, Are24b, HW24, Are24c, AJW24].

In the course of this investigation, we found *pseudographs* to be a unifying theme. These are nonpositively curved cube complexes whose hyperplanes are contractible (see Definition 3.1). They arise naturally in the work of Caprace and Sageev on rank rigidity, and they are readily used as relators in a cubical small-cancellation quotient. The flow of ideas in the text is then:

$$\begin{gathered} X \text{ is a finite-dimensional non-product} \\ \Downarrow \\ X \text{ contains a rank 2 pseudograph} \\ \Downarrow \\ X \text{ has a rich family of cubical small-cancellation quotients.} \end{gathered}$$

In view of the above, Theorem 1.1 is a consequence of the following.

Theorem 1.4. [Theorem 5.10 and Theorem 7.12] Let X be a nonpositively curved cube complex that admits a local isometry of a rank 2 superconvex pseudograph. For every $\alpha \leq \frac{1}{16}$ there exists a pseudograph $Y \looparrowright X$ with $\pi_1 Y \neq 1$ such that $\langle X \mid Y\rangle$ is a cubical $C'(\alpha)$ small-cancellation presentation. Moreover, we can choose Y so that $\pi_1 X/\langle\langle \pi_1 Y\rangle\rangle$ acts freely on a CAT(0) cube complex $\mathcal{C}$.

Furthermore, if X is compact, then Y can be chosen so that $\pi_1 X/\langle\langle \pi_1 Y\rangle\rangle$ acts properly and cocompactly on $\mathcal{C}$.

Organization. In Section 2 we discuss various results and notions related to rank rigidity, and use them to give a characterization of non-product nonpositively curved cube complexes. In Section 3 we introduce pseudographs, and explain the relation between pseudographs and non-products. In Section 4 we review the cubical small-cancellation terminology. In Section 5 we explain how to use pseudographs to produce cubical small-cancellation quotients of non-products. In Section 6 we review disc diagrams, state the cubical versions of Greendlinger's Lemma and the Ladder Theorem, and describe the more sophisticated $B(6)$ cubical small-cancellation condition and related results that lead to cubulation. In Section 9.a we prove that certain disc diagrams arising from collections of intersecting walls in a $B(6)$ presentation are uniformly thin. In Section 7 we begin the proof of Theorem 1.4. Namely, π_1 of a suitably chosen $B(6)$ cubical presentation X^* acts on the non-product CAT(0) cube complex dual to the $B(6)$ wallspace structure on $\widetilde{X}^*$. In Section 8 we show that X^* can moreover be chosen so that the action is free. In Section 9 we complete the proof of Theorem 1.4 by verifying cocompactness of the action on the dual cube complex obtained in Section 7.

Related work. It is known that for any non-product X, there exists a nontrivial quotient of $\pi_1 X$, as $\pi_1 X$ is acylindrically hyperbolic. Indeed, Caprace and Sageev [CS11] showed that every group acting properly and cocompactly on a geodesically complete non-product CAT(0) cube complex contains a rank one isometry. Osin showed every non virtually cyclic group acting properly on a proper CAT(0) space and containing a rank one isometry is acylindrically hyperbolic [Osi16]. In particular, the action of $\pi_1 X$ on its contact graph, which is hyperbolic, is acylindrical when X admits a factor system [BHS17]. Acylindrically hyperbolic groups have many proper quotients, e.g. they are SQ-universal [DGO17]. There exists a version of small-cancellation theory for acylindrical hyperbolic groups which also produces nontrivial quotients [Hul16]. Acylindrical hyperbolicity has been applied to cube complexes to show that the growth-rate of $\pi_1 X$ is strictly larger than the growth-rate of its hyperplane stabilizers [DFW19]. However, none of these results provide cubical geometry for the quotient group.

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2. CAT(0) cube complexes

We assume familiarity with the basic background on CAT(0) cube complexes, for which the reader can consult [BSV14, Wis12, Wis21], for instance.

2.a. **Halfspaces.** A *hyperplane* U in a CAT(0) cube complex $\widetilde{X}$ is a non-empty connected subspace such that $U \cap c$ is a midcube of c for each cube c of $\widetilde{X}$. Its two associated *halfspaces* are the closures of the components of $\widetilde{X} - U$ and are denoted U^+, U^-. While halfspaces are not subcomplexes of $\widetilde{X}$, we can define associated subcomplexes as follows. The *carrier* $N(U)$ of a hyperplane U is the convex subcomplex consisting of the union of all cubes intersecting U. A *minor halfspace* is the closure of a component of the complement of the interior of $N(U)$. A *major halfspace* is the closure of the complement of a minor halfspace.

In the next few sections we describe some background, mostly from Caprace-Sageev [CS11].

2.b. **Properties of actions.** Let G act on a CAT(0) cube complex $\widetilde{X}$. A hyperplane $U \subseteq \widetilde{X}$ is G-essential for any (equivalently every) basepoint $v \in \widetilde{X}$, each halfspace of U contains points of Gv lying arbitrarily far from U. The action of G on $\widetilde{X}$ is *essential* if each hyperplane of $\widetilde{X}$ is G-essential. The *G-essential core* of $\widetilde{X}$ is the cube complex dual to all the G-essential hyperplanes of $\widetilde{X}$. The action of G on its G-essential core is essential. When G either acts on $\widetilde{X}$ cocompactly or without fixed point at infinity, then the G-essential core of $\widetilde{X}$ is unbounded if and only if G has no fixed points in $\widetilde{X}$. In that case the G-essential core embeds as a convex G-invariant subcomplex of the cubical subdivision of $\widetilde{X}$ [CS11, Prop 3.5]. Thus, in the setting of Theorem 1.1, up to possibly passing from $\widetilde{X}$ to its $\pi_1 X$-essential core, we can assume that the action of $\pi_1 X$ on $\widetilde{X}$ is essential. Indeed, if X is a non-product then so is the $\pi_1 X$-essential core of $\widetilde{X}$. It is worth noting that when $\pi_1 X$ is not trivial or cyclic, the $\pi_1 X$-essential core is a non-product if and only if it is not a direct product of unbounded cube complexes.

A group G acts on a CAT(0) space $\widetilde{X}$ *without a fixed point at infinity*, if there does not exist $\xi \in \partial X$ where ∂X is the visual boundary of X, with $g\xi = \xi$ for all $g \in G$. By [Gen25, Thm 1.9 and Clm 4.9], if G acts on a finite dimensional CAT(0) cube complex $\widetilde{X}$, and for every finite index subgroup $H \subseteq G$ the commutator subgroup $[H, H]$ is infinite, then G acts on $\widetilde{X}$ without a fixed point at infinity.

Remark 2.1. When X is a finite-dimensional non-product, then $G = \pi_1 X$ satisfies those assumptions. Indeed, otherwise G is virtually finite-by-abelian, so in particular virtually abelian, but this contradicts the assumption that G is the π_1 of a non-product.

By Remark 2.1, in the setting of Theorem 1.1 the action of $\pi_1 X$ on $\widetilde{X}$ has no fixed point at infinity. Consequently, we can apply Proposition 3.4 to deduce Theorem 1.1 from Theorem 1.4.

2.c. **Characterization of non-products.** Two hyperplanes U_1, U_2 are *strongly separated* if there does not exist a hyperplane V which intersects both U_1, U_2.

Proposition 2.2 ([CS11, Prop 5.1])**.** Let $\widetilde{X}$ be a finite-dimensional unbounded CAT(0) cube complex such that $\text{Aut}(\widetilde{X})$ acts essentially without a fixed point at infinity. Then the following conditions are equivalent:

(1) $\widetilde{X}$ is a non-product.
(2) There is a pair of strongly separated hyperplanes.
(3) For each halfspace U^+ there is a pair of halfspaces U_1, U_2 such that $U_1^+ \subseteq U^+ \subseteq U_2^+$ and the hyperplanes U_1, U_2 are strongly separated.

Lemma 2.3 (Flipping Lemma [CS11, Thm 4.1])**.** Let $\widetilde{X}$ be a finite-dimensional CAT(0) cube complex, and let $G \subseteq \text{Aut}(\widetilde{X})$ act essentially, without a fixed point at infinity. For each halfspace U^+, there exists $\gamma \in G$ with $U^- \subsetneq \gamma U^+$.

Lemma 2.4 (Double Skewering Lemma [CS11])**.** Let $\widetilde{X}$ be a finite-dimensional CAT(0) cube complex and $G \subseteq \text{Aut}(\widetilde{X})$ be a group acting essentially without a fixed point at infinity. Then for any two half-spaces $U_1^+ \subsetneq U_2^+$, there exists $g \in G$ such that $U_2^+ \subsetneq gU_1^+$.

A hyperplane U *separates* sets $A, B \subseteq \widetilde{X}$ if $A \subseteq U^+$ and $B \subseteq U^-$, or vice-versa. A *facing triple of hyperplanes* is a collection of three disjoint hyperplanes U_1, U_2, U_3 such that no U_i separates the other two hyperplanes. In particular, we can always choose the halfspaces U_1^-, U_2^-, U_3^- of these hyperplanes so that the intersection $U_1^- \cap U_2^- \cap U_3^-$ is non-empty. A *wedge* corresponding to a facing triple U_1, U_2, U_3 in $\widetilde{X}$ is the intersection of the associated minor halfspaces of a facing triple, i.e. a maximal subcomplex contained $U_1^- \cap U_2^- \cap U_3^-$.

Theorem 2.5 ([CS11, Thm 7.2])**.** Let $\widetilde{X}$ be a finite-dimensional CAT(0) cube complex such that $\text{Aut}(\widetilde{X})$ acts essentially and satisfies at least one of the following conditions:

(1) $\text{Aut}(\widetilde{X})$ has no fixed points at infinity.
(2) $\text{Aut}(\widetilde{X})$ acts cocompactly and X is locally compact.

Then $\text{Aut}(\widetilde{X})$ stabilizes some Euclidean flat if and only if there is no facing triple of hyperplanes in $\widetilde{X}$.

Proposition 2.6. Let $\widetilde{X}$ be a finite-dimensional CAT(0) cube complex, and let $G \subseteq \text{Aut}(\widetilde{X})$ act essentially, without a fixed point at infinity. If $\widetilde{X}$ is a non-product, then there exists a facing triple of pairwise strongly separated hyperplanes in $\widetilde{X}$.

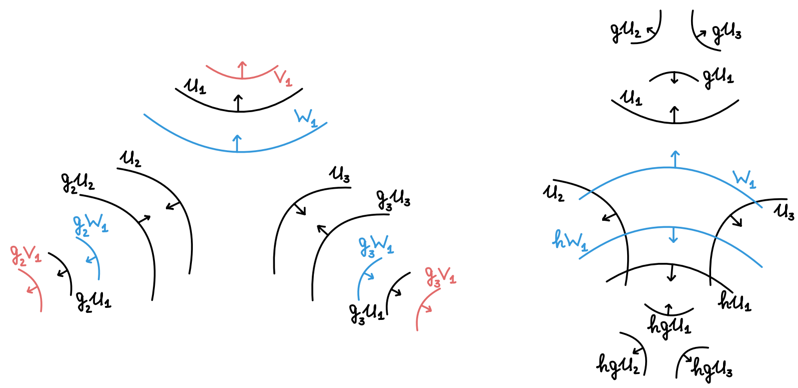

FIGURE 1. Left: Strongly separated facing triple $\{V_1, g_2V_1, g_3V_1\}$, when $U_2, U_3 \subseteq W_1^-$. Right: The facing triple $\{U_1, hgU_2, hgU_3\}$ plays the role of $\{U_1, U_2, U_3\}$ on the left.

Proof. By Theorem 2.5 there exists a facing triple U_1, U_2, U_3 of hyperplanes in $\widetilde{X}$. Let U_i^+ be the halfspace of U_i which does not contain the other two hyperplanes. By Proposition 2.2 there exists a pair of strongly separated hyperplanes V_1, W_1 such that $V_1^+ \subseteq U_1^+ \subseteq W_1^+$.

First consider the case where $U_2, U_3 \subseteq W_1^-$, as in the left of Figure 1. For $i = 2, 3$, let $g_i \in G$ be an element flipping U_1, provided by Lemma 2.3, i.e. $g_iU_i^- \subseteq U_i^+$. We claim that $\{V_1, g_2V_1, g_3V_1\}$ is a strongly separated facing triple. For symmetry, let $g_1 = id$, so we can write $V_1 = g_1V_1$. Then for $\{i, j\} \subseteq \{1, 2, 3\}$ we have

$$g_iV_1^+ \subseteq g_iW_1^+ \subseteq g_iU_i^- \subseteq g_jU_j^+ \subseteq g_jW_j^- \subseteq g_jV_j^-.$$

This shows that g_iV_i and g_jV_j are strongly separated, since g_iV_1 and g_1W_1 are.

Now consider the remaining case, see the right diagram in Figure 1. We will construct a new triple of hyperplanes $\{U_1, U_2', U_3'\}$ where $U_2', U_3' \subseteq W_1^-$. First let $g \in G$ be an element that flips U_1, i.e. $gU_1^- \subseteq U_1^+$. We also have $gU_i^+ \subseteq gU_1^- \subseteq U_1^+$ for $i = 2, 3$. Now let $h \in G$ be an element flipping W_1, i.e. $hW_1^+ \subseteq W_1^-$. We claim that $\{U_1, hgU_2, hgU_3\}$ is a facing triple of hyperplanes such that $hgU_i^+ \subseteq W_1^-$. Clearly hgU_2, hgU_3 are disjoint as images of disjoint hyperplanes U_2, U_3 under hg. Moreover, for $i = 2, 3$ we have $hgU_i^+ \subseteq hgU_1^- \subseteq hU_1^+ \subseteq hW_1^+ \subseteq W_1^- \subseteq U_1^-$. □

To summarize, we have the following.

Corollary 2.7. Let X be a finite-dimensional nonpositively curved cube complex such that $\pi_1 X$ is nontrivial and acts essentially without a fixed point at infinity. The following are equivalent.

(1) X is a non-product,

(2) there is pair of strongly separated hyperplanes in $\widetilde{X}$.
(3) there is a facing triple of pairwise strongly separated hyperplanes in $\widetilde{X}$.

Proof. The implication (3) $\to$ (2) is obvious, the implication (1) $\to$ (3) holds by Proposition 2.6, and the implication (2) $\to$ (1) is part of Proposition 2.2. □

3. Superconvex pseudographs

A map $\varphi : Y \to X$ between nonpositively curved cube complexes is a *local isometry* if φ is locally injective, maps open cubes homeomorphically to open cubes, and whenever a, b are concatenable edges of Y, if $\varphi(a)\varphi(b)$ is a subpath of the attaching map of a 2-cube of X, then ab is a subpath of a 2-cube in Y. If $Y \to X$ is a local isometry, then its lift $\widetilde{Y} \to \widetilde{X}$ is an embedding.

Definition 3.1 (Superconvex Pseudograph)**.** A *pseudograph* is a nonpositively curved cube complex whose immersed hyperplanes are contractible. Note that $\pi_1 Y$ is free when Y is a compact pseudograph [Wis21, Lem 9.9]. A compact pseudograph Y has *rank* n if $\pi_1 Y$ is a free group of rank n. We will always assume that pseudographs are compact without stating it explicitly.

A local isometry $Y \to X$ is *superconvex* if $\widetilde{Y}$ is convex and for any bi-infinite geodesic $\tilde{\gamma}$ in $\widetilde{X}$, if $\tilde{\gamma} \subset \mathcal{N}_r(\widetilde{Y})$ for some $r > 0$ then $\tilde{\gamma} \subset \widetilde{Y}$, where $\mathcal{N}_r(\widetilde{Y})$ denotes the closed r-neighborhood of $\widetilde{Y}$ in $\widetilde{X}$. When X is locally-finite and Y is compact, this is equivalent to having an upper bound on the length ℓ of a combinatorial strip $[0,1] \times [0,\ell] \subseteq \widetilde{X}$ such that $\{0\} \times [0,\ell] \subseteq \widetilde{Y}$ but $[0,1] \times [0,\ell] \not\subseteq \widetilde{Y}$. See [Wis21, Lem 2.40].

Non-example 3.2. Let X be a product of non-tree finite graphs $Y_1 \times Y_2$ with the natural cubical structure. Then $Y_i \to X$ is a local isometry of a pseudograph but is not superconvex.

Remark 3.3. Given a superconvex rank 2 pseudograph $Y \to X$, any local isometry $W \to Y$ provides a superconvex pseudograph $W \to X$. See Lemma 5.1.

The goal of this section is the following:

Proposition 3.4. Let X be a finite-dimensional nonpositively curved cube complex, where $\pi_1 X$ is nontrivial and acts essentially, without a fixed point at infinity. The following statements are equivalent:

(1) X is a non-product.
(2) $\widetilde{X}$ has a pair of strongly separated hyperplanes.
(3) $\widetilde{X}$ has a facing triple of strongly separated hyperplanes.
(4) There is a local isometry $Y \to X$ of a superconvex rank 1 pseudograph.
(5) There is a local isometry $Y \to X$ of a superconvex rank 2 pseudograph.

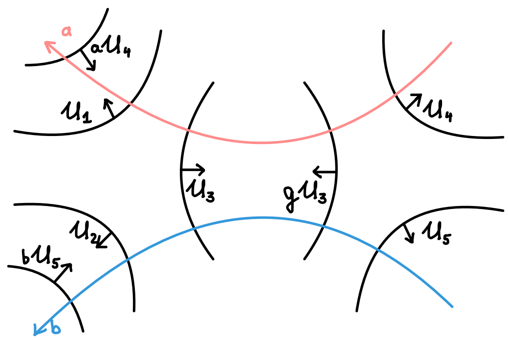

FIGURE 2. Isometries a, b form a free group.

The equivalence of (1), (2), and (3) is Corollary 2.7. We will prove (3) ⇒ (5) and (4) ⇒ (2). The implication (5) ⇒ (4) uses Remark 3.3 and the following observation: if Y is a compact cube complex and $\pi_1 Y$ is hyperbolic, then every quasiconvex, in particular cyclic, subgroup of $\pi_1 Y$ has a cocompact core in Y [Hag08].

Proof of Proposition 3.4 (3) ⇒ (5). Let $\{U_1, U_2, U_3\}$ be a strongly separated facing triple of hyperplanes in $\widetilde{X}$, and let W be the associated wedge contained in $U_1^- \cap U_2^- \cap U_3^-$.

By Lemma 2.3 applied to U_3^+ there exists $g \in G$ such that $U_3^- \subseteq gU_3^+$. Let $U_4 = gU_1$ and $U_5 = gU_2$. See Figure 2. Note that $\{U_4, U_5, U_3\}$ is also a strongly separated facing triple of hyperplanes, and let W' be the associated wedge contained in $U_4^- \cap U_5^- \cap U_3^+$. By applying Lemma 2.4 to the pairs of halfspaces $U_1^+ \subseteq U_4^-$ and $U_2^+ \subseteq U_5^-$, we obtain two isometries a, b in G, such that $aU_4^- \subseteq U_1^+$ and $bU_5^- \subseteq U_2^+$. See Figure 2. By the ping-pong lemma, $F = \langle a, b \rangle$ is a rank 2 free subgroup of G.

Let p, q be points in the wedges W, W' respectively, and let $A_{p,q}$ be a geodesic from p to q. Similarly, let $A_{p,aq}$ and $A_{p,bq}$ be geodesics from p to aq and bq respectively. Let T be an F-cocompact tree that is the union of all the F-translates of $A_{p,q}$, $A_{p,aq}$, and $A_{p,bq}$.

Let $\widetilde{Y}$ be the convex hull of T. Note that $\widetilde{Y}$ is the intersection of minor half-spaces of $\widetilde{X}$ that contain T. It is clear that $\widetilde{Y}$ is F-invariant. We claim that $\widetilde{Y}$ is F-cocompact. Firstly, note that any hyperplane of $\widetilde{Y}$ is dual to an edge of T, as otherwise, T would lie in one of its associated minor halfspaces. Secondly, each hyperplane of $\widetilde{Y}$ might intersect at most one F-translate of the wedge W' as otherwise it would have two intersect translates (by the same element from G) of more than one of the hyperplanes U_1, U_2, U_3, violating their strong separation. Let r be the maximal length of $A_{p,q}, A_{p,aq}, A_{p,bq}$. Observe that any collection of pairwise-crossing hyperplanes of $\widetilde{Y}$ are dual to edges that lie in a $2r$-ball in T. Thus $\widetilde{Y}$ has finitely many F-orbits of maximal cubes. Consequently $\widetilde{Y}$ is F-cocompact. Let $Y = \widetilde{Y}/F$, then $Y \to X$ is a local isometry. We now claim that $Y \to X$ is superconvex. An upper bound on the length ℓ of a rectangle $[0,1]\times[0,\ell]$ such that $\{0\}\times[0,\ell] \subset \widetilde{Y}$ but $[0,1]\times[0,\ell] \not\subset \widetilde{Y}$ can be taken to be $\ell = 2r$, again by the strong separation of U_1, U_2, U_3. Finally, strong separation of U_1, U_2, U_3 also ensures that all the hyperplanes of $\widehat{Y}$ are compact. In particular, the hyperplanes of Y must be simply connected, and therefore contractible. This proves that Y is a pseudograph. □

The proof of (4) ⇒ (2) uses the Helly property of CAT(0) cube complexes. This was originally stated in [Ger98], but see, for instance, [Wis21, Lem 2.10].

Lemma 3.5 (Helly property)**.** Let $\{Y_i\}$ be a finite collection of convex subcomplexes of a CAT(0) cube complex. If $Y_i \cap Y_j \neq \emptyset$ for each i, j, then $\bigcap_i Y_i \neq \emptyset$.

Proof of Proposition 3.4 (4) ⇒ (2). Let $Y \to X$ be a superconvex rank 1 pseudograph. Since the immersed hyperplanes of Y are contractible and compact, their diameter is bounded by some constant K. This also bounds the diameters of hyperplanes of $\widetilde{Y}$. Let L be the upper bound from the definition of superconvexity, i.e. for every $\ell > L$ if $[0,1] \times [0,\ell] \subseteq \widetilde{X}$ and $\{0\} \times [0,\ell] \subseteq \widetilde{Y}$, then $[0,1] \times [0,\ell] \subseteq \widetilde{Y}$. We set $M = \max\{K, L\}$. By Lemma 3.5, disjoint hyperplanes of $\widetilde{Y}$ extend to disjoint hyperplanes in $\widetilde{X}$. Thus any two hyperplanes in $\widetilde{Y}$ that are at distance greater than M in $\widetilde{Y}$ are strongly separated in $\widetilde{X}$. □

The following direct proof of (5) ⇒ (3) is illustrative.

Proof of Proposition 3.4 (5) ⇒ (3). Let $Y \to X$ be a superconvex rank 2 pseudograph. Let K be the upper bound on the diameters of hyperplanes of $\widetilde{Y}$. Let L be the upper bound from the definition of superconvexity. We set $M = \max\{K, L\}$.

By Lemma 3.5, disjoint hyperplanes of $\widetilde{Y}$ extend to disjoint hyperplanes in $\widetilde{X}$. There exists a finite cover of Y where all hyperplanes are embedded. By possibly passing to a superconvex subcomplex of that finite cover, we can assume Y is a rank 2 superconvex pseudograph with embedded hyperplanes.

Let U be a non-separating hyperplane of Y, let $Y' = Y - N(U)$ and note that $\pi_1 Y' \cong \mathbb{Z}$. Consider a cyclic cover $\widehat{Y}'$ of Y' whose degree m is sufficiently large, and let N_0 be a lift of $N(U)$.

We claim there is a non-separating hyperplane U' of $\widehat{Y}'$ (i.e. so that $\pi_1(\widehat{Y}' - U') = 1$) such that $\mathsf{d}_{\widetilde{Y}}(U', N_0) > M$. Indeed Y' and therefore also the universal cover $\widetilde{Y}'$ are locally finite, but the diameter of $\widetilde{Y}'$ is infinite. Thus if the degree $m > M$, then there exist hyperplanes at distance greater than M away from N_0, any such non-separating hyperplane can be picked as U'. In particular U' does not intersect N_0.

Let $\widehat{Y}$ be the cover of Y corresponding to $\widehat{Y}'$, i.e. so that $\mathrm{Aut}(\widehat{Y} \to Y)$ is naturally isomorphic to $\mathrm{Aut}(\widehat{Y}' \to Y')$. Consider lifts of U, U' to $\widehat{Y}$, where $\partial N(U)$ naturally corresponds to N_0. Then the lift of $\widehat{Y} - U - U'$ to $\widetilde{X}$ provides a facing quadruple of hyperplanes in $\widetilde{X}$. Indeed, $\widehat{Y} - U - U'$ embeds in $\widetilde{Y}$ and is bounded by a quadruple of disjoint hyperplanes U_1, U_2, U_1', U_2'. These hyperplanes extend to disjoint hyperplanes of $\widetilde{X}$, which we continue to denote by U_1, U_2, U_1', U_2'. We now show that U_1, U_1', U_2' are pairwise strongly separated. Note that the distance between any two of U_1, U_1', U_2' is bounded below by M.

Let V and V' be two hyperplanes that intersect $\widetilde{Y}$. We claim that if there exists a hyperplane W in $\widetilde{X}$ that intersects both V and V', then V and V' are already intersected by another hyperplane in $\widetilde{Y}$.

Consider a disc diagram D containing segments of hyperplanes V, V', W whose boundary path is a concatenation of four paths: one lying in $\widetilde{Y}$, and three paths in the carriers $N(V), N(V'), N(W)$ respectively. We assume that D has minimal area among all the choices of such disc diagrams. By convexity of $N(W), N(V), N(V'), \widetilde{Y}$ and minimality of D, we can deduce that D is a grid. Indeed any hyperplane in D starting in $\widetilde{Y}$ must exit D in $N(W)$, and vice-versa, as otherwise we can pick a disc diagram of smaller area. This implies that for the hyperplane W' which intersects V and V closest to $\widetilde{Y}$, the intersection $D \cap N(W')$ is of the form $[0,1] \times [0,\ell]$, where $\{0\} \times [0,\ell] \subseteq \widetilde{Y}$. By superconvexity of $\widetilde{Y}$, either W' intersects V, V' in $\widetilde{Y}$, or $\ell \leq L$.

Thus if $\mathsf{d}_{\widetilde{X}}(V, V') > L$, then V, V' are either strongly separated, or they are both intersected by some hyperplane of $\widetilde{Y}$. Since the diameter of any hyperplane of $\widetilde{Y}$ is bounded above by K, so the hyperplanes V, V' must be strongly separated, if we also have $\mathsf{d}_{\widetilde{X}}(V, V') > K$. □

4. Cubical small-cancellation review

4.a. **Cubical presentations and small-cancellation conditions.** A *cubical presentation* $\langle X \mid \{Y_i\} \rangle$ consists of the following data:

(1) A connected nonpositively curved cube complex X.
(2) A collection of local isometries of connected nonpositively curved cube complexes $Y_i \xrightarrow{\varphi_i} X$.

Local isometries of nonpositively curved cube complexes are π_1-injective, and we define the *fundamental group* of $\langle X \mid \{Y_i\} \rangle$ as $\pi_1 X / \langle\langle \{\pi_1 Y_i\} \rangle\rangle$. Note that this group is isomorphic to the fundamental group of the space X^* obtained by coning off each Y_i in X. Throughout, we write $X^* = \langle X \mid \{Y_i\} \rangle$ and identify a cubical presentation with its coned-off space. A cubical presentation X^* is *compact* if both X and all Y_i's are compact. The universal cover $\widetilde{X}^*$ of the coned-off space X^* is $\hat{X} \cup \bigcup Cone(gY_i)$ where $\hat{X}$ is the covering space of X

associated to the kernel of the map $\pi_1 X \to \pi_1 X^*$, and $g \in \pi_1 X^*$. We refer to $\hat{X}$ as the *cubical part* of $\widetilde{X}^*$.

An *abstract contiguous cone-piece* of Y_j in Y_i is an intersection $\widetilde{Y}_j \cap \widetilde{Y}_i$ where either $i \neq j$ or where $i = j$ but $\widetilde{Y}_j \neq \widetilde{Y}_i$. A *cone-piece* of Y_j in Y_i is a path $p \to P$ in an abstract contiguous cone-piece of Y_j in Y_i. An *abstract contiguous wall-piece* of Y_i is a non-empty intersection $N(H) \cap \widetilde{Y}_i$ where $N(H)$ is the carrier of a hyperplane H that is disjoint from $\widetilde{Y}_i$. A *wall-piece* of Y_i is a path $p \to P$ in an abstract contiguous wall-piece of Y_i. A *piece* is either a cone-piece or a wall-piece.

Recall that the *systole* of Y, denoted by $\operatorname{sys}(Y)$, is the infimum of lengths of essential combinatorial paths in Y. For a constant $\alpha > 0$, the cubical presentation $X^* = \langle X \mid \{Y_i\} \rangle$ satisfies the $C'(\alpha)$ *small-cancellation* condition if $\operatorname{diam}(P) < \alpha \operatorname{sys}(Y_i)$ for every piece P in Y_i.

As in the case of classical small-cancellation theory, when α is small, the $C'(\alpha)$ condition provides control over $\pi_1 X^*$. This is explained in [Wis21] at $\alpha = \frac{1}{12}$, and in [Jan17] at $\alpha = \frac{1}{8}$.

Let $Y \to X$ be a local isometry. $\operatorname{Aut}_X(Y)$ is the group of automorphisms $\psi : Y \to Y$ such that the diagram below is commutative:

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & Y \\ & \searrow & \downarrow \\ & & X \end{array}$$

If Y is simply connected, then $\operatorname{Aut}_X(Y)$ is equal to $\operatorname{Stab}_{\pi_1 X}(Y)$.

We adopt the convention that two elevations $\widetilde{Y}_j, \widetilde{Y}_i$ with $i = j$ are regarded as equal if they differ only by an element of $\operatorname{Aut}_X(Y)$.

5. Small-cancellation via the pseudograph

In this section we prove the first part of Theorem 1.4. Throughout this section we assume X is a nonpositively curved cube complex, which admits a local isometry $Y \to X$ of a superconvex rank 2 pseudograph.

The strategy of the proof is as follows. In Section 5.b, we construct a further superconvex rank 2 pseudograph whose fundamental group is malnormal in G. Then in Section 5.c we use it to pick a pseudograph so that the cubical presentation with the pseudograph as a relator satisfies the small-cancellation conditions, which also uses the uniform bound on the size of wall-pieces obtained in Section 5.a. The proof of the first part of Theorem 1.4, combining these ingredients, is in Section 5.d.

5.a. **Uniform bound on the size of wall-pieces.**

Lemma 5.1. Let $Y \to X$ be a superconvex pseudograph. There is $M' > 0$ such that for any local-isometry $W \to Y$, any wall-piece of $W \to X$ has diameter at most M'. Hence, $W \to X$ is itself superconvex.

Proof. For $W \to Y$, any wall-piece of $W \to X$ maps to either a hyperplane of $\widetilde{Y}$ or to a wall-piece of $Y \to X$. Both of these have uniformly bounded diameter.

Indeed, in the first case, the wall-piece is uniformly bounded because Y is a pseudograph (in particular, compact) and therefore has finitely many hyperplanes, all of which are contractible. In the second case, the wall-piece is uniformly bounded because Y is compact and superconvex. □

5.b. **Passing to a malnormal free subgroup.**

Definition 5.2 (fiber-product)**.** Let $f : Y \to X$ be a map. The *fiber-product* $Y \otimes_X Y$ is the subspace of $Y \times Y$ consisting of the preimage of the diagonal $\{(x, x) : x \in X\}$ using the map $f \times f : Y \times Y \to X \times X$. The *diagonal* component of $Y \otimes_X Y$ is the subspace $\{(y, y) : y \in Y\}$. The projection maps $Y \leftarrow Y \times Y \to Y$ induce projection maps $Y \leftarrow Y \otimes_X Y \to Y$.

When f is a local-isometry map between nonpositively curved cube complexes, $Y \otimes_X Y$ is a nonpositively curved cube complex, and the projection maps are local-isometries. Concretely, the i-cubes of $Y \otimes_X Y$ are pairs of i-cubes in Y that map to the same i-cube of X.

Lemma 5.3. No non-diagonal component of $Y \otimes_X Y$ is a finite cover of Y.

Proof. Suppose $\widehat{Y}$ is a non-diagonal component of $Y \otimes_X Y$ that is a finite cover of Y. Let x the basepoint of X, and let y be a point of Y mapping to x. Let (y, y') be a point of $\widehat{Y}$ that left-projects to y. Since (y, y') is not in the diagonal, we see that $y' \neq y$. However, both y and y' project to x. Let κ be a path in Y from y to y'. Then $[\kappa] \in \pi_1 X$ stabilizes $\widetilde{Y}$, but $[\kappa] \notin \pi_1 Y$ since its based lift to Y is not closed. Thus $F \subsetneq \bar{F} = \langle F, [\kappa] \rangle$. Let $\bar{Y} = \bar{F} \backslash \widetilde{Y}$. Then there is a proper covering map $Y \to \bar{Y}$. Let d be the degree of this proper covering map, and note $1 < d$ by properness and $d < \infty$ since Y is compact. However, $-1 = \chi(Y) = d \cdot \chi(\bar{Y}) \neq -1$, which is impossible. □

Proposition 5.4. Let $Y \to X$ be a superconvex rank 2 pseudograph. There exists a local isometry $W \to Y$ where W is a superconvex rank 2 pseudograph, such that no nontrivial element of $\pi_1 W$ is conjugate to an element in π_1 of a non-diagonal component of $Y \otimes_X Y$.

Moreover, we can assume that $\pi_1 W$ is malnormal in $\pi_1 Y$.

Proof. Let $K_1, \ldots, K_m$ be the fundamental groups of the non-diagonal components of $Y \otimes_X Y$. Each K_i is finitely generated by compactness. By Lemma 5.3, no non-diagonal component of $Y \otimes_X Y$ is a finite cover of Y. Thus each K_i is an infinite index subgroup of the free group $\pi_1 Y$. Let $H \subset \pi_1 Y$ be the subgroup provided by Lemma 5.5. Let $W \to Y$ be a local-isometry with W compact such that $\pi_1 W = H$. Then $W \to Y$ satisfies the statement of the lemma. □

Above we used the following statement about subgroups of free groups. This was also noted in [Kap99, Thm D].

Lemma 5.5 (A free group lemma)**.** Let F be a free group on at least two generators, and let $K_1, \ldots, K_n$ be finitely generated infinite index subgroups. There exists a subgroup $H \subseteq F$ isomorphic to F_2 such that no nontrivial element of H is conjugate to an element of $\cup_i K_i$.

Moreover, we can assume H is malnormal in F.

We will use the graphical viewpoint on subgroups of free groups popularized by Stallings [Sta83].

Proof. Regard F as $\pi_1 B$ where B is a bouquet of circles. For each K_i let $B_i \to B$ be an immersion of a finite based graph with $\pi_1 B_i$ mapping to K_i.

We will produce two immersed cycles $\sigma_1 \to B$ and $\sigma_2 \to B$, such that $\sigma_1 \vee \sigma_2$ maps to B by an immersion, and such that the based path $\sigma_j \to B$ does not have a lift to B_i for any i, j. The result follows letting $H = \sigma_1 \vee \sigma_2$.

It suffices to produce one such path σ, since we can precompose and postcompose it with edges to produce another immersed cycle with the same property, and such that their wedge immerses in B.

Let $C = \cup_i B_i$. Let $\{v_1, \ldots, v_s\}$ be the vertices of C. Let $C^+ \to B$ be the covering map obtained by adding finitely many trees to C. A path μ *exits* C, if μ traverses an edge of one of these trees.

We produce σ through the following recursion: Let $\mu_0 \to B$ be the trivial path. For each $0 \leq m \leq s-1$, consider the lift of μ_m at v_m, and let λ_{m+1} be a (possibly trivial) path such that $\mu_{m+1} := \mu_m \lambda_{m+1}$ exits C. The path $\sigma = \mu_s$ has the desired property by construction.

For the malnormality, suppose U, V are closed paths in B such that $U \vee V$ immerses in B. Let $U' = UVUV^2UV^3 \cdots UV^nU$ and $V' = VUVU^2 \cdots VU^nV$. Then $U' \vee V'$ immerses in B, and $U' \vee V' \to B$ is small-cancellation and hence malnormal for $n \gg 0$. See [Wis01, Thm 2.14]. □

5.c. **Induced small-cancellation conditions.**

Proposition 5.6. Let $W \to X$ be a superconvex pseudograph such that the non-diagonal components of $W \otimes_X W$ are contractible (equivalently, $\pi_1 W$ is malnormal in $\pi_1 X$).

Let B be a graph and $B \to W$ be a combinatorial homotopy equivalence.

There exist $\kappa, \epsilon > 0$, and $\beta = \beta(\kappa, \epsilon) > 0$, such that for any immersion $A \to B$, there exists a local isometry $Z \to W$ such that

- $\text{sys}(Z) \geq \frac{1}{\kappa}\,\text{sys}(A) - \epsilon$,
- if $\langle B \mid A \rangle$ is $C'(\alpha)$ then $\langle X \mid Z \rangle$ is $C'(\beta\alpha)$.

We use the notation $\mathcal{N}_r(S)$ for the closed r-neighborhood of S.

Proof. The map $B \to W$ lifts to a (κ, ϵ)-quasiisometry $\phi : \widetilde{B} \to \widetilde{W}$ for some $\kappa, \epsilon > 0$. The correspondence between (compact) immersions $A \to B$ and (compact) immersions $Z \to W$ arises by letting Z be the quotient of $\widetilde{Z}$ by the action of $\pi_1 A$, where $\widetilde{Z}$ is the convex hull of $\phi(\widetilde{A})$. We note that there is a uniform constant r with $\widetilde{Z} \subset \mathcal{N}_r(\phi(\widetilde{A}))$. We refer to [Hag08, Thm 2.28] and [SW15, Prop 3.3], and note that the constant μ in the proof of the second reference is uniform since $\phi(\widetilde{A})$ is uniformly quasiconvex.

There is a uniform linear relationship between the systole of Z and the systole of A, namely $\frac{1}{\kappa}\operatorname{sys}(A) - \epsilon \leq \operatorname{sys}(Z) \leq \kappa \operatorname{sys}(A) + \epsilon$. By Lemma 5.1 there is a uniform bound on diameters of wall-pieces of $Z \to X$.

A cone-piece U in $Z \to X$ corresponds to a pair of maps $Z \leftarrow U \to Z$ which composes to a pair of maps $W \leftarrow Z \leftarrow U \to Z \to W$. The universal property of the fiber-product provides a lift of U to $W \otimes_X W$. The non-diagonal components of $W \otimes_X W$ have uniformly bounded diameter, since they are compact and contractible by assumption. Thus there is a uniform bound on the diameter of pieces U mapping to the non-diagonal components of $W \otimes_X W$.

Suppose U maps to the diagonal component of $W \otimes_X W$ and so U lifts to a component of $Z \otimes_W Z$. Then U is a piece in $Z \to W$, i.e. a component of the intersection between translates $g\widetilde{Z}$ and $\widetilde{Z}$ in $\widetilde{W}$. There exists a corresponding piece V in $A \to B$, which is the intersection between $g\widetilde{A}$ and $\widetilde{A}$ in $\widetilde{B}$. We now claim that $\operatorname{diam}(V) \geq \kappa' \operatorname{diam}(U) - \epsilon'$ for some $\kappa', \epsilon' > 0$.

For $a_1, a_2 \in \widetilde{A}$, if $\mathsf{d}(\phi(a_1), \phi(a_2)) < r$ then $\mathsf{d}(a_1, a_2) < \kappa r + \epsilon$. Suppose $\operatorname{diam}(\widetilde{Z} \cap g\widetilde{Z}) = M$, then there are points w_1, w_2 inside with $\mathsf{d}(w_1, w_2) = M$. Let a_i be a point in $\widetilde{A}$ with $\mathsf{d}(\phi(a_i), w_i) < r$, and let a_i' be a point in $g\widetilde{A}$ such that $\mathsf{d}(\phi(a_i'), w_i) < r$. In particular, $\mathsf{d}(a_i, a_i') \leq \kappa(2r) + \epsilon\kappa$. On the other hand $\mathsf{d}(\phi(a_1), \phi(a_2)) \geq M - 2r$, so $\mathsf{d}(a_1, a_2) \geq \frac{1}{\kappa}(M - 2r) - \frac{\epsilon}{\kappa}$.

Note that a_1, a_2 belong to the convex subset $\widetilde{A}$ of the tree $\widetilde{B}$, and so the unique path $[a_1, a_2] \subseteq \widetilde{A}$. Similarly $[a_1', a_2'] \subseteq g\widetilde{A}$. Since points a_i, a_i' are close for both $i = 1, 2$, and points a_1, a_2 are far away, it follows that there exist points $b_i \in [a_i, a_i']$ with $[b_1, b_2] \subseteq \widetilde{A} \cap g\widetilde{A}$. We have $\mathsf{d}(b_i, a_i) \leq \kappa(2r) + \epsilon\kappa$, and therefore

$$\mathsf{d}(b_1, b_2) \geq \mathsf{d}(a_1, a_2) - \mathsf{d}(b_1, a_1) - \mathsf{d}(b_2, a_2) \geq \frac{1}{\kappa}(M - 2r) - \frac{\epsilon}{\kappa} - 2(\kappa(2r) + \epsilon\kappa).$$

Thus $\operatorname{diam}(V) \geq \kappa' \operatorname{diam}(U) - \epsilon'$ for $\kappa' = \frac{1}{\kappa}$, and $\epsilon' = \frac{2r+\epsilon}{\kappa} + 2\kappa(2r + \epsilon)$.

To summarize, every piece U in W either has diameter bounded by some universal constant, or satisfies

$$\operatorname{diam}(U) \leq \frac{1}{\kappa'}\operatorname{diam}(V) + \frac{\epsilon'}{\kappa'} \leq \frac{1}{\kappa'}\alpha \operatorname{sys}(A) + \frac{\epsilon'}{K'} \leq \frac{1}{\kappa'}(\alpha\kappa \operatorname{sys}(W) + \epsilon) + \frac{\epsilon'}{\kappa'}.$$

In either case, we get that $\operatorname{diam}(U) \leq \beta\alpha \operatorname{sys}(W)$ for some $\beta > 0$ which depends on $W \to X$ and $B \to W$ only. □

Remark 5.7. For each $\alpha > 0$, there exists $A \to B$ where $\langle B \mid A \rangle$ is $C'(\alpha)$ and $\operatorname{rank}(A) = 2$. For instance, if B is a bouquet of circles labelled by a and b, choose A associated to $\langle aba^2b \cdots a^m b, bab^2a \cdots b^n a \rangle$ for sufficiently large m, n.

Proposition 5.6 and Remark 5.7 imply the following.

Corollary 5.8. Let $W \to X$ be a superconvex pseudograph where the non-diagonal components of $W \otimes_X W$ are contractible. For any $\alpha > 0$ there is a local isometry of superconvex rank 2 pseudograph $Z \to W \to X$ such that $\langle X \mid Z \rangle$ is $C'(\alpha)$. Moreover, for any $R > 0$ we can choose Z so that $\operatorname{sys}(Z) \geq R$.

To finish this section we note that Proposition 5.6 does not assume that A and Z are connected. Thus we can thus deduce the following improvement, by choosing appropriate $A_1, A_2 \to B$ as in Remark 5.7 whose cancellations with one another are small.

Corollary 5.9. Let $W \to X$ be a superconvex pseudograph such that the non-diagonal components of $W \otimes_X W$ are contractible. Then for every $\alpha, R > 0$ there exists a pair of superconvex rank 2 pseudographs $Z_1 \to W \to X$ and $Z_2 \to W \to X$ with $\operatorname{sys}(Z_i) \geq R$ and $\langle X \mid Z_1, Z_2 \rangle$ satisfying $C'(\alpha)$.

5.d. **Proof of a proper $C'(\alpha)$ small-cancellation quotient.** We are now ready to prove the first part of our main theorem.

Theorem 5.10 (Theorem 1.4 without cubulation)**.** Let X be a nonpositively curved cube complex that admits a local isometry of a rank 2 superconvex pseudograph. Then for every $\alpha > 0$ there exists $Y \looparrowright X$ with $\pi_1 Y \neq 1$ such that $\langle X \mid Y \rangle$ is a cubical $C'(\alpha)$ small-cancellation presentation.

Proof. Proposition 5.4 provides a local isometry $W \to Y$ so that all non-contractible components of $W \otimes_X W$ map to the diagonal component of $Y \otimes_X Y$. Since we can choose W such that $\pi_1 W$ is malnormal in $\pi_1 Y$, we can assume that all non-diagonal components of $W \otimes_X W$ are contractible. By Corollary 5.8 there is $Z \to W$ such that $\langle X \mid Z \rangle$ is a $C'(\alpha)$ small-cancellation quotient. □

Recall that by possibly passing to the $\pi_1 X$-invariant subcomplex of the cubical subdivision of $\widetilde{X}$, we can assume $\pi_1 X$ acts essentially on $\widetilde{X}$ by [CS11, Prop 3.5]. Then a rank 2 superconvex pseudograph $Y \to X$ exists by Proposition 3.4.

Similarly as in Theorem 5.10, we deduce a version of SQ-universality for the fundamental groups of non-product nonpositively curved cube complexes.

Proposition 5.11 (Subgroup Quotient Universal)**.** Fix $\alpha > 0$. There exists $R = R(\alpha)$ such that the following holds. Suppose X admits a local isometry from a rank 2 pseudograph $Z \to X$ such that

- wall pieces are uniformly bounded,
- non-diagonal components of $Z \otimes_X Z$ are contractible,
- $\text{sys}(Z) \geq R$.

Then for every rank 2 group H, there is a $C'(\alpha)$ cubical small-cancellation quotient $X^* = \langle X \mid \widehat{Z} \rangle$ with $H \hookrightarrow \pi_1 X^*$ where $\widehat{Z} \to Z$ is a covering map.

Compactness is not assumed for $\widehat{Z}$. The existence of Z is ensured by Lemma 5.1, Proposition 5.4, and Corollary 5.8. The same proof works for rank m, but anyhow, every countable group is a subgroup of a quotient of a rank 2 free group.

Proof. Choose a regular cover $\widehat{Z} \to Z$ with $H \cong \text{Aut}(\widehat{Z} \to Z)$. Every cone-piece of $\widehat{Z} \to X$ is a cone-piece of $Z \to X$. But $\text{sys}(\widehat{Z}) \geq \text{sys}(Z)$. Finally, since $\widehat{Z} \subset \widetilde{X}^*$, then $H \cong \text{Aut}(\widehat{Z} \to Z) \subset \text{Aut}(\widetilde{X}^* \to X) = \pi_1 X^*$. Indeed every automorphism of $\widehat{Z} \to Z$ extends to an automorphism of $\widetilde{X}^* \to X$. □

6. Disc diagrams and more cubical small-cancellation

A *disc diagram* D is a compact contractible combinatorial 2-complex, together with an embedding $D \hookrightarrow S^2$. The *boundary path* ∂D is the attaching map of the 2-cell at infinity.

Similarly, an *annular diagram* A is a compact combinatorial 2-complex homotopy equivalent to S^1, together with an embedding $A \hookrightarrow S^2$, which induces a cellular structure on S^2. The *boundary paths* $\partial_{int} A$ and $\partial_{out} A$ of A are the attaching maps of the two 2-cells in this cellulation of S^2 that do not correspond to cells of A.

A *disc (resp. annular) diagram in* X^* is a combinatorial map $(D, \partial D) \to (X^*, X^1)$ of a disc diagram (resp. $(A, \partial A) \to (X^*, X^1)$). A *square disc (resp. annular) diagram* is a disc (resp. annular) whose image is entirely contained in X (without cones).

The 2-cells of a disc diagram D in X^* are of two kinds: squares mapping onto squares of X, and triangles mapping onto cones over edges contained in Y_i. The vertices in D which are mapped to the cone-vertices of X^* are also called the *cone-vertices*. Triangles in D are grouped into cyclic families meeting around a cone-vertex. We refer to such families as *cone-cells*, and treat a whole such family as a single 2-cell.

Let D be a disc diagram in X^*. The *square part* $D_\square$ of D is the union of all the squares in D that are not contained in any cone-cells.

The *complexity* of a disc diagram D in X^* is defined as

$$\text{Comp}(D) = (\#\text{cone-cells in } D, \#\text{squares in } D_\square).$$

We say that D has *minimal complexity* if $\text{Comp}(D)$ is minimal in the lexicographical order among disc diagrams with the same boundary path as D. A disc diagram D in X^* is *degenerate* if $\text{Comp}(D) = (0, 0)$. A disc diagram D, in X^* is *singular* if D is not homeomorphic to a closed ball in $\mathbb{R}^2$. This is equivalent to D either being a single vertex or an edge, or containing a cut vertex. In particular, every degenerate disc diagram is singular.

Definition 6.1 (n-collared)**.** A disc diagram $D \to X^*$ is *collared* by an annular diagram $A \to D \to X^*$ if there is a subdiagram $D_0 \subseteq D$ such that

$$D = A \bigsqcup_{\partial_{in} A = \partial D_0} D_0.$$

Note that if $\partial_{in} A$ does not embed in D, then D_0 is singular. Also $\partial D = \partial_{out} A$.

If A decomposes as a union of n ladders $L_1, \ldots, L_n$, and each cone-cell of each L_i intersects $\partial_{out} A$ and $\partial_{in} A$ nontrivially, then D is *n-collared* by A. We refer to A as the *collar* of D, to $L_1, \ldots, L_n$ as *collaring ladders*.

6.a. **Greendlinger's Lemma.** A cone-cell C in a disc diagram D is a *boundary cone-cell* if C intersects the boundary ∂D along at least one edge. A non-disconnecting boundary cone-cell C is a *shell of degree k* if $\partial C = RQ$ where Q is the maximal subpath of ∂C contained in ∂D, and k is the minimal number such that R can be expressed as a concatenation of k pieces. We refer to R as the *innerpath* of C and Q as the *outerpath* of C. A shell of degree ≤ 4 is an *exposed shell*.

A *corner of a square* in a disc diagram D is a vertex v in ∂D of valence 2 in D that is contained in some square of D. A *cornsquare* is a square c and a pair of dual curves emanating from consecutive edges a, b of c that terminate on consecutive edges a', b' of ∂D. We abuse notation and refer to the common vertex of a', b' as a cornsquare as well. A *spur* is a vertex in ∂D of valence 1 in D. If D contains a spur or a cut-vertex, then D is *singular*.

A *pseudo-grid* between paths μ and ν is a square disc diagram E where the boundary path ∂E is a concatenation $\mu\rho\overline{\nu}\overline{\eta}$ where each $\mu, \rho, \overline{\nu}, \overline{\eta}$ is a called a *side*, and such that

(1) each dual curve starting on μ ends on ν, and vice versa,
(2) no pair of dual curves starting on μ cross each other,
(3) no pair of dual curves cross each other twice.

If a pseudo-grid E is degenerate then either $\mu = \nu$ or $\rho = \eta$. A *grid* is a pseudogrid isometric to the product of two intervals.

A *ladder* is a disc diagram $(D, \partial D) \to (X^*, X^0)$ which is an alternating union of cone-cells and/or vertices $C_0, C_2 \ldots, C_{2n}$ and (possibly degenerate) pseudo-grids $E_1, E_3 \ldots, E_{2n-1}$, with $n \geq 0$, in the following sense:

(1) the boundary path ∂D is a concatenation $\lambda_1\overline{\lambda_2}$ where the initial points of λ_1, λ_2 lie in C_0, and the terminal points of λ_1, λ_2 lie in C_{2n},
(2) $\lambda_1 = \alpha_0\rho_1\alpha_2 \cdots \alpha_{2n-2}\rho_{2n-1}\alpha_{2n}$ and $\lambda_2 = \beta_0\eta_1\beta_2 \cdots \beta_{2n-2}\eta_{2n-1}\beta_{2n}$,

(3) the boundary path $\partial C_i = \nu_{i-1}\alpha_i\overline{\mu_{i+1}}\overline{\beta_i}$ for some ν_{i-1} and μ_{i+1} (where ν_{-1} and μ_{2n+1} are trivial), and
(4) the boundary path $\partial E_i = \mu_i\rho_i\overline{\nu_i\eta_i}$.

The cubical version of Greendlinger's Lemma will be used in Section 9.a. See [Wis21, Thm 3.46] and [Jan17, Thm 2] for the proof.

Lemma 6.2 (Cubical Greendlinger's Lemma)**.** Let $X^* = \langle X \mid Y_1, \ldots, Y_s\rangle$ be a cubical presentation satisfying the $C(9)$ condition. Let $D \to X^*$ be a minimal complexity disc diagram. Then one of the following holds:

- D is a ladder, or
- D has at least three exposed shells and/or corners of squares and/or spurs.

Theorem 6.3 (The Ladder Theorem)**.** Let $X^* = \langle X \mid Y_1, \ldots, Y_s\rangle$ be a cubical presentation satisfying the $C(9)$ condition and let $D \to X^*$ be a minimal complexity disc diagram in X^*. If D has exactly two exposed shells, then D is a ladder.

6.b. **The** $B(6)$ **condition.** We now introduce a set of conditions that provides a wallspace structure on $\widetilde{X}^*$ so that $\pi_1 X$ acts on a CAT(0) cube complex:

Definition 6.4. A cubical presentation $\langle X \mid \{Y_i\}\rangle$ satisfies the *B(6) condition* if the following conditions are satisfied:

(1) (Small-cancellation) $\langle X \mid \{Y_i\}\rangle$ satisfies the $C'(\frac{1}{\alpha})$ condition for $\alpha \geq 14$.
(2) (Wallspace Cones) Each Y_i is a wallspace where each wall in Y_i is the union $\sqcup H_j$ of a collection of disjoint embedded 2-sided hyperplanes in Y_i, and there is an embedding $\sqcup N(H_j) \to Y_i$ of the disjoint union of their carriers into Y_i. Each such collection separates Y_i. Each hyperplane in Y_i lies in a unique wall.
(3) (Hyperplane Convexity) If $P \to Y_i$ is a path that starts and ends on vertices on 1-cells dual to a hyperplane H of Y_i and P is the concatenation of at most 7 pieces, then P is path homotopic in Y_i to a path $P \to N(H) \to Y_i$.
(4) (Wall Convexity) Let S be a path in Y_i that starts and ends with 1-cells dual to the same wall of Y_i. If S is the concatenation of at most 7 pieces, then S is path-homotopic into the carrier of a hyperplane of that wall.
(5) (Equivariance) The wallspace structure on each cone over Y_i is preserved by $\mathrm{Aut}_X(Y)$.

Definition 6.5 (k-Wall Convexity)**.** We will consider a variant of Definition 6.4.(4) where 7 is replaced with k. We call it *k-Wall Convexity*.

This condition will be used in the proof of cocompactness of the action of $\pi_1 X^*$.

7. Cubulating

The purpose of this section is to set up the proof of the remaining part of Theorem 1.4, i.e. that Y can be chosen so that $\pi_1 X^*$ is cubulated, and prove some additional properties of the cubulation (Theorem 7.12). In Section 7.a we describe how to obtain the cubical structure and some of its properties. In Section 7.b we discuss induced cubical presentations, which are used in some parts of the proof of Theorem 7.12. In Section 8, we prove that Y can be chosen so that $\pi_1 X^*$ acts freely on its dual. Section 9 contains a result that implies cocompactness of the action on the dual under appropriate assumptions on X.

7.a. **Wallspace structures on pseudographs.** Let $X^* = \langle X \mid \{Y_i\}\rangle$ be a cubical presentation where each Y_i is a rank 1 pseudograph. There is a standard wallspace structure that can be put on each Y_i to obtain a wallspace structure on $\widetilde{X}^*$, which we now describe.

Construction 7.1 (Wallspace structure on rank 1 pseudographs). Let $Y \to X$ be a rank 1 pseudograph and let $\sigma \to Y$ be a closed-geodesic in Y realising the systole, so $|\sigma| = sys(Y)$. After potentially barycentrically subdividing X, we may assume that $|\sigma| = 2n$ for some n, so that there is a well-defined equivalence relation on the hyperplanes of Y where two hyperplanes that are dual to edges of σ are equivalent if and only if they are dual to antipodal edges of σ. If a hyperplane is not dual to an edge of σ, it is its own equivalence class. This equivalence relation then defines a wallspace structure on Y.

Lemma 7.2. Let $X^* = \langle X \mid \{Y_i\}\rangle$ be a cubical presentation satisfying the $C'(\frac{1}{n})$ condition for $n \geq 16$ and where each Y_i is a rank 1 pseudograph. Then X^* satisfies the $B(6)$ condition with each Y_i endowed with the wallspace structure in Construction 7.1. Thus $\pi_1 X^*$ acts on the CAT(0) cube complex dual to the wallspace structure on X^* obtained by extending the wallspace structures on the Y_i.

Versions of Lemma 7.2 are proven in [Are24c, FW24, Wis21].

With a bit of care, the ideas in Construction 7.1 can be extended to work for higher-rank pseudographs; we explain this in detail for the rank 2 case below.

Construction 7.3 (Wallspace structure on rank 2 pseudographs). Let Y be a rank 2 superconvex pseudograph and let $\pi_1 Y = \langle a, b\rangle$. Consider rank 1 pseudographs Z_1 and Z_2 corresponding to the subgroups $\langle aba^2b\cdots a^m b\rangle$ and $\langle bab^2a\cdots b^n\rangle$ for some appropriately large n, m.

There exists $r = r(Y, a, b) > 0$ such that the local isometry $Y' \to X$ associated to $\langle aba^2b\cdots a^m b, bab^2a\cdots b^n\rangle$ has the following form: There is a contractible locally convex subcomplex $K \subset Y'$ with

(1) $\mathrm{diam}(K) \leq r$ such that:

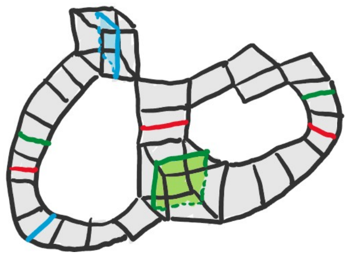

FIGURE 3. Some walls in the wallspace of Construction 7.3.

(2) Y' is the union $Z_1 \cup K \cup Z_2$.
(3) $Z_1 \cap Z_2 \subset K$.

As in the rank 1 case in Construction 7.1, we may subdivide X and consider the antipodal wallspace structure on each of Z_1 and Z_2 with respect to some choice of $\sigma_i \to Z_i$ where $|\sigma_i| = \operatorname{sys} Z_i$. We extend this to a wallspace structure on Y' as follows:

(1) Each wall of Z_1 disjoint from K is a wall of Y'.
(2) Each wall of Z_2 disjoint from K is a wall of Y'.
(3) Each hyperplane U of K extends to a wall of Y' consisting of: U, the hyperplane of Z_1 that is antipodal to $Z_1 \cap U$ (if non-empty), and the hyperplane of Z_2 that is antipodal to $Z_2 \cap U$ (if non-empty).

See Figure 3.

We prove in Theorem 5.10 that for each $\alpha > 0$ and for all for $n, m \gg 1$, $\langle X \mid Y' \rangle$ is $C'(\alpha)$, in Theorem 7.12.(3) that $\langle X \mid Y' \rangle$ satisfies the $B(6)$ condition, and in Theorem 7.12.(4) that $\langle X \mid Y' \rangle$ satisfies 25-wall convexity. In Lemma 8.4 that $\langle X \mid Y' \rangle$ satisfies the hypotheses of Theorem 8.1 with this wallspace structure on Y'. Finally we prove in 8.5 that $\langle X \mid Y' \rangle$ has torsion-free fundamental group.

Remark 7.4. The choice of $\langle aba^2b \cdots a^mb \rangle$ and $\langle bab^2a \cdots b^n \rangle$ is rather arbitrary, and the construction and proof work quite generally for a pair of small-cancellation words.

Remark 7.5. Note that in Construction 7.3 for any C we can choose Y' such that $\operatorname{sys}(Y') \geq C \operatorname{diam}(K)$. Indeed, $\operatorname{diam}(K)$ is proportional to the overlap between the words defining Z_1, Z_2.

Remark 7.6. Note that $\operatorname{Aut}_X(Y')$ is trivial in Construction 7.3. Indeed, as Y is compact, $|\operatorname{Aut}_X(Y')| < \infty$. Thus, $[\operatorname{Stab}_X(\widetilde{Y}') : \pi_1 Y'] < \infty$ and if $\operatorname{Stab}_X(\widetilde{Y}') \neq \pi_1 Y'$, then there is a local isometry $Y_0 \to X$ such that Y' is a proper, regular, finite-degree covering of Y_0. This contradicts the choice of Y'.

7.b. **Induced Cubical Presentations.**

Definition 7.7 (Induced cubical presentation)**.** Given a cubical presentation $X^* = \langle X \mid \{Y_i\}\rangle$ and a local-isometry $f : E \to X$ there is an *induced cubical presentation* $E^* = \langle E \mid \{E \otimes_X Y_i\}\rangle$. For each $i \in I$ there is an induced map $f_i : E \otimes_X Y_i \to Y_i$ such that the following diagram commutes.

$$\begin{array}{ccc} E \otimes_X Y_i & \xrightarrow{f_i} & Y_i \\ \downarrow & & \downarrow \\ E & \xrightarrow{f} & X \end{array}$$

A *map of cubical presentations* is the data (X^*, E^*, f). Note that f induces a combinatorial map $f^* : E^* \to X^*$ that sends cone-points to cone-points and the cubical part of E^* to the cubical part of X^*.

Definition 7.8 (Liftable shells in induced presentations)**.** Let X^* be a cubical presentation. Let $f : E \to X$ be a local isometry. We say that the induced presentation $f^* : E^* \to X^*$ has *liftable shells* if the following holds:

Let R be a non-replaceable (i.e. whose boundary path is not null-homotopic within its associated cone over Y_j) shell in a minimal complexity diagram $D \to X^*$ with $\partial R = QS$, where Q is its outerpath and S is its innerpath. If Q lifts to E, then so does S.

The above condition is relevant due to the following theorem.

Theorem 7.9 ([Wis21, Thm 3.68])**.** Let $f : E^* \to X^*$ be a map of cubical presentations, where X^* satisfies $C'(\frac{1}{12})$. If f has liftable shells then f is π_1-injective, and $\widetilde{E}^* \to \widetilde{X}^*$ is injective on the cubical part.

The liftable shells property in induced presentations is guaranteed by:

Lemma 7.10 ([Wis21, Lem 3.67])**.** Let $\langle X \mid \{Y_i\}\rangle$ be a $C'(\frac{1}{14})$ cubical presentation. Let $A \to X$ be a local-isometry and let A^* be the associated induced presentation. Suppose that for each i, each component of $A \otimes_X Y_i$ is either a copy of Y_i or is a contractible complex K with $\operatorname{diam}(K) \leq 12 \operatorname{sys}(Y_i)$. Then the natural map $A^* \to X^*$ has liftable shells.

The next lemma guarantees convexity.

Lemma 7.11 ([Wis21, Lem 3.74])**.** Let X^* be a $C'(\frac{1}{14})$ cubical presentation. Suppose $E^* \to X^*$ has no missing shells. Let $2\alpha + \beta \leq \frac{1}{2}$ where $\alpha, \beta > 0$. Suppose $|P|_{Y_i} < \alpha \operatorname{sys}(Y_i)$ whenever P is a cone-piece of a translate of $\widetilde{Y}_j$ in a translate of $\widetilde{Y}_i$ (so one is not contained in the other). Suppose that for any path P in the intersection of translates $\widetilde{Y}_i, \widetilde{E}$ in $\widetilde{X}$, either $|P|_{Y_i} < \beta \operatorname{sys}(Y_i)$ or $Y_i \subset E$. Then $\widetilde{E}^* \to \widetilde{X}^*$ embeds as a convex subcomplex.

7.c. **Cubulated quotients.** Let X be a nonpositively curved cube complex that is a non-product and where $\pi_1 X$ acts without a fixed point at infinity on $\widetilde{X}$. In Section 5.d we proved that for every $\alpha > 0$ there exists a local isometry $Y \hookrightarrow X$ of a superconvex pseudograph with $\pi_1 Y = F_2$ such that $\langle X \mid Y \rangle$ is a cubical $C'(\alpha)$ small-cancellation presentation. We now prove the following, which implies Theorem 1.4.

Theorem 7.12. Let X be a nonpositively curved cube complex that admits a local isometry of a rank 2 superconvex pseudograph. For every $\alpha \leq \frac{1}{16}$ there is a superconvex rank 2 pseudograph $Y \to X$ such that $X^* = \langle X \mid Y \rangle$ is $C'(\alpha)$.

The complex Y can be chosen so that the following properties hold:

(1) If S is a finite set of nontrivial elements in $\pi_1 X$, then the image $\bar{s} \in \pi_1 X^*$ is nontrivial for each $s \in S$.
(2) $\pi_1 X^*$ is not virtually cyclic.
(3) $X^* = \langle X \mid Y \rangle$ is $B(6)$. Thus, $\pi_1 X^*$ acts on the CAT(0) cube complex $\mathcal{C}$ dual to the associated wallspace on $\widetilde{X}^*$.
(4) $X^* = \langle X \mid Y \rangle$ satisfies 11-Wall Convexity.
(5) $\pi_1 X^*$ acts freely on $\mathcal{C}$.
(6) $\pi_1 X^* \backslash \mathcal{C}$ is a non-product.
(7) If X is compact then $\pi_1 X^*$ acts cocompactly on $\mathcal{C}$.

Below we prove Theorem 7.12.(1)-(4) and (6). Theorem 7.12.(5) follows from Theorem 8.5, and Theorem 7.12.(7) follows from Theorem 9.1.

Proof of Theorem 7.12.(1). Let $S = \{g_1, \ldots, g_m\}$ be nontrivial elements. Let $\tilde{x} \in \widetilde{X}$ be a basepoint. For each k, let $J_k \to \widetilde{X}$ be the convex hull of the lift $[\tilde{x}, g_k \tilde{x}]$. Since S is finite, there is $n \in \mathbb{N}$ with $S \subset B_n(\tilde{x})$ where $B_n(\tilde{x})$ is the radius n ball at $\tilde{x}$. Choosing Y so that $\text{sys}(Y) > n$ guarantees that if $X^* = \langle X \mid Y \rangle$, then each lift $J_k \to \widetilde{X}^*$ is embedded. Hence g_k is nontrivial in $\pi_1 X^*$. □

Proof of Theorem 7.12.(2). This holds by the following proposition. □

Proposition 7.13. Let $Y \to X$ be a superconvex rank 2 pseudograph. Then for any $\alpha \leq \frac{1}{14}$ there exist $Y_1, Y_2 \to Y$ such that

- the presentation $X^* = \langle X \mid Y_1 \rangle$ is $C'(\alpha)$,
- $Y_2 \to X^*$ is π_1-injective, and
- $\widetilde{Y}_2 \to \widetilde{X}^*$ is an embedding onto a convex subcomplex.

Proof. By Proposition 5.4 we can assume that the non-diagonal components of $Y \otimes_X Y$ are contractible. Let $Y_1, Y_2 \to Y \to X$ be the two superconvex pseudographs of rank 2 as in Proposition 5.9, i.e. $\langle X \mid Y_1, Y_2 \rangle$ satisfies $C'(\alpha)$. The fiber-product $Y_1 \otimes_X Y_2$ consists of contractible components. Let C be the maximal diameter of a connected components of $Y_1 \otimes_X Y_2$. By possibly replacing Y_1 with a further $Y_1' \to Y_1 \to X$ we can assume $\mathrm{sys}(Y_1) > 12C$. The presentation $X^* = \langle X \mid Y_1 \rangle$ still satisfies $C'(\alpha)$. The induced map of cubical presentations $Y_2 = Y_2^* \to X^*$ has liftable shells by Lemma 7.10 (hence has no missing shells). By Theorem 7.9 the map $Y_2 \to X^*$ is π_1-injective and $\widetilde{Y}_2 \to \widetilde{X}^*$ is an embedding. Lemma 7.11 with $\beta = \alpha$ implies $\widetilde{Y}_2 \to \widetilde{X}^*$ is an embedding of a convex subcomplex. □

Proof of Theorem 7.12.(3). We verify the $B(6)$ condition for the wallspace structure described in Construction 7.3.

Wallspace structure: By construction, each wall is a union of disjoint hyperplanes that separate Y, and all hyperplanes are 2-sided because they are contractible. Each wall is embedded because $\mathrm{diam}(N(U)) < \frac{1}{2}\,\mathrm{sys}(Y)$ for each hyperplane U in Y. Assuming moreover that $\mathrm{diam}(N(U)) < \frac{1}{4} sys(Y)$, it also follows that the disjoint union of the hyperplane carriers corresponding to hyperplanes in the same wall embeds in Y.

Hyperplane convexity: Let $P \to Y$ be a path that starts on a vertex p and ends on a vertex q, so that both p and q lie on 1-cells dual to a hyperplane U of Y, and let $\tau \to N(U)$ be a path starting on q and ending on p. Since $\langle X \mid Y \rangle$ satisfies the $C'(\frac{1}{16})$ condition, the concatenation $P\tau$ is either nullhomotopic or P is the concatenation of at least 17 pieces. In particular, if P is the concatenation of at most 7 pieces, then P is path homotopic in Y to a path $P \to N(u) \to Y$.

Wall convexity: Suppose that $\mathrm{diam}(N(U)) \leq \frac{1}{33}\,\mathrm{sys}(Y)$ and $\mathrm{diam}(K) \leq \frac{1}{33}\,\mathrm{sys}(Y)$ (which can be ensured by Remark 7.5). Let P be a path in Y that starts and ends with 1-cells dual to the same wall U of Y. If P is the concatenation of at most 7 pieces, then $|P| \leq 7 \cdot \frac{1}{16}\,\mathrm{sys}(Y)$.

If P intersects more than one hyperplane in U. First suppose that $P \to Z_i$ where Z_i is as in Construction 7.3. Then $|P| \geq \frac{1}{2}\,\mathrm{sys}(Y) - \frac{2}{33}\,\mathrm{sys}(Y)$ since the first and last edges of P are dual to hyperplanes that are also dual to antipodal edges of σ_i where σ_i is a closed path in of Z_i realizing its systole (as in Construction 7.3). Now suppose that the first edge of P belongs to Z_1 but not Z_2, and the last edge of P belongs to Z_2 but not Z_1. Then P has a an initial subpath P' that ends in K. Then, similarly as above $|P| \geq |P'| \geq \frac{1}{2}\,\mathrm{sys}(Y) - \frac{2}{33}\,\mathrm{sys}(Y)$.

Thus $\frac{1}{2} - \frac{2}{33} < 7 \cdot \frac{1}{16}$, which is a contradiction. Thus P starts and ends on the same hyperplane u in U, and by hyperplane convexity (see above), P is path-homotopic into $N(u)$.

Equivariance: This holds since $\mathrm{Aut}_X(Y)$ is trivial by the choice made in Construction 7.3. See Remark 7.6. □

Proof of Theorem 7.12.(4). The proof of wall convexity in 7.12.(3) generalizes to an arbitrary k assuming that $\alpha < \frac{1}{2k}$ and $\mathrm{sys}(Y)$ is sufficiently large compared to $\mathrm{diam}(N(U))$ and $\mathrm{diam}(K)$. □

Proof of Theorem 7.12.(6). By Proposition 7.13 there exist superconvex rank 2 pseudographs $Y_1, Y_2 \to X$ such that $Y_2 \to X^*$ is π_1-injective where $X^* = \langle X \mid Y_1 \rangle$, and $\widetilde{Y}_2 \to \widetilde{X}^*$ is an embedding as a convex subcomplex. By Part (3) we can assume X^* is $B(6)$. We will show the dual $\mathcal{C}$ of X^* is not a product. By Proposition 2.2, it suffices to show $\mathcal{C}$ has a pair of strongly separated hyperplanes.

Recall that by Lemma 5.1, the diameter of wall-pieces of Y is bounded by some constant M'. Since $Y_1 \otimes_X Y_2$ has contractible components, there exists a constant M'' bounding the diameter of the cone pieces between Y_1 and Y_2. Let $M = \max\{M', M''\}$. Note that for any $\Sigma \to Y_2$ the cone-pieces between Σ and Y_1 are bounded by M. Since Y us a compact pseudograph, there is also a bound B on the diameter of hyperplanes of Y.

Let U_1, U_2 be strongly separated hyperplanes in $\widetilde{Y}_2$ at distance greater than the maximum of $7\max\{\frac{sys(Y_1)}{\alpha}, M\}$ and $2M + B$. Consider the hyperplanes in $\widetilde{X}^*$ extending U_1, U_2, and continue to denote them by U_1, U_2. We will prove that U_1, U_2 lie in distinct walls W_1, W_2 are strongly separated (meaning W_1, W_2 do not cross and no other wall crosses both of them).

Let σ be a geodesic path connecting U_1 to U_2 in $\widetilde{X}^*$. As $\widetilde{Y}_2$ is convex, $\sigma \to \widetilde{X}^*$ lies in $\widetilde{Y}_2$. Let Σ be the cubical convex hull of σ in $\widetilde{X}$. Note that Σ is contractible. By possibly replacing Y_1 by another superconvex rank 2 pseudograph mapping into Y_1, we can assume $\mathrm{sys}(Y_1) > \alpha\,\mathrm{diam}(\Sigma)$.

Consider the following wallspace on Σ: there is a noteworthy wall U consisting of the two hyperplanes $U_1 \cap \Sigma$ and $U_2 \cap \Sigma$, but every other wall consists of a single hyperplane of Σ. The cubical presentation $X^*_\Sigma = \langle X \mid Y_1, \Sigma \rangle$ has the same π_1 as X^*, but has a coarser wallspace structure. We use X^*_Σ to facilitate the proof of the strong separation of W_1, W_2. We will denote the wall of X^*_Σ containing U by W.

Since X^* satisfies $C'(\alpha)$, so does X^*_Σ. Indeed, since $\mathrm{sys}(Y_1) > \alpha\,\mathrm{diam}(\Sigma)$, then $\alpha\,\mathrm{diam}(Y_1 \cap \Sigma) < \mathrm{sys}(Y_1)$, and the pieces coming from X^* still satisfy this condition in X^*_Σ. The $B(6)$ condition (using the wallspace on Σ described above, and the standard structure on Y_1) is also satisfied by X^*_Σ. Indeed, Y_1 already satisfies all the hypothesis of $B(6)$; using the contractibility of Σ, it is straightforward that the choice of walls on Σ satisfies Hypothesis (2) of the $B(6)$ condition, and, again since Σ is contractible, hyperplane convexity and Hypothesis (4) of the $B(6)$ condition follow vacuously. To finish verifying the $B(6)$ condition, we need to check the Wall Convexity condition only for the walls in Σ, since this condition was verified for the walls in Y_1 in the proof of Part (3) of this theorem. Let S be a path in Σ that starts and ends with 1-cells dual to the same wall of Σ. Suppose S is the concatenation of at most 7 pieces. The diameter of a piece in Σ is at most $\max\{\frac{sys(Y_1)}{\alpha}, M\}$ where M' is the bound on wall-pieces from Lemma 5.1. Since U_1, U_2 were chosen at distance

$> 7\max\{\frac{sys(Y_1)}{\alpha}, M\}$, it follows that the path S starts and ends on the same hyperplane of Σ, and is thus nullhomotopic.

Let $\mathcal{C}_\Sigma$ be the cube complex dual to the wallspace on $\widetilde{X}^*_\Sigma$ described above and note that there is a natural projection $\mathcal{C} \to \mathcal{C}_\Sigma$. By construction of $\mathcal{C}_\Sigma$ the hyperplanes of $\mathcal{C}$ dual to W_1, W_2 project to a single hyperplane of $\mathcal{C}_\Sigma$. Since a hyperplane in a CAT(0) cube complex cannot self-cross, we deduce that W_1, W_2 cannot cross in $\widetilde{X}^*$.

We now argue that W_1 and W_2 are strongly separated in $\widetilde{X}^*$. Suppose there exist a wall W' in $\widetilde{X}^*$ that crosses both walls W_1 and W_2. We claim that W' crosses Σ in a hyperplane that we denote by U'. Let V_1, V_2 be hyperplanes of cone-cells of the wall W (which is the wall of $\widetilde{X}^*_\Sigma$ "containing" W_1, W_2) crossed by W'. By Proposition 7.15, every cone of W between V_1 and V_2 is crossed by W'. In particular W' crosses Σ which separate W_1, W_2 in W. Moreover, considering the Σ cone-cell in the ladder, we see that $U' \cap \Sigma$ and $U_i \cap \Sigma$ either cross each other (which happens if the first or last cone-cell maps to Σ), or are dual to edges in a single piece of $\langle X \mid Y_1, \Sigma\rangle$ which is either a wall-piece of Σ, or a cone-piece of Σ with Y_1. Thus, $\mathsf{d}(U_1, U_2) \leq \mathsf{d}(U_1, U') + \mathsf{d}(U_2, U') + \operatorname{diam}(U') \leq 2M + B$. But by our choice of U_1, U_2 at distance $\mathsf{d}(U_1, U_2) > 2M + B$, which yields a contradiction.

Finally, $\mathcal{C}$ is not a quasiline by Part (2). □

Proposition 7.15 is the remaining ingredient needed in the proof of Theorem 7.12.(6). It engages with the following notions:

A W*-ladder* L is a ladder mapping to $\widetilde{X}^*$ such that

- L is the union of 2-cells $C_1, C_2, \ldots, C_m$,
- each C_i is a square or a cone-cell,
- $C_i \cap C_{i+1}$ is a 1-cube e_i, and
- each e_i is dual to a hyperplane of W.

Remark 7.14 (Squares as cones). Let $X^* = \langle X \mid Y_1, Y_2, \ldots, \rangle$ be a cubical presentation that satisfies $C'(\alpha)$. Let s be a 2-cube of X. If s does not lie in any Y_i, then we can add s to the cubical presentation to obtain $\langle X \mid s, Y_1, Y_2, \ldots, \rangle$ which satisfies the same $C'(\alpha)$ condition. Indeed, $\operatorname{sys}(s) = \infty$, and if s is not in any Y_i, any piece between s and any Y_j is already a wall piece. It follows that we can add as many squares as needed to X^* to form a $C'(\alpha)$ cubical presentation $X^*_\square$ where every square lies in a cone. This will facilitate the use of the Ladder Theorem in Proposition 7.15.

Proposition 7.15 (Crossing Intermediate Cones). Let X^* be $B(6)$. Let W, W' be walls in $\widetilde{X}^*$. Assume there are distinct hyperplanes or cones V_1, V_2 of W that are crossed by W'. Then every cone Y of W between V_1 and V_2 is crossed by W'.

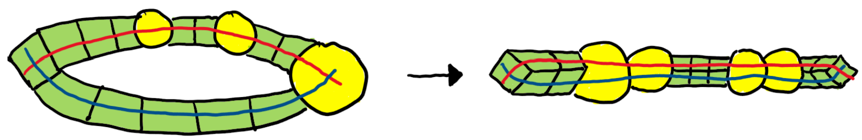

FIGURE 4. A 2-collared diagram exhibiting two crossing between walls W and W' on the left, and the corresponding quasi-2-collared ladder on the right.

Moreover, there is a ladder L in $X^*_\square$, containing a W-ladder K, and a W'-ladder K'. And K, K' start and end on the first and last cells of L. And L is *quasi-2-collared* in the sense that every external cell of L is a cell of K or K' or both. See Figure 4.

Remark 7.16. In the proof of Proposition 7.15 we use the notion of a reduced diagram. A diagram is *reduced* if it cannot be simplified by performing any of the following moves: removing square bigons, combining cone-cells, absorbing a square into a cone-cell, absorbing a cornsquare into a cone-cell, and replacing internal non-essential cone-cells with square diagrams. See the discussion following [Wis21, Def 3.11].

Proof. Let K be a reduced W-ladder in $\widetilde{X}^*$ that starts and ends with a square or cone-cell containing the intersection at V_1 and V_2.

Viewing K as a W-ladder in $\widetilde{X}^*_\square$, we break K into a sequence of sub-W-ladders $K_1, K_2, \ldots, K_m$ where for each i the first square/cone-cell of K_{i+1} is the last square/cone-cell of K_i, and where the squares/cone-cells of K_i crossed by W' are precisely the first and last ones. Since we are working in $\widetilde{X}^*_\square$, we treat the first and last cells of each K_i as cone-cells.

We claim that for each i there is a diagram L_i that is 2-collared by K_i and a W'-ladder K'_i. Let K'_i be any W'-ladder starting and ending at the first and last cone-cells of K_i. Let J_i be the union of K_i and K'_i along their first and last cone-cells. We can assume without loss of generality that the combined cone-cells of K_i, K'_i are the same, since any two paths in a cone are subpaths of a closed path in the cone. Thus J_i is an annulus or Möbius strip.

Let D_i be a diagram whose boundary path P_i is an immersed path in J_i generating $\pi_1 J_i$ and where P_i does not traverse any edge of K_i dual to W_i. Moreover choose K'_i and D_i so that D_i has minimal complexity among all such choices. If P_i traversed an edge of K'_i dual to W'_i, then following this dual curve within D_i, we see that it cannot exit on the K_i side by our non-intersection assumption on K_i, and hence it exits on the K'_i side. This contradicts the minimality of D_i. Thus P_i is a boundary cycle of J_i, and J_i is not a Möbius strip. The union $L_i = J_i \cup_{P_i} D_i$ is a disc diagram, proving the claim.

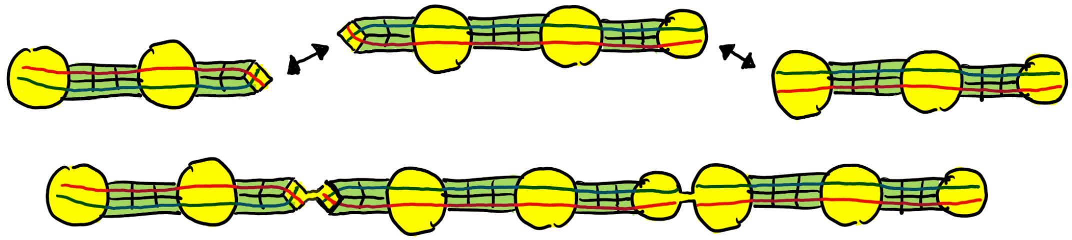

FIGURE 5. Combining ladders.

We now amalgamate $L_1, L_2, \ldots, L_m$ by combining cone-cells mapping to the same cone, using that any two paths in a cone are subpaths of a closed path. See Figure 5.

Let $\bar{L}$ be the combination of $L_1, L_2, \ldots, L_m$. We obtain L from $\bar{L}$ by reducing to obtain a disc diagram using the reduction moves in Remark 7.16. These moves preserve the quasi-2-collaring by W and W'. Moreover, our hypothesis that W' only crosses at the first and last cone-cells of each K_i ensures that K is transformed to a reduced ladder in $X^*_\square$ visiting the same sequence of cones as the original W-ladder. □

8. Freeness

In this section, we prove Part (5) of Theorem 7.12, i.e. that the pseudograph $Y \to X$ can be chosen so that the action of $\pi_1 X^*$ on $\mathcal{C}$ is free. We start by recalling a criterion that is often useful for proving properness in the setting of cubical small-cancellation.

A hyperplane U is *m-proximate* to a 0-cube v if there is a path $P = P_1, \ldots, P_m$ such that each P_i is either a single edge or a piece, v is the initial vertex of P_1 and H is dual to an edge in P_m. A wall is *m-proximate* to v if it has a hyperplane that is m-proximate to v.

A hyperplane U of a cone over Y is *piecefully convex* if the following holds: For any path $\tau\rho \to Y$ with endpoints on $N(U)$, if τ is a geodesic and ρ is trivial or lies in a piece of Y containing an edge dual to U, then $\tau\rho$ is path-homotopic in Y to a path $\mu \to N(U)$.

Theorem 8.1 ([Wis21, Thm 5.44 and Cor 5.45]). Let $X^* = \langle X \mid \{Y_i\} \rangle$ be a $B(6)$ cubical presentation that satisfies the following conditions:

(1) Each hyperplane H of each cone over Y_i is piecefully convex.
(2) Let $\kappa \to Y \in \{Y_i\}$ be a geodesic with endpoints p, q. Let U_1 and U_1' be distinct hyperplanes in the same wall W_1 of Y. Suppose κ traverses a 1-cell dual to U_1, and either U_1' is 1-proximate to q or κ traverses a

1-cell dual to U_1'. Then there is a wall W_2 in Y that separates p, q but is not 2-proximate to p or q.

(3) Each infinite order element of $\text{Aut}_X(Y_i)$ is cut by a wall.

Then $\pi_1 X^*$ acts on the dual CAT(0) cube complex $\mathcal{C}$ with torsion stabilizers.

The following result shows that if $Z \to X$ is homotopy equivalent to a circle, then Z can be chosen so that the cubical presentation $X^* = \langle X \mid Z \rangle$ satisfies the conditions of Theorem 8.1. We state it in a way that will simplify the proof of Lemma 8.4 (stated below).

Theorem 8.2 ([FW24, Thm 3.5])**.** Let $X^* = \langle X \mid Z \rangle$ be a cubical presentation with Z homotopy equivalent to a circle, and satisfying the $C'(\alpha)$ condition for $\alpha \leq \frac{1}{20}$ and where $\text{diam}(N(U)) < \alpha\,\text{sys}(Z)$ for every hyperplane of Z. Then $X^* = \langle X \mid Z \rangle$ satisfies the conditions of Theorem 8.1. Moreover, for every geodesic $\kappa \to Z$:

$$|\kappa| < \frac{1}{2}\,\text{sys}(Z) + 2\alpha\,\text{sys}(Z).$$

Remark 8.3. Theorem 8.2 as stated in [FW24] assumes moreover that $Y - U$ is contractible for each hyperplane U of Y. The reason for this assumption is that then the wallspace structure in Construction 7.1 is slightly simplified, since all hyperplanes in Y must then cross the geodesic σ in the construction. However, this hypothesis is unnecessary for the calculations therein, so we omit it.

Lemma 8.4. Let $X^* = \langle X \mid Y \rangle$ be a cubical presentation that satisfies the $C'(\alpha)$ condition for $\alpha \leq \frac{1}{16}$, and where Y is a superconvex rank 2 pseudograph that is the union of two rank 1 pseudographs Z_1 and Z_2, which are locally convex in Y. Let $M = \min_i\{sys(Z_i)\}$ and $\text{diam}(Z_1 \cap Z_2) \leq \beta M$ where $\alpha + \beta < \frac{1}{8}$. Suppose $\text{diam}(N(U)) \leq \alpha M$ for each hyperplane U of Y.

Consider the wallspace structure on Y obtained by combining the antipodal wall structure on Z_1 and Z_2 as in Construction 7.3. Then this wallspace structure satisfies the hypotheses of Theorem 8.1. Therefore, the action of $\pi_1 X^*$ on the dual $\mathcal{C}$ to this wallspace structure has torsion stabilizers.

Proof. The $B(6)$ condition is verified in Part 3 of Theorem 7.12. We now verify the rest of the hypotheses in Theorem 8.1.

Condition 1 (Pieceful-convexity): Let $\xi\rho \to Y$ be a path with endpoints on $N(U)$, and where ξ is a geodesic and ρ is either trivial or lies in a piece of Y containing an edge dual to U. The $C'(\alpha)$ condition implies that the diameter of pieces appearing in essential paths in Y is $< \alpha M$. Let $\tau \to N(U)$ be a geodesic path joining the endpoints of $\xi\rho$ in $N(U)$. Since $\text{diam}(N(U)) \leq \alpha M$, and $|\xi| < \frac{1}{2}M + 2\alpha M$ by Theorem 8.2, then $|\xi\rho\tau| < \frac{1}{2}M + 4\alpha M < M$ provided that we choose α so that $4\alpha < \frac{1}{2}$. Since the concatenation $\xi\rho\tau$ is shorter than the systole of Y, then $\xi\rho\tau$ must be nullhomotopic, so $\xi\rho$ is homotopic into the carrier of U, as claimed.

Condition 2 (Existence of non-proximate separating wall): Let κ be a geodesic path from p to q as in Theorem 8.1.(2) and consider distinct hyperplanes U_1, U_1' of W_1 where κ traverses an edge e dual to U_1 and either κ traverses an edge dual to U_1', or q is 1-proximate to an edge dual to U_1'. If U_1 and U_1' are hyperplanes of the same Z_i, then the result follows from Theorem 8.2.

Otherwise, the wall W_1 consists of hyperplanes U_1, U_1' and U_1'' where $U_1'' \subset Z_1 \cap Z_2$. Without loss of generality assume that $U_1 \subseteq Z_1$ and $U_1' \subseteq Z_2$. We claim that $\kappa \cap Z_1 \cap Z_2 \neq \emptyset$. This is clear if κ traverses an edge dual to U_1'. If q is 1-proximate to an edge dual to U_1', then q is at distance at least $\frac{1}{2}M - \alpha M$ from U_1''. Thus q is at distance at least $\frac{1}{2}M - \alpha M - \beta M$ from $Z_1 \cap Z_2$, which is greater than $\frac{3}{8}M$, since $\alpha + \beta < \frac{1}{8}$. This proves that κ must intersect $Z_1 \cap Z_2$.

Let $o \in \kappa \cap Z_1 \cap Z_2$. Note that p is at distance at least $\frac{3}{8}M$ from o by a similar argument to the above. Let e_2 be an edge on κ halfway between o and p and let U_2 be the hyperplane dual to e_2. Then the distance from U_2 to each of p, q is at least $\frac{3}{16}M$, in particular U_2 is not 2-proximate to p or q. The same holds for the hyperplane U_2' dual to the edge antipodal to e_2 in Z_1. Thus the wall W_2 containing U_2 satisfies the condition.

Condition 3 (Cutting infinite-order automorphisms): This holds since Y is compact so $\mathrm{Aut}_X(Y)$ is finite (and in our case, trivial by Remark 7.6). □

We can now prove freeness of the action of $\pi_1 X^*$ on $\mathcal{C}$.

Theorem 8.5. Let X be a nonpositively curved cube complex that admits a local isometry $Y \to X$ of a superconvex rank 2 pseudograph, where $X^* = \langle X \mid Y \rangle$ satisfies $C'(\alpha)$ for some $\alpha \leq \frac{1}{16}$ and satisfies $B(6)$. Then there exist superconvex rank 2 pseudographs $Y' \to Y$ such that $\pi_1 X^{**}$ acts freely on the dual $\mathcal{C}$ of the wallspace structure on $\widetilde{X}^{**}$ where $X^{**} = \langle X \mid Y' \rangle$.

Proof. By Construction 7.3 there exist $Y' \to Y$ which is the union of two rank 1 pseudographs Z_1 and Z_2, and a contractible subcomplex $K \subset Y'$, which all are locally convex in Y', and such that $Z_1 \cap Z_2 \subset K$. To simplify the proof, we assume that $Z_1 \cap Z_2 = K$ (which can be done by replacing Z_i with $Z_i \cup K$).

Since Y is compact, there exists a constant $D > 0$ such that for each hyperplane U in Y, $\operatorname{diam} N(U) < D$. The same follows for Y'. We note that in Construction 7.3 the systoles $\mathrm{sys}(Z_i)$ can be chosen to arbitrarily large, so we can assume that $\mathrm{sys}(Z_i) > 16 \max\{D, \mathrm{diam}(Z_1 \cap Z_2)\}$. This ensures that the assumptions of Lemma 8.4 are satisfied with $\alpha = \beta \leq \frac{1}{16}$, so $\pi_1 X^{**}$ acts on the dual cube complex $\mathcal{C}$ with torsion stabilizers.

We now explain that $\pi_1 X^{**}$ is torsion-free and therefore the cell-stabilizers are trivial. This can be deduced in two ways. It follows from [Wis21, Thm 4.2 and Rmk 4.3] that if X^{**} is $C'(\frac{1}{20})$, then every torsion element in $\pi_1 X^{**}$ has to be conjugated into $\mathrm{Aut}_X(Y')$, which is trivial by Remark 7.6, so the cell stabilizers in the dual are trivial. Alternatively, using the $C(9)$ condition, and

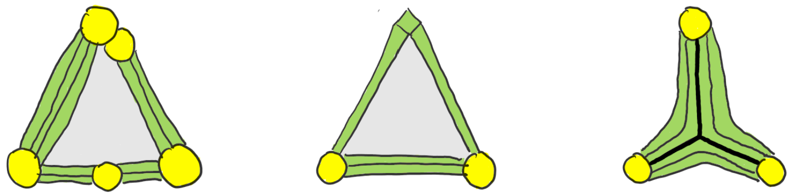

FIGURE 6. Three wall-triangles.

$\mathrm{Aut}_X(Y') = \{1\}$, it follows from the homology formula in [Are24b, Cor 1.6] that $\pi_1 X^{**}$ is torsion-free. ☐

9. Cocompactness

The goal of this section is the following which is proven in Section 9.e:

Theorem 9.1. Let $X^* = \langle X \mid Y \rangle$ be a compact cubical presentation that satisfies the $C'(\frac{1}{12})$ and $B(6)$ conditions, 11-wall convexity, and where Y is a superconvex rank 2 pseudograph. Then $\pi_1 X^*$ acts cocompactly on the dual $\mathcal{C}$ of the wallspace structure on $\widetilde{X}^*$.

The proof of Theorem 9.1 involves thin triangles (Section 9.a), induced subwallspaces and cubical presentations (Section 7.b), hemiwallspaces (Section 9.b), and relative hyperbolicity (Sections 9.c and 9.d).

9.a. **Wall-triangles.** This section introduces wall-triangles, which are disc diagrams determined by triples of pairwise intersecting walls. We prove a technical result, Proposition 9.3, which is a variant of the Ladder Theorem for $B(6)$ cubical presentations. Corollary 9.4, which follows from Proposition 9.3, will be used to prove Theorem 9.1.

Let W_1, W_2, W_3 be walls in $\widetilde{X}^*$. A *wall-triangle collared by* W_1, W_2, W_3 is a 3-collared disc diagram D (see Definition 6.1) where the collaring ladders L_1, L_2, L_3 carry dual curves of W_1, W_2, W_3. We refer to L_i as the W_i*-collar* of D. The ladders L_1, L_2, L_3 intersect at *corner-cells* C_{12}, C_{23}, C_{31}, which are cone-cells or squares. See Figure 6. The wall-triangle D is *minimal* if D has minimal complexity among all wall-triangles collared by W_1, W_2, W_3.

Let U_1 be a hyperplane of W_1. Assume $L_1 \to N(U_1)$ is a square ladder. We say D is U_1*-minimal*, if D has a minimal complexity among all wall-triangles collared by W_1, W_2, W_3 where the W_1-collar is a square ladder mapping to $N(U_1)$. We do not claim that a U_1-minimal wall-triangle is necessarily minimal in the sense above, but the statements that we prove below also apply to U_1-minimal wall triangles, and will be used in the proof of Theorem 9.1.

Lemma 9.2. If three walls pairwise cross in $\widetilde{X}^*$, then there is a wall-triangle $D \to \widetilde{X}^*$ collared by these walls.

Proof. Let W_1, W_2, W_3 be pairwise crossing walls. Consider associated ladders L_1, L_2, L_3, carrying a dual curve in each of W_1, W_2, W_3, and with the crossings occurring at the first and last cells of each L_i. Consider the union $A = L_1 \cup L_2 \cup L_3$. By construction A is homotopy equivalent to a cycle.

Let $P \to A$ be a combinatorial embedding of a closed cycle P, that generates $\pi_1 A$. Consider a disc diagram D_0 with $\partial D_0 = P$. If P traverses an edge dual to any of L_1, L_2, L_3, then there is a proper subdiagram $D_0' \subseteq D_0$ with $\partial D_0'$ contained in a union of ladders L_1', L_2', L_3' carrying dual curves of W_1, W_2, W_3. Note that P must traverse such an edge if A is a Möbius strip. Thus, if D_0 has minimal complexity among all the disc diagrams obtained this way for W_1, W_2, W_3, i.e. among all choices of L_1, L_2, L_3, P, and D, then P does not traverse any such edge. In particular, A is an annular diagram, and so $D = D_0 \cup A$ is a wall-triangle collared by W_1, W_2, W_3. □

Proposition 9.3. Let $X^* = \langle X \mid \{Y_i\} \rangle$ be a cubical presentation satisfying the $C'(\frac{1}{12})$ and $B(6)$ conditions, and 11-wall convexity.

(1) Each cone-cell in a minimal (or U_1-minimal) wall-triangle is a corner-cell.
(2) If X^* is compact, then there exists $R > 0$ such that for any walls W_1, W_2, W_3 in $\widetilde{X}^*$, the minimal (or U_1-minimal) wall-triangle collared by W_1, W_2, W_3 has at most R hyperplanes.

Proof of (1). Let D be a minimal wall-triangle collared by W_1, W_2, W_3. Let L_1, L_2, L_3 be its collaring ladders, and let C_{12}, C_{23}, C_{31} be its corner-cells.

In the degenerate cases where D is a cone-cell or a ladder, the statement is clear. We focus on the non-degenerate case where the corner-cells are distinct.

Suppose that D contains an interior cone-cell C, i.e. C is disjoint from ∂D. Consider the cyclic order on the dual curves starting at ∂C and travelling away from C. By $C'(\frac{1}{12})$ and Lemma 6.2 no dual curve returns to ∂C, and therefore must end on ∂D. Any two dual curves that intersect in D must start in a single piece in ∂C. Thus by $C'(\frac{1}{12})$ and the pigeon-hole principle, there exists a pair of dual curves which end at the same side ladder L_i such that any subpath of ∂C containing both of them has at least 5 pieces. Thus the subdiagram D' bounded by a subladder of L_i, the carriers of these two dual curves, and C, is a disc diagram with only two exposed cells, and therefore by Lemma 6.2, D' must be a ladder. Hence C lies in L_i, which contradicts the assumption that C is an interior cone-cell. Thus there are no interior cone-cells in D.

Now suppose that D contains a non-interior cone-cell C, which is different from C_{12}, C_{23}, C_{31}. Without loss of generality, assume that C is contained in L_2. By 11-wall convexity the innerpath S of ∂C consists of at least 11 pieces. Again we consider all the dual curves that start at S and again none of them return to S. Any dual curve that ends in L_2 must belong to the initial or

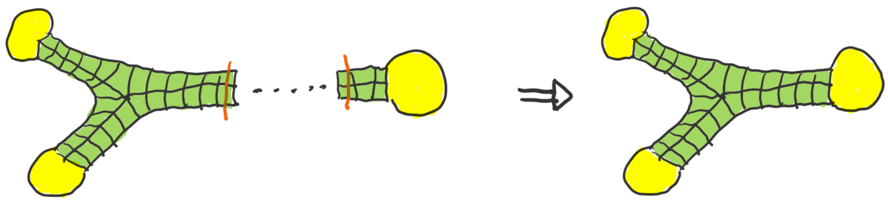

FIGURE 7. Repeated hyperplane allows us to produce smaller D.

terminal piece of S. By the pigeon-hole principle (applied to the second, sixth, and tenth pieces of S) there exists a pair of dual curves that both end on L_i for $i = 1$ or 3, such that the subpath of S containing both of them has at least 5 pieces. Thus the subdiagram D' bounded by a subladder of L_i, the carriers of these two dual curves, and C, is a disc diagram with only two exposed cells. By Lemma 6.2, D' is a ladder. Hence C belongs to both L_2 and L_i. This contradicts the minimality of D, as the subdiagram of D with cone-cells C replacing C_{12} or C_{31} depending on whether $i = 1$ or 3 is a smaller diagram collared by W_1, W_2, W_3.

For the U_1-minimal case, we must have $i = 3$ in the above argument, and we get a contradiction with the U_1-minimality. □

Proof of (2). Let D be a minimal (or U_1-minimal) wall-triangle collared by W_1, W_2, W_3. By Part (1) D does not have any cone-cells that are not corner-cells. Consequently, the collaring ladders L_i decompose as the union $C_{(i-1)i} \cup G_i \cup C_{i(i+1)}$ where G_i is a square ladder. The hyperplanes of D can be partitioned according to which corner-cells C_{12}, C_{23}, C_{31} and/or square ladders G_1, G_2, G_3 they cross.

The number of hyperplanes intersecting corner-cells $C_{(i-1)i}$ and $C_{i(i+1)}$ is bounded by the length of a piece, which is uniformly bounded since X^* is compact. By the minimal complexity of D hyperplanes intersecting $C_{(i-1)i}$ cannot cross G_{i-1} or G_i, and the number of hyperplanes intersecting $C_{(i-1)i}$ and crossing G_{i+1} is again bounded by the length of a piece.

Finally, consider the collection $\mathcal{H}_{(i-1)i}$ of hyperplanes crossing both G_{i-1} and G_i. By the minimality of D no two such hyperplanes cross in D. Since X is compact, there is a bound M on the number of immersed hyperplanes in X. If $|\mathcal{H}_{(i-1)i}| > M$, then by the pigeon-hole principle $\mathcal{H}_{(i-1)i}$ contains at least two hyperplanes U, gU that are $\pi_1 X^*$ translates of one another, i.e. $g \in \pi_1 X^*$. This yields a contradiction since cutting D along U and gU, removing the part between, and gluing back together along g, gives a lower complexity disc diagram collared by W_1, W_2, W_3.

In the U_1-minimal case, we observe that the cutting and gluing procedure preserves U_1-minimality, so the above argument is valid. □

We will use the following consequence of Proposition 9.3.(2). The *r-neighborhood* of a wall Z is the union of r-neighborhoods of its hyperplanes.

Corollary 9.4. Let $X^* = \langle X \mid \{Y_i\}\rangle$ be a compact cubical presentation that satisfies the $C'(\frac{1}{12})$ and $B(6)$ conditions, and 11-wall convexity.

(1) There exists r such that for every wall Z in $\widetilde{X}^*$, if any two other walls W, W' intersect Z and each other, then W, W' intersect each other in the r-neighborhood $\mathcal{N}_r(Z) \subseteq \widetilde{X}^*$ of Z.

(2) There exists r' such that for every wall Z in $\widetilde{X}^*$ and every hyperplane U of Z, if any two other walls W, W' intersect each other and intersect U in hyperplanes V, V' respectively, then $V \cap N(U), V' \cap N(U)$ are at distance r' in $N(U)$.

9.b. **Hemiwallspaces.** For a wallspace $(X, \mathcal{W})$, an associated *hemiwallspace* is a subcollection $\mathcal{H}$ of halfspaces of $\mathcal{W}$ such that for each wall $W = \{W^+, W^-\}$ at least one of W^+, W^- is contained in $\mathcal{H}$. Every hemiwallspace has a dual, which admits a natural embedding as a convex subcomplex of the dual of $(X, \mathcal{W})$. See [HW14, Sec 3.4] for more about hemiwallspaces. For a subset $S \subseteq X$, the collection of all halfspaces intersecting S nontrivially is a hemiwallspace, which we denote by $\mathcal{H}(S)$. The dual of $\mathcal{H}(S)$ is denoted by $\mathcal{C}(\mathcal{H}(S))$.

Let Z be a wall in $\widetilde{X}^*$. For a set $S \subseteq \widetilde{X}^*$, the collection of halfspaces of walls crossing Z that also intersect S is equal to $\mathcal{H}(Z) \cap \mathcal{H}(S)$, which is still a hemiwallspace.

Lemma 9.5. Let $W, Z \in \mathcal{W}$ be two crossing walls of a wallspace $(X, \mathcal{W})$, and let $\mathcal{C}$ be the convex subcomplex of the dual $\mathcal{C}(\mathcal{H}(Z))$, corresponding to $\mathcal{H}(Z) \cap \mathcal{H}(W)$. Then $\mathcal{C}$ is a hyperplane carrier in $\mathcal{C}(\mathcal{H}(Z))$

Proof. Ignore the halfspaces of $\mathcal{H}(Z) \cap \mathcal{H}(W)$ whose complements are not in $\mathcal{H}(Z) \cap \mathcal{H}(W)$, as these halfspaces only play a role in situating $\mathcal{C} \subset \mathcal{C}(\mathcal{H}(Z))$. Let V be the hyperplane of $\mathcal{C}(\mathcal{H}(Z))$ corresponding to W. There is a natural bijection between halfspaces of $N(V) \subseteq \mathcal{C}(\mathcal{H}(Z))$ and halfspaces of $\mathcal{H}(Z) \cap \mathcal{H}(W)$, and it preserves non-empty intersection. Thus the duals are isomorphic. □

Lemma 9.5 will be applied to two walls in a cubical presentation X^* satisfying the $B(6)$ condition.

9.c. **Relative hyperbolicity and local relative quasiconvexity.** We follow the approach to relative hyperbolicity using Bowditch's fine hyperbolic graphs [Bow12]. Recall that G is *hyperbolic relative to* $\{G_v\}$ if G acts cocompactly on a fine hyperbolic graph Γ with finite edge stabilizers, each G_v stabilizes a vertex, and each vertex stabilizer is either finite or conjugate to some G_v. In this setting, the condition that H is *relatively quasiconvex* is that there is an

H-cocompact quasiconvex subgraph $\Upsilon \subset \Gamma$, in some (and in fact any) action of G on such a fine hyperbolic graph Γ [MPW11].

Lemma 9.6. Let G split as a finite graph of groups with finite edge groups, then G is hyperbolic relative to its (possibly infinite) vertex groups.

Proof. Consider the action of G on the associated Bass-Serre tree T. Note that T is obviously 0-hyperbolic and fine. The quotient is compact, and the action has finite edge stabilizers. Thus by Bowditch's criterion, G is hyperbolic relative to its vertex stabilizers. $\square$

The following generalizes the local quasiconvexity of free groups:

Lemma 9.7 (Local relative quasiconvexity)**.** Let G split as a finite graph of groups with finite edge groups and vertex groups $\{G_v\}$. So G is hyperbolic relative to $\{G_v\}$ by Lemma 9.6.

Let H be a subgroup generated by a finite set, together with finitely many subgroups of conjugates of the $\{G_v\}$. Then H is relatively quasiconvex.

In particular, G is locally relatively quasiconvex.

In our application, G splits as a finite bipartite graph of groups where the left vertex groups are finite and the right vertex groups are arbitrary.

Proof. Let T be the Bass-Serre tree associated to the finite graph of groups for G. Let H be a subgroup generated by a finite set of elements $\{g_1, \ldots, g_k\}$ and a finite set of elliptic subgroups $\{K_{v_1}, \ldots, K_{v_\ell}\}$, i.e. K_{v_j} is a subgroup of the stabilizer of a vertex v_j in T.

Without loss of generality, we assume each g_i is loxodromic. For $1 \leq i \leq k$, let γ_i be a fundamental domain for the action of $\langle g_i \rangle$ on its axis.

Let J be a finite tree containing each γ_i and each v_i. Then $\Upsilon = HJ$ is an H-cocompact subtree of T. Finally, $\Upsilon \subset T$ is obviously quasiconvex. $\square$

9.d. **Relative cocompactness.**

Lemma 9.8 (Relatively-hyperbolic wall-stabilizers)**.** Let X^* satisfy $B(6)$. For a wall Z in $\widetilde{X}^*$, the group $\text{Stab}(Z)$ acts on a bipartite tree whose vertices correspond to Z-cones and Z-hyperplanes, and whose edges correspond to incidence.

In particular, $\text{Stab}(Z)$ is hyperbolic relative to $\{\text{Stab}(U_i)\}_i$ where U_i are the hyperplanes of Z.

Proof. The graph of groups whose vertices are stabilizers of hyperplanes and cone-cells of Z is defined [Wis21, Def 5.16], and shown to be a tree in [Wis21, Thm 5.17/5.20]. The result then holds by Lemma 9.6. Indeed, $\text{Aut}_X(Y)$ is finite since Y is compact. Thus the stabilizer of the cone over Y is finite in $\pi_1 X^*$ as a quotient of $\text{Aut}_X(Y)$, and the stabilizer of any cone-cell is contained in the stabilizer of the cone over Y. $\square$

Sageev proved cocompactness of the dual when the wallspace is cocompact, the group is hyperbolic, and the wall-stabilizers are quasiconvex [Sag97]. We use the following generalization [HW14, Thm 7.12]:

Proposition 9.9 (Relative Cocompactness)**.** Let $(X, \mathcal{W})$ be a wallspace such that X is also a length space. Let G act properly and cocompactly on X preserving both its metric and wallspace structures. Suppose G is hyperbolic relative to $\mathbb{P}$, and for each $P \in \mathbb{P}$ let $Y = Y(P) \subset X$ be a non-empty P-invariant P-cocompact subspace. Suppose that the action on $\mathcal{W}$ has only finitely many G-orbits of walls, and $\text{Stab}(W)$ is relatively quasiconvex and acts cocompactly on W for each $W \in \mathcal{W}$.

Then there exists a compact K with GK connected such that $\mathcal{C}(X) = GK \cup_{PK} G\mathcal{C}(\mathcal{H}(Y))$. In particular, $\mathcal{C}(X)$ is G-cocompact, if $\mathcal{C}(\mathcal{H}(Y))$ is P-cocompact for each $P \in \mathbb{P}$ and $Y = Y(P)$.

9.e. **Proof of cocompactness.**

Lemma 9.10. Let $\mathcal{C}$ be a G-cocompact CAT(0) cube complex with finite vertex stabilizers. There is a uniform upper bound on degrees of vertices of $\mathcal{C}$. Thus each vertex of $\mathcal{C}$ lies in uniformly finitely many hyperplane carriers.

Proof of Lemma 9.10. It suffices to prove the statement for a group G acting on a graph Γ (the 1-skeleton of $\mathcal{C}$). For each vertex v of Γ, let $e_1, \ldots, e_n$ be $\text{Stab}(v)$-representatives of the edges at v. Then $\deg(v) = \sum_i \text{Stab}(v)/\text{Stab}(e_i)$. The statement follows by using $\max\{\deg(v)\}$ as v ranges over the finitely many G-orbits of vertices in Γ.

If $v \in N(U)$ then U is dual to an edge at v. Hence $\deg(v)$ is uniformly bounded as above. □

The proof of Theorem 9.1 uses the following cocompactness characterization.

Remark 9.11. Let $\mathcal{C}$ be the CAT(0) cube complex dual to a wallspace with a G-action. Then the following statements are equivalent

(1) $\mathcal{C}$ is G-cocompact,
(2) there are finitely many G-orbits of maximal cubes in $\mathcal{C}$,
(3) there are finitely many G-orbits of collections of pairwise crossing walls in the wallspace.

Lemma 9.12. Let $\widetilde{X}$ be a G-cocompact CAT(0) cube complex, and $U \subseteq \widetilde{X}$ a hyperplane. Then U is $\text{Stab}_G(U)$-cocompact.

Proof. We use characterization (2) of cocompact action from Remark 9.11. Since G acts cocompactly on X, there are finitely many G-orbits of midcubes in $\widetilde{U}$ under the action of G. We claim that each such orbit is invariant under the action of $\text{Stab}_G(\widetilde{U})$. Indeed, if $g \in G$ sends one midcube of $\widetilde{U}$ to another, then it must stabilize $\widetilde{U}$, since $\widetilde{U}$ is uniquely determined by any of its midcubes, and g sends hyperplanes to hyperplanes as a cubical automorphism. □

Proof of Theorem 9.1. We use Remark 9.11.(3). Throughout this proof, we write $\text{Stab}(\cdot)$ for $\text{Stab}_{\pi_1(X^*)}(\cdot)$.

We will show that for each wall Z in $\widetilde{X}^*$, there are finitely many $\text{Stab}(Z)$-orbits of collections of pairwise-crossing walls that include Z. Since $\widetilde{X}^*$ is cocompact, there are finitely many orbits of walls in $\widetilde{X}^*$. Thus we can conclude that there are finitely many orbits of collections of pairwise-crossing walls.

We will first show that the dual $\mathcal{C}(\mathcal{H}(Z))$ of the hemiwallspace $\mathcal{H}(Z)$ is $\text{Stab}(Z)$-cocompact. Let U be a hyperplane of Z. Note that U is embedded and simply connected by Lemma 6.2. See [Are24b, Lem 3.11(i)+(ii)] for details. Since X is compact, U is $\text{Stab}(U)$-cocompact by Lemma 9.12.

We first prove that the dual $\mathcal{C}(\mathcal{H}(U))$ is $\text{Stab}(U)$-cocompact. Let $W, W' \in \mathcal{H}(U)$ be intersecting walls that cross U, and let V, V' be the hyperplanes in $\widetilde{X}^*$ contained in W, W' respectively, which cross U. Let $r' \in \mathbb{N}$ satisfy Corollary 9.4.(2). Let $\mathcal{C}$ and $\mathcal{C}'$ be the subcomplexes of $N(U)$ that are the r'-*thickened* carriers of the hyperplanes $V \cap N(U), V' \cap N(U)$ of $N(U)$. These are convex subcomplexes containing r'-neighborhoods of the hyperplane carriers of $V \cap N(U), V' \cap N(U)$ [HW12]. By Corollary 9.4.(2), the hyperplanes V, V' are at distance at most r', so in particular $\mathcal{C} \cap \mathcal{C}' \neq \emptyset$. Thus every collection $\{W_i\}_{i \in I}$ of U-crossing, pairwise-crossing walls in $\mathcal{H}(U)$ corresponds to a collection of pairwise-intersecting r'-thickened hyperplane carriers $\{\mathcal{C}_i\}_{i \in I}$ in $N(U)$. In particular, for every finite collection $\{W_i\}_{i \in I}$ of U-crossing, pairwise-crossing walls, $\bigcap_{i \in I} \mathcal{C}_i \neq \emptyset$ by Lemma 3.5. There is a uniform bound on the number of $\text{Stab}(U)$-orbits of collections of r'-thickened hyperplane carriers containing a 0-cube, by the cocompactness of U (and hence of $N(U)$). By compactness of X there is a uniform upper bound on the number of r'-thickened hyperplane carriers containing any given 0-cube of X, which implies that there is a uniform upper bound on the size of a collection of pairwise-intersecting r'-thickened hyperplane carriers in X, hence also in $N(U)$. This implies that $\mathcal{C}(\mathcal{H}(U))$ is $\text{Stab}(U)$-cocompact.

Let Z be a wall which is a union of hyperplanes $\{U_i\}$. Then $\text{Stab}(Z)$ is hyperbolic relative to $\{\text{Stab}(U_i)\}$ by Lemma 9.8. Let $r \in \mathbb{N}$ be provided by Corollary 9.4.(1), i.e. for walls W and W' that cross each other and cross Z, we have that W, W' cross each in $\mathcal{N}_r(Z)$. By compactness of X^*, the action of $\text{Stab}(Z)$ on $\mathcal{N}_r(Z)$ is proper and cocompact, and has finitely many orbits of walls from $\mathcal{H}(Z)$.

We now apply Proposition 9.9 with $G = \text{Stab}(Z)$, $(X, \mathcal{W}) = (\mathcal{N}_r(Z), \mathcal{H}(Z))$, $\mathbb{P} = \{\text{Stab}(U_i)\}$, and U_i playing the role of subspaces Y. To view $\mathcal{H}(Z)$ as a wallspace, we ignore the halfspaces whose complements are not contained in $\mathcal{H}(Z)$. For each $W \in \mathcal{H}(Z)$, the stabilizer $\text{Stab}_{\text{Stab}(Z)}(W)$ is relatively-quasiconvex in $\text{Stab}(Z)$ by Lemma 9.7, and W is $\text{Stab}_{\text{Stab}(Z)}(W)$-cocompact by compactness of X^*. Thus, $\mathcal{C}(\mathcal{H}(Z))$ is $\text{Stab}(Z)$-cocompact by Proposition 9.9.

To complete the proof, we show that $\text{Stab}(Z)$-cocompactness of $\mathcal{C}(\mathcal{H}(Z))$ implies the finiteness of the $\text{Stab}(Z)$-orbits of collections of pairwise-crossing walls that include Z. The argument is similar to the earlier argument showing the $\text{Stab}(U)$-cocompactness of $\mathcal{C}(\mathcal{H}(U))$, we include it for completeness.

Let W, W' be intersecting walls that cross Z, and let $\mathcal{C}, \mathcal{C}'$ be the convex subcomplexes of the dual $\mathcal{C}(\mathcal{H}(Z))$, corresponding to $\mathcal{H}(Z) \cap \mathcal{H}(W)$ and $\mathcal{H}(Z) \cap \mathcal{H}(W')$, and which are hyperplane carriers in $\mathcal{C}(\mathcal{H}(Z))$ by Lemma 9.5. Then $\mathcal{C} \cap \mathcal{C}' \neq \emptyset$, as otherwise there is a hyperplane V in $\mathcal{C}(\mathcal{H}(Z))$ that separates $\mathcal{C}$ from $\mathcal{C}'$, which is dual to a wall that separates W from W', contradicting that W, W' intersect. Thus, every collection $\{W_i\}_{i \in I}$ of Z-crossing, pairwise-crossing walls corresponds to the pairwise-intersecting collection $\{\mathcal{C}_i\}_{i \in I}$. If I is finite, then $\bigcap_{i \in I} \mathcal{C}_i \neq \emptyset$ by Lemma 3.5. Each 0-cube lies in finitely many $\text{Stab}(Z)$-orbits of hyperplane carriers in $\mathcal{C}(\mathcal{H}(Z))$ by Lemma 9.10. By $\text{Stab}(Z)$-cocompactness of $\mathcal{C}(\mathcal{H}(Z))$, there are finitely many $\text{Stab}(Z)$-orbits of 0-cubes in $\mathcal{C}(\mathcal{H}(Z))$. Thus each collection of pairwise-crossing walls corresponds to one of these finitely many $\text{Stab}(Z)$-orbits of collections of mutually intersecting hyperplane carriers. □

References

[Ago13] Ian Agol. The virtual Haken conjecture. *Doc. Math.*, 18:1045–1087, 2013. With an appendix by Agol, Daniel Groves, and Jason Manning.

[AH22] Goulnara N. Arzhantseva and Mark F. Hagen. Acylindrical hyperbolicity of cubical small cancellation groups. *Algebr. Geom. Topol.*, 22(5):2007–2078, 2022.

[AJW24] Macarena Arenas, Kasia Jankiewicz, and Daniel T. Wise. Hyperbolicity in non-metric cubical small-cancellation. *Bulletin of the London Mathematical Society*, 56(6):2036–2052, 2024.

[Are24a] Macarena Arenas. Asphericity of cubical presentations: the 2-dimensional case. *Int. Math. Res. Not.*, 2024(7):5524–5547, 2024.

[Are24b] Macarena Arenas. Asphericity of cubical presentations: the general case, 2024.

[Are24c] Macarena Arenas. A cubical Rips construction. *Algebr. Geom. Topol.*, 24(8):4353–4372, 2024.

[BHS17] Jason Behrstock, Mark F. Hagen, and Alessandro Sisto. Hierarchically hyperbolic spaces, I: Curve complexes for cubical groups. *Geom. Topol.*, 21(3):1731–1804, 2017.

[BM97] Marc Burger and Shahar Mozes. Finitely presented simple groups and products of trees. *C. R. Acad. Sci. Paris Sér. I Math.*, 324(7):747–752, 1997.

[Bow12] B. H. Bowditch. Relatively hyperbolic groups. *Internat. J. Algebra Comput.*, 22(3):1250016, 66, 2012.

[BSV14] Mladen Bestvina, Michah Sageev, and Karen Vogtmann, editors. *Geometric group theory*, volume 21 of *IAS/Park City Mathematics Series*. American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 2014.

[CS11] Pierre-Emmanuel Caprace and Michah Sageev. Rank rigidity for CAT(0) cube complexes. *Geometric And Functional Analysis*, 21:851–891, 2011. 10.1007/s00039-011-0126-7.

[DFW19] François Dahmani, David Futer, and Daniel T. Wise. Growth of quasiconvex subgroups. *Math. Proc. Cambridge Philos. Soc.*, 167(3):505–530, 2019.

[DGO17] F. Dahmani, V. Guirardel, and D. Osin. Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces. *Mem. Amer. Math. Soc.*, 245(1156):v+152, 2017.

[FW24] David Futer and Daniel T. Wise. Cubulating random quotients of hyperbolic cubulated groups. *Trans. Amer. Math. Soc. Ser. B*, 11:622–666, 2024.

[Gen25] Anthony Genevois. Cyclic hyperbolicity in CAT(0) cube complexes, 2025.

[Ger98] V. Gerasimov. Fixed-point-free actions on cubings [translation of *algebra, geometry, analysis and mathematical physics (russian) (novosibirsk, 1996)*, 91–109, 190, Izdat. Ross. Akad. Nauk Sibirsk. Otdel. Inst. Mat., Novosibirsk, 1997; MR1624115 (99c:20049)]. *Siberian Adv. Math.*, 8(3):36–58, 1998.

[Hag08] Frédéric Haglund. Finite index subgroups of graph products. *Geom. Dedicata*, 135:167–209, 2008.

[Hul16] Michael Hull. Small cancellation in acylindrically hyperbolic groups. *Groups Geom. Dyn.*, 10(4):1077–1119, 2016.

[HW12] Frédéric Haglund and Daniel T. Wise. A combination theorem for special cube complexes. *Ann. of Math. (2)*, 176(3):1427–1482, 2012.

[HW14] G. C. Hruska and Daniel T. Wise. Finiteness properties of cubulated groups. *Compos. Math.*, 150(3):453–506, 2014.

[HW24] Jingyin Huang and Daniel T. Wise. Virtual specialness of certain graphs of special cube complexes. *Math. Ann.*, 388(1):329–357, 2024.

[Jan17] Kasia Jankiewicz. The fundamental theorem of cubical small cancellation theory. *Trans. Amer. Math. Soc.*, 369(6):4311–4346, 2017.

[JW22] Kasia Jankiewicz and Daniel T. Wise. Cubulating small cancellation free products. *Indiana Univ. Math. J.*, 71(4):1397–1409, 2022.

[Kap99] Ilya Kapovich. A non-quasiconvexity embedding theorem for hyperbolic groups. *Math. Proc. Cambridge Philos. Soc.*, 127(3):461–486, 1999.

[MPW11] Eduardo Martínez-Pedroza and Daniel T. Wise. Relative quasiconvexity using fine hyperbolic graphs. *Algebr. Geom. Topol.*, 11(1):477–501, 2011.

[Osi16] D. Osin. Acylindrically hyperbolic groups. *Trans. Amer. Math. Soc.*, 368(2):851–888, 2016.

[Sag97] Michah Sageev. Codimension-1 subgroups and splittings of groups. *J. Algebra*, 189(2):377–389, 1997.

[Sta83] John R. Stallings. Topology of finite graphs. *Invent. Math.*, 71(3):551–565, 1983.

[SW15] Michah Sageev and Daniel T. Wise. Cores for quasiconvex actions. *Proc. Amer. Math. Soc.*, 143(7):2731–2741, 2015.

[Wis01] Daniel T. Wise. The residual finiteness of positive one-relator groups. *Comment. Math. Helv.*, 76(2):314–338, 2001.

[Wis12] Daniel T. Wise. *From riches to raags: 3-manifolds, right-angled Artin groups, and cubical geometry*, volume 117 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2012.

[Wis21] Daniel T. Wise. *The structure of groups with a quasiconvex hierarchy*, volume 209 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, [2021] ©2021.

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