

EMBEDDING COMPLEXES INTO PSEUDOMANIFOLDS

KASIA JANKIEWICZ AND KEVIN SCHREVE

ABSTRACT. We show that for $d \geq 2$ every finite d -dimensional simplicial complex is a deformation retract of a $(2d - 1)$ -dimensional pseudomanifold with boundary. Moreover, it embeds as a retract in a closed $(2d - 1)$ -dimensional pseudomanifold.

1. INTRODUCTION

By general position, every finite d -dimensional simplicial complex embeds into \mathbb{R}^{2d+1} . This dimension is known to be sharp, for instance van Kampen showed that the d -fold join of 3 points does not embed into \mathbb{R}^{2d} [vK33]. On the other hand, Stallings showed that every finite d -dimensional CW-complex is homotopy equivalent to a complex which embeds into \mathbb{R}^{2d} [Sta65, DR93]. By taking a regular neighborhood of the image, this shows that every d -dimensional CW-complex is homotopy equivalent to a $2d$ -dimensional manifold with boundary.

In this paper, we are interested in thickenings of simplicial complexes into pseudomanifolds, possibly with boundary. There are many different definitions of pseudomanifolds in the literature. The pseudomanifolds we construct are simplicial complexes with the links of vertices PL-homeomorphic to $(d - 1)$ -dimensional manifolds (possibly with boundary). We call such a complex a *pseudomanifold with isolated singularities*. A pseudomanifold with isolated singularities is *closed* if all links of vertices are PL-homeomorphic to closed manifolds. Our analogue of Stallings theorem in this setting is:

Theorem 1.1. For $d \geq 2$, every finite d -dimensional simplicial complex X is a deformation retract of an orientable $(2d - 1)$ -dimensional pseudomanifold with boundary $P(X)$ with isolated singularities.

More concretely, $P(X)$ is obtained from a certain $(2d - 1)$ -dimensional manifold with boundary by coning off disjoint codimension-zero submanifolds of the boundary. Of course, the theorem does not hold if $d = 1$.

Via simplicial approximation, every finite d -dimensional CW-complex is homotopy equivalent to a finite d -dimensional simplicial complex [Hat02, Thm 2C.5], so we obtain the following.

Corollary 1.2. Every finite d -dimensional CW-complex X is homotopy equivalent to an orientable $(2d - 1)$ -dimensional pseudomanifold with boundary and with isolated singularities.

We are also interested in embedding simplicial complexes into closed pseudomanifolds. Combining Theorem 1.1 with a general version of Davis's reflection group trick implies

Theorem 1.3. Every finite d -dimensional simplicial complex X embeds as a retract into a closed $(2d - 1)$ -dimensional pseudomanifold $Q(X)$ with isolated singularities. If X is aspherical, then $Q(X)$ can be chosen to be aspherical.

The idea of the proof of Theorem 1.1 is simpler than Stallings in the sense that we do not have to homotope the simplicial complex L . Instead, we consider the punctured simplicial complex $L - L^{(0)}$, where $L^{(0)}$ is the vertex set of L , which deformation retracts onto an $(n - 1)$ -dimensional subcomplex K spanned by barycenters of simplices of dimension > 0 . Since K is $(n - 1)$ -dimensional, by general position it embeds into \mathbb{R}^{2n-1} . We construct this embedding in a precise way so that it is induced from a general position map from $L \rightarrow \mathbb{R}^{2n-1}$. This implies that a suitable regular neighborhood M of K in \mathbb{R}^{2n-1} intersects the image of L precisely in a regular neighborhood of K in L . The boundary of this regular neighborhood of K in L is a disjoint union of links of original vertices of L , which by construction are embedded subcomplexes of ∂M . Thickening these subcomplexes to manifolds inside of ∂M and coning each of them off gives a pseudomanifold with boundary which is homotopy equivalent to L .

Here is an example which motivated Theorem 1.1 and Theorem 1.3.

Example 1.1. Let W_L be a right-angled Coxeter group with nerve L a flag simplicial graph. The commutator subgroup is finite index in W_L and is the fundamental group of a locally CAT(0) cube complex P_L with all links isomorphic to L . If L is planar, then we can assume L is a full subcomplex of a flag triangulation of S^2 , and the Davis complex for the associated right-angled Coxeter group W_{S^2} is a contractible 3-manifold with a proper, cocompact W_{S^2} -action. Then $P_L \subset P_{S^2}$ thickens to an aspherical 3-manifold with boundary.

If L is not planar, then it is usually the case that W_L is not virtually the fundamental group of an aspherical 3-manifold [DHW23]. However, every such L embeds as a full subcomplex into some closed higher genus surface Σ_g . The Davis complex corresponding to a regular neighborhood of L is a contractible 3-pseudomanifold with isolated singularities and nonempty boundary, and the Davis complex associated to Σ_g is a contractible 3-pseudomanifold with isolated singularities and proper W_L -action.

In higher dimensions one can do a similar construction using the fact that k -dimensional simplicial complexes immerse into \mathbb{R}^{2k} , and pulling back a regular neighborhood gives an embedding into a $(2k)$ -dimensional manifold.

Remark 1.4. Bestvina-Kapovich-Kleiner defined the *action dimension* of a finitely generated group G as the minimal dimension of contractible manifold with a proper G -action [BKK02]. We propose the *pseudomanifold action dimension* of a finitely generated group as the minimal dimension of contractible pseudomanifold with a proper G -action. It is a corollary of Stallings' theorem that if G admits a finite $K(G, 1)$, then the action dimension of G is $\leq 2 \operatorname{gdim}(G)$. Similarly, Theorem 1.1 implies the following.

Corollary 1.5. Let G be a group that admits a finite $K(G, 1)$. Then the pseudomanifold action dimension of a group G is $\leq 2 \operatorname{gdim}(G) - 1$.

Bestvina-Kapovich-Kleiner show that a coarsened version of the van Kampen obstruction gives a lower bound to the action dimension. It would be interesting to find a similar lower bound to the pseudomanifold action dimension. Note that Davis-Okun have conjectured that if P is a closed aspherical n -pseudomanifold with isolated singularities then the L^2 -homology of the universal cover \tilde{P} vanishes above dimension $\lceil \frac{n}{2} \rceil$ [DO01, Conjecture 0.6].

Since 2-dimensional pseudomanifolds are pinched surfaces, there are many examples of 2-dimensional groups whose pseudomanifold action dimension has to be equal to the bound $2\text{gdim}(G) - 1$. In higher dimensions, we do not know how to establish lower bounds. We also do not know whether the notion of the pseudomanifold action dimension does depend on whether we require only isolated singularities.

Question 1.6. Does there exist a group G whose pseudomanifold action dimension differs depending on whether pseudomanifold is required to have isolated singularities?

Related work. The statement of Theorem 1.1 and its proof was inspired by a talk that Jason Manning gave at BIRS in Fall 2024 on recent work with Ruffoni [MR26]. They construct closed, locally $\text{CAT}(-1)$ 3-pseudomanifolds whose fundamental groups are not cubulable; their method is also to thicken a certain 2-dimensional seed complex (whose fundamental group has Property (T) and is hyperbolic) to a 3-pseudomanifold with boundary and then apply a version of the reflection group trick. Our construction for thickening 2-dimensional simplicial complexes is essentially the same as theirs, though with a general seed complex. Therefore, we have not attempted to control any geometry of our complex. On the other hand, our construction allows for more flexible inputs, for instance, it follows from Theorem 1.3 that there are closed, aspherical 3-pseudomanifolds whose fundamental groups have unsolvable word problem [CM99].

Thickenings of 2-complexes to 3-pseudomanifolds have also been studied earlier. For instance, Quinn showed that every finite 2-dimensional complex is homotopy equivalent to a 3-pseudomanifold with boundary, whose links of vertices are homeomorphic to \mathbb{D}^2 , \mathbb{S}^2 , or $\mathbb{R}P^2$ (see e.g. [HAM93]). Since in the Quinn's construction the links can be homeomorphic to $\mathbb{R}P^2$, the pseudomanifold is not orientable in general. Also, there does not need to be a retraction from the pseudomanifold to the complex.

Acknowledgements. The first author is supported by NSF grants DMS-1926686 and DMS-2238198, and an NSERC Discovery Grant. The second author is supported by NSF grants DMS-2203325 and DMS-2505290.

2. SIMPLICIAL COMPLEXES AND PSEUDOMANIFOLDS

2.1. Simplicial Complexes in \mathbb{R}^n . We will mainly follow Bryant's notes on PL topology [Bry02].

A k -simplex σ in \mathbb{R}^n (with $n \geq k$) is the convex hull of $k + 1$ affinely independent points $\{v_0, \dots, v_k\}$ in \mathbb{R}^k , i.e. where the k vectors $v_1 - v_0, \dots, v_k - v_0$ are linearly independent. The value k is the *dimension* of σ . A *face* of a simplex σ is a simplex spanned by any proper subset of the vertices of σ . A *facet* of σ is a codimension-one face.

A *simplicial complex* X in \mathbb{R}^n is a collection of simplices in \mathbb{R}^n satisfying:

- for every simplex $\sigma \in X$ every face τ of σ belongs to X .
- any non-empty intersection of two simplices $\sigma_1, \sigma_2 \in X$ is a face of each of σ_1, σ_2 .

The union $|X|$ of all simplices of X is a subspace of \mathbb{R}^n . We abuse the notation and denote by X both this topological space and its combinatorial structure. When we write $|X| = |Y|$ we mean that the simplicial complexes X and Y determine the same subspaces of \mathbb{R}^n but do not necessarily have the same combinatorial structures. We say X is a *simplicial complex* if X is a simplicial complex in \mathbb{R}^n for some $n \in \mathbb{N}$. A simplicial complex X viewed combinatorially

has a natural decomposition into $X^{(0)} \cup X^{(1)} \cup \dots \cup X^{(i)} \cup \dots$ where $X^{(i)}$ is the set of all simplices of dimension i .

A *simplicial map* $f : X \rightarrow Y$ is a map which restricted to each simplex of X is a linear map onto a simplex of Y . Two simplicial complexes X, Y are *isomorphic* if there exists a simplicial map $f : X \rightarrow Y$ which is a homeomorphism of subsets of \mathbb{R}^n .

Two simplicial complexes X, Y are *PL-homeomorphic* if there exists subdivisions (see Section 2.3 for the definition of a subdivision) of X and Y which are isomorphic. A triangulation of a manifold M is a simplicial complex X and a homeomorphism $|X| \rightarrow M$. We also say a simplicial complex X is *PL-homeomorphic* to a manifold M if X is PL-homeomorphic triangulation of M .

A *free face* of a simplicial complex X is a simplex τ which is a face of exactly one simplex. The *boundary* ∂X of X is the union of its free faces, and the *interior* X° of X is $X - \partial X \subseteq \mathbb{R}^n$. In particular, the boundary $\partial\sigma$ of a simplex σ is the union of its facets.

Let v be a vertex of X . The *star* of v in X , denoted $\text{St}_X(v)$, is the subcomplex $\{\sigma \in X \mid v * \sigma \in X\}$ where $v * \sigma$ is a simplex which is the convex hull of $v \cup \sigma$, i.e. $v * \sigma$ is spanned by v and the vertices of σ . The *link* of v in X , denoted $\text{Lk}_X(v)$, is the subcomplex of $\text{St}_X(v)$ consisting of the simplices of $\text{St}_X(v)$ which do not contain v .

2.2. Abstract simplicial complexes. At certain points, it is more convenient to work with abstract simplicial complexes. Given a vertex set V , a simplicial complex X is a collection of subsets of V which is closed under inclusion and contains all singletons. It is naturally a poset under inclusion. If the cardinality of V is n , then the geometric realization of X is a simplicial complex in \mathbb{R}^n sitting inside the standard simplex Δ^{n-1} , where we include faces of Δ^{n-1} exactly corresponding to subsets in X .

If X is a simplicial complex and L a subcomplex, then we let $\text{Cone}_X(L)$ denote the simplicial complex obtained from X by adding a new vertex v and the subsets $\tau \cup v$ for $\tau \in L$. If X is a simplicial complex in \mathbb{R}^n , then $\text{Cone}_X(L)$ is naturally a simplicial complex in \mathbb{R}^{n+1} .

We also record the following easy fact:

Lemma 2.1. Let L, N be simplicial complexes in \mathbb{R}^n , and let $H : N \times I \rightarrow N$ be a strong deformation retraction of N onto L . Then there exists a strong deformation retraction of $\text{Cone}_v(N)$ onto $\text{Cone}_v(L)$ whose restriction to N is H .

2.3. Barycentric subdivision and spine. Let X, Y be simplicial complexes in \mathbb{R}^n . Then Y is a *subdivision* of X if $|X| = |Y|$ and every simplex of Y is contained in a simplex of X .

For every simplex σ of X , let v_σ be a point in the interior of σ when $\dim \sigma > 0$ and let $v_\sigma = \sigma$ when $\dim \sigma = 0$. The corresponding *barycentric subdivision* of X is the simplicial complex $B(X)$ in \mathbb{R}^n with

$$B(X)^{(0)} = \{v_\sigma : \sigma \text{ is a simplex of } X\}$$

and where vertices $v_{\sigma_0}, \dots, v_{\sigma_i}$ span an i -simplex if and only if $\sigma_0 \subseteq \sigma_1 \subseteq \dots \subseteq \sigma_i$. We will also denote $B(X)$ by X' .

Remark 2.2. To make such a choice canonical, we can use barycentric coordinates; if σ is the span of $\{v_0, v_1, \dots, v_k\}$, we set v_σ to $\sum_{i=0}^k \frac{1}{k+1} v_i$. However, in our construction we will require some flexibility in the choice of v_σ

We will need the following:

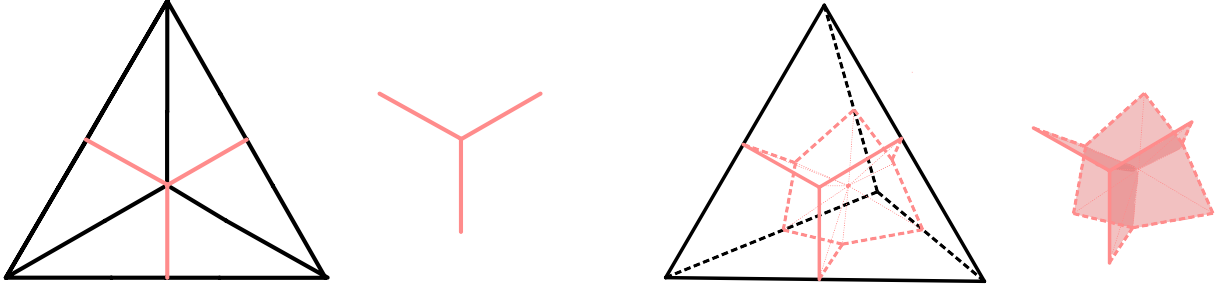


FIGURE 1. The spine $K(\sigma)$ for a 2- and 3-simplex, viewed as a subcomplex of the barycentric subdivision $B(\sigma)$ or as its own simplicial complex.

Lemma 2.3 ([Zee63, Chap 1, Lem 4]). Let K, X be simplicial complexes in \mathbb{R}^n such that $|K| \subseteq |X|$. Then there exists an r -fold barycentric subdivision $B^r(X)$ of X such that some subdivision \bar{K} of K is a subcomplex of $B^r(X)$.

Definition 2.4 (Spine $K(X)$). Given a simplicial complex X and its barycentric subdivision $B(X)$, we define *the spine $K(X)$ of $X - X^{(0)}$* as the subcomplex of $B(X)$ spanned by the vertices $\{v_\sigma : \sigma \in X^{(k)} \text{ for } k \geq 1\}$, i.e. all vertices of $B(X)$ except for those corresponding to $X^{(0)}$.

For short, we will omit “of $X - X^{(0)}$ ”, and refer to $K(X)$ as just the *spine*. In the literature $K(X)$ is sometimes referred to as the *dual of the 0-skeleton of X* (see e.g. [Bry02]).

Lemma 2.5. The spine $K(X)$ is a deformation retract of $X - X^{(0)}$.

Proof. First note that $K(X) = \bigcup_{v \in X^{(0)}} \text{Lk}_{B(X)}(v)$ and $B(X) = \bigcup_{v \in X^{(0)}} \text{St}_{B(X)}(v)$. The different stars possibly intersect only in their links. For each vertex $v \in X^{(0)}$, there is a deformation retraction

$$r_v : \text{St}_{B(X)}(v) - \{v\} \rightarrow \text{Lk}_{B(X)}(v).$$

Thus we can define $r : [0, 1] \times (X - X^{(0)}) \rightarrow K(X)$ given piecewise as $r(p) = r_v(p)$ if $p \in \text{St}_{B(X)}(v)$. \square

2.4. Relative barycentric subdivision. Let $K \subseteq X$ be a subcomplex. For each simplex σ in X not contained in K , let v_σ be a point in the interior of σ when $\dim \sigma > 0$, and let $v_\sigma = \sigma$ when $\dim \sigma = 0$. The corresponding *barycentric subdivision of X relative to K* is the simplicial complex $B(X, K)$ in \mathbb{R}^n with

$$B(X, K)^{(0)} = \{v_\sigma : \sigma \text{ is a simplex in } X \text{ not contained in } K\} \cup K^{(0)}$$

and where vertices $v_{\sigma_1}, \dots, v_{\sigma_i}, w_1, \dots, w_j$ span a simplex if w_1, \dots, w_j span a simplex τ in K , and $\tau \subseteq v_{\sigma_1} \subseteq \dots \subseteq v_{\sigma_i}$. See Figure 2.

2.5. Regular neighborhoods. Let X be a simplicial complex in \mathbb{R}^n . A subcomplex $K \subseteq X$ is *full* if a simplex σ of X belongs to K if and only if all of its vertices belong to K . Let K be a full subcomplex of a simplicial complex X . The *simplicial neighborhood $N(K, X)$ of K in X* is the subcomplex

$$\bigcup_{v \in K} \text{St}_X(v).$$

In other words, $N(K, X)$ is the smallest subcomplex of X containing every simplex which intersects L . The *boundary* of $N(K, X)$, denoted $\dot{N}(K, X)$, is the subcomplex of $N(K, X)$ containing simplices which do not intersect K .

A *regular neighborhood of K in X* is the simplicial neighborhood $N(K, X')$ for X' a barycentric subdivision of X relative to K . See Figure 2.

Lemma 2.6. Let $K = K(X)$ be the spine of a simplicial complex X . Then the boundary \dot{N} of the regular neighborhood N of K in X is the disjoint union of $\text{Lk}_{X''}(v)$ over all $v \in X^{(0)}$.

Proof. A vertex in X'' corresponds to a chain of simplices in X . The vertices in $\text{Lk}_{X''}(v)$ for $v \in X^{(0)}$ are chains with minimal term a simplex τ properly containing v .

Similarly the vertices in $B(X'', K)^{(0)} - K^{(0)}$ correspond to simplices in X' not contained in K , i.e. chains of simplices with minimal element a simplex not in K . One of these spans an edge with a vertex $k \in K^{(0)}$ if and only if k is contained in the minimal element, i.e. the minimal element is a simplex τ properly containing some vertex $v \in X^{(0)}$. Therefore, the vertices of $\bigcup \text{Lk}_{X''}(v)$ and $\dot{N}(K, X'')$ are the same.

We claim that $\dot{N}(K, X'')$ is a full subcomplex, since $\bigcup \text{Lk}_{X''}(v)$ is also full we will be done. A simplex σ in X'' corresponds to nested chains of simplices of X . Since the chains are nested, each minimal element of a chain in $\sigma^{(0)}$ contains a simplex σ_0 of X . If the vertices of σ are in $\dot{N}(K, X'')$, then by the above σ_0 is not a vertex of v . Therefore, σ and $(\sigma - \{v\})$ spans a simplex in $N(K, X')$, and hence $\sigma \subset \dot{N}(K, X')$ \square

Any two regular neighborhoods corresponding to barycentric subdivisions X'_1 and X'_2 relative to K are canonically homeomorphic via a homeomorphism that fixes K . Regular neighborhoods are also preserved under taking subdivisions:

Lemma 2.7 ([Bry02, Prop 3.1]). Suppose that K is a full subcomplex of a simplicial complex X in \mathbb{R}^n , and suppose X_1 is a subdivision of X inducing a subdivision K_1 of K . Then there is a regular neighborhood N of K in X and a regular neighborhood N_1 of K_1 in X_1 so that $|N| = |N_1|$.

A *combinatorial d -disc* is a simplicial complex PL-homeomorphic to a d -simplex, and a *combinatorial d -sphere* is a simplicial complex PL-homeomorphic to the boundary of a $(d + 1)$ -simplex. A simplicial complex X is a *combinatorial d -manifold* if the link of each p -simplex is either a combinatorial $(d - p - 1)$ -sphere, or a combinatorial $(d - p - 1)$ -disc for $p = 0, \dots, d$. The simplices with the links of the second kind form a subcomplex called the *boundary* ∂X of X . This subcomplex is itself a combinatorial $(d - 1)$ -manifold without boundary. If X is a combinatorial d -manifold, then $|X|$ is obviously a topological d -manifold with (possibly empty) boundary $\partial|X| = |\partial X|$.

Theorem 2.8 ([Bry02, Thm 3.6]). Suppose K is a subcomplex of a combinatorial manifold M . Then a regular neighborhood N of K in M is a combinatorial manifold. If K is in the interior of M then $\partial|N| = |\dot{N}|$.

Definition 2.9. Let K be a full subcomplex of X . Given $\epsilon > 0$, the *regular ϵ -neighborhood* N_ϵ of K in X is a simplicial neighborhood constructed as follows. Since K is full in X , the simplicial map $f : X \rightarrow [0, 1]$ which sends the vertices of K to 0 and the other vertices to 1 has the property that $f^{-1}(0) = K$. For any simplex σ of $N(K, X)$ which is not a simplex of K , we choose $v_\sigma \in \sigma^\circ \cap f^{-1}(\epsilon)$. For any other simplex σ of X which is not in K choose $v_\sigma \in \sigma^\circ$ arbitrarily. This yields a barycentric subdivision X' of $X \bmod K$. Let $N_\epsilon = N(K, X')$.

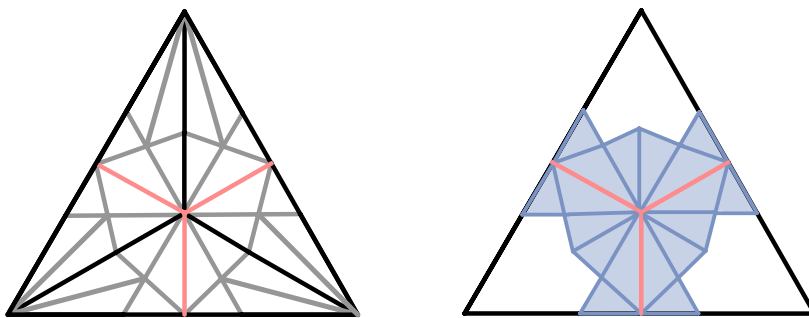


FIGURE 2. A barycentric subdivision of a 2-simplex σ relative the spine $K(\sigma)$, and the corresponding regular neighborhood $N(K(\sigma))$.

2.6. Pseudomanifolds. There are various definitions of pseudomanifolds (with and without boundary) in the literature. The weakest definition is the following.

Definition 2.10. A finite connected d -dimensional simplicial complex P is a d -dimensional pseudomanifold with boundary if

- (1) P is *pure*, i.e. P is a union of its d -simplices, and
- (2) each $(d - 1)$ -simplex is a face of one or two d -dimensional simplices.

In this case, the *boundary* ∂P is the union of the $(d - 1)$ -simplices which are faces of single d -dimensional simplices. When $\partial P = \emptyset$, then we say P is a *closed pseudomanifold*.

It follows that $\text{Lk}_P(\sigma)$ is a disjoint union of $(d - k - 1)$ -dimensional pseudomanifolds for every k -simplex σ . The definition often includes the assumption that a pseudomanifold is *gallery connected* ([Dav08], also referred to as *strongly connected* in other sources, e.g. [Bjö95]): for every pair of d -simplices σ and σ' in P , there is a sequence of d -simplices $\sigma = \sigma_0, \sigma_1, \dots, \sigma_k = \sigma'$ such that the intersection $\sigma_i \cap \sigma_{i+1}$ is an $(d - 1)$ -simplex for all $i = 0, \dots, k - 1$. One downside is a link in a gallery connected pseudomanifold does not need to be gallery connected. The pseudomanifold that we construct in Theorem 1.1 is gallery connected, so our main results remains unchanged if we include this assumption.

Definition 2.11. A d -dimensional pseudomanifold P with boundary has *isolated singularities* if for every k -simplex σ with $k > 0$, $\text{Lk}_P(\sigma)$ is PL-homeomorphic to the boundary of a $(d - k)$ -simplex when $\sigma \not\subseteq \partial P$, and $\text{Lk}_P(\sigma)$ is PL-homeomorphic to a $(d - k - 1)$ -simplex when $\sigma \subseteq \partial P$.

It follows that in a pseudomanifold with isolated singularities P , each vertex link $\text{Lk}_P(v)$ is a combinatorial manifold:

Lemma 2.12. A pseudomanifold has isolated singularities if and only if the vertex links are combinatorial manifolds.

Proof. Given a vertex $v \in P$ and $\sigma \in \text{Lk}_P(v)$, we have that $\text{Lk}_P(\sigma * v)$ is isomorphic to $\text{Lk}_{\text{Lk}_P(v)}(\sigma)$. The conclusion immediately follows. \square

An *orientation* of a k -simplex $\sigma = \{v_0, \dots, v_k\}$ is an equivalence class of orderings of the vertices of σ , where two orderings are equivalent if they differ by an even permutation. We

write (v_0, \dots, v_k) to represent the orientation given by the listed ordering, and denote the opposite orientation by $-(v_0, \dots, v_k)$.

An orientation (v_0, \dots, v_k) of k -simplex induces the orientation of its facets: the facet excluding the vertex v_i has *induced orientation* $(-1)^i(v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_k)$

Definition 2.13. A d -dimensional pseudomanifold (with or without a boundary) P is *orientable* if there is an orientation of each d -simplex such that the induced orientations on each $(d-1)$ -simplex contained in two d -simplices are opposite. Equivalently, $H_d(P, \partial P; \mathbb{Z}) \neq 0$.

Note that the links in an orientable pseudomanifold are orientable.

3. EMBEDDING SIMPLICIAL COMPLEXES INTO PSEUDOMANIFOLDS

In this section, we prove Theorem 1.1.

3.1. Embedding a simplicial complex in Euclidean space. A countable set S of points in \mathbb{R}^n is said to be *in general position*, if every subset $\{v_0, v_1, \dots, v_k\}$ of S with $k \leq n$ spans a k -simplex. More generally, for a simplicial complex X we say that a map $f : X \rightarrow \mathbb{R}^n$ which is linear on simplices is *a general position map*, if f restricted to the 0-skeleton $X^{(0)}$ is an embedding and the set $f(X^{(0)}) \subseteq \mathbb{R}^n$ is in general position.

Suppose $f : X \rightarrow \mathbb{R}^n$ is a continuous piecewise linear function. The *singular set* of f is the subset $S(f) = \text{Cl}\{x \in X : f^{-1}(f(x)) \neq \{x\}\}$. If X is compact, then $S(f)$ intersects each simplex of X in a finite union of linear subspaces. We define $\dim S(f)$ to be the largest dimension of such a subspace.

Lemma 3.1. Let X be a simplicial complex, and let $f : X \rightarrow \mathbb{R}^n$ be a general position map. Suppose that σ_1, σ_2 are two simplices of X of dimension d_1 and d_2 , and let $d_3 = \dim \sigma_1 \cap \sigma_2$ ($d_3 = -1$ when $\sigma_1 \cap \sigma_2 = \emptyset$). Then

$$\dim S(f|_{\sigma_1 \cup \sigma_2}) \leq d_1 + d_2 - n \quad \text{if} \quad d_1 + d_2 - d_3 \geq n$$

and $S(f|_{\sigma_1 \cup \sigma_2}) = \emptyset$ otherwise.

Proof. Let σ_1 be the simplex spanned by vertices v_0, \dots, v_{d_1} , and σ_2 be the simplex spanned by vertices w_0, \dots, w_{d_2} , where $v_i = w_i$ for $i = 0, \dots, d_3$.

If $|\sigma_1 \cup \sigma_2| = d_1 + d_2 - d_3 + 1 \leq n + 1$, then the vertices of $\sigma_1 \cup \sigma_2$ are mapped by f to a set spanning an n -dimensional simplex in \mathbb{R}^n , so $f|_{\sigma_1 \cup \sigma_2}$ is an embedding. In particular, $S(f|_{\sigma_1 \cup \sigma_2}) = \emptyset$.

If $|\sigma_1 \cup \sigma_2| = d_1 + d_2 - d_3 + 1$ is greater or equal to $n + 1$, then $\dim(\text{span } f(\sigma_1 \cup \sigma_2)) = n$, and so

$$\begin{aligned} \dim(S(f|_{\sigma_1 \cup \sigma_2})) &= \dim(\text{Cl}(f(\sigma_1^\circ) \cap f(\sigma_2^\circ))) \leq \dim(\text{span}\{f(\sigma_1^\circ) \cap f(\sigma_2^\circ)\}) = \\ &= \dim f(\sigma_1) + \dim f(\sigma_2) - \dim(\text{span } f(\sigma_1 \cup \sigma_2)) = d_1 + d_2 - n. \end{aligned}$$

□

We have the following consequences of Lemma 3.1.

Corollary 3.2. Let X be a finite d -dimensional simplicial complex, and let $f : X \rightarrow \mathbb{R}^{2d+1}$ be a general position map. Then f is an embedding.

Corollary 3.3. Suppose X is a finite d -dimensional simplicial complex and $f : X \rightarrow \mathbb{R}^n$ is a general position map. Then $\dim(S(f)) \leq 2d - n$.

3.2. Embedding the spine in Euclidean space. Given two affine subspaces $V, W \subseteq \mathbb{R}^k$, V and W *intersect transversely* if V, W together span \mathbb{R}^k . In particular, when $\dim V + \dim W = k$, then their intersection is a single point, and \mathbb{R}^k can be expressed as the direct sum $V \oplus W$.

Similarly, given convex subsets $A, B, C \subseteq \mathbb{R}^k$, we say that the intersection of A and B is transversal in C , if $A \cap B$ is nonempty and $\text{span}(A)$ and $\text{span}(B)$ intersect transversely in $\text{span}(C)$.

Lemma 3.4. Let $\sigma \subseteq \mathbb{R}^k$ be a k -simplex with a choice of $K(\partial\sigma)$ inside of $\partial\sigma$. Let $\mathcal{P} = \bigcup\{p_1, \dots, p_s\}$ be a finite collection of points in the interior of σ , and $\mathcal{L} = \bigcup\{\ell_1, \dots, \ell_t\}$ be a finite collection of line segments in σ not contained in $\partial\sigma$. For $v \in \sigma^\circ$, let $K_v = \text{Cone}_v(K(\partial\sigma)) \subseteq \mathbb{R}^k$. Then the set

$$\{v \in \sigma^\circ \mid K_v \cap \mathcal{P} \neq \emptyset \text{ or } K_v \text{ intersects } \mathcal{L} \text{ not transversely in } \sigma\}$$

is contained in a union of finitely many codimension one subspaces of \mathbb{R}^k .

Proof. The set

$$\{v \in \mathbb{R}^k \mid K_v \cap \mathcal{P} \neq \emptyset \text{ or } K_v \text{ intersects } \mathcal{L} \text{ non-transversely in } \sigma\}$$

is a union of finitely many sets of the form

$$X_{p,\tau} := \{v \in \mathbb{R}^k \mid p \in \text{Cone}_v(\tau)\}$$

for $p \in \mathcal{P}$, and a face τ of $K(\partial\sigma)$ and finitely many sets of the form

$$Y_{\ell,\tau} := \{v \in \mathbb{R}^k \mid \ell \text{ intersects } \text{Cone}_v(\tau) \text{ non-transversely in } \sigma\}$$

for $\ell \in \mathcal{L}$, and a face τ of $K(\partial\sigma)$. We show that each of those sets is contained in a codimension one subspace of \mathbb{R}^k .

Fix $p \in \mathcal{P}$ and a face τ of $K(\partial\sigma)$. The set of $v \in \mathbb{R}^k$ such that $p \in \text{Cone}_v(\tau)$ is contained in the set $\{v \in \mathbb{R}^k \mid p \in \text{span}(\{v\} \cup h(\tau))\}$. But the condition that $p \in \text{span}(\{v\} \cup h(\tau))$ is equivalent to the condition that $v \in \text{span}(\{p\} \cup \tau)$. Since by assumption $p \notin \text{span } \tau$ and $\dim \tau \leq k - 2$, we have $\dim(\text{span}(\{p\} \cup \tau)) = \dim \tau + 1 \leq k - 1$.

Similarly, if $\text{Cone}_v(\tau)$ intersects a line l non-transversely, then v is in the codimension-one hyperplane spanned by τ and l . \square

Remark 3.5. In our application of Lemma 3.4 in the proof of Proposition 3.6 $\mathcal{P} \subseteq \mathcal{L}$.

Proposition 3.6. Let X be a finite d -dimensional complex and $f : X \rightarrow \mathbb{R}^{2d-1}$ be a general position map. Then there exists a choice of spine $K(X) \subset X$ such that f restricted to $K(X)$ is an embedding.

Moreover, there exists $\epsilon > 0$ such that the embedding extends to the regular ϵ -neighborhood N_ϵ of $K(X)$ in X .

Proof. For $i = 1, \dots, k$, let $\sigma_1, \dots, \sigma_k$ be the d -simplices of X . In this proof, there are various objects that we denote by a pair of subscripts ij where $i, j \in \{1, \dots, k\}$. We follow a convention that any such object whose first subscript is i is a subset of a simplex σ_i .

We first show how to choose $K(X)$ such that f restricted to $K(X)$ is an embedding. The spine $K(X)$ is the union

$$K(X) = K(X^{(d-1)}) \cup \bigcup_{i=1}^k K(\sigma_i),$$

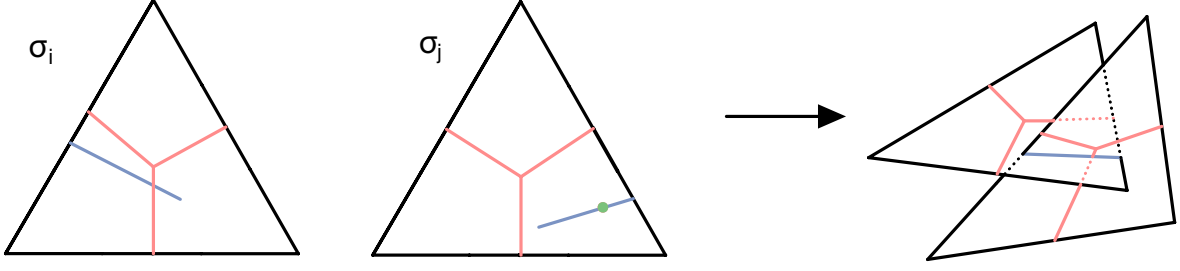


FIGURE 3. Two simplices σ_i, σ_j map to \mathbb{R}^{2d-1} with an intersection along a line segment. The blue line segments are $l_{ij} \subseteq \sigma_i$ and $l_{ji} \subseteq \sigma_j$. The distinguished green point in l_{ji} forms \mathcal{P}_{ji} . The spine $K(\sigma_j)$ is always chosen so that $K(\sigma_j) \cap \mathcal{P}_{ji} = \emptyset$.

and each $K(\sigma_i)$ is naturally isomorphic to $\text{Cone}_{v_{\sigma_i}}(K(\partial\sigma_i))$. By Corollary 3.2 the restriction $f|_{X^{(d-1)}}$ to the $(d-1)$ -skeleton of X is an embedding. By Corollary 3.3 $\dim S(f) \leq 1$. In particular, for any σ_i, σ_j with $i \neq j$ such that $f(\sigma_i^\circ) \cap f(\sigma_j^\circ) \neq \emptyset$, the intersection $f(\sigma_i^\circ) \cap f(\sigma_j^\circ)$ consists of either a single vertex or a line segment intersecting the faces of σ_1 and σ_2 transversely. Let l_{ij} be its preimage in σ_i under $f|_{\sigma_i}$. See Figure 3. If $f(\sigma_i^\circ) \cap f(\sigma_j^\circ) = \emptyset$, let $l_{ij} = \emptyset$. Let $\mathcal{L}_i = \bigcup_{j \neq i} l_{ij} \subseteq \sigma_i$, and $\mathcal{L} = \bigcup \mathcal{L}_i$. Note that \mathcal{L} is a finite union.

We now choose a barycenter v_σ for each simplex σ of X with $\dim \sigma > 0$. If $\dim \sigma < d$, we can choose v_σ to be an arbitrary point in the interior of σ such that $f(v_\sigma)$ is disjoint from \mathcal{L} . We pick v_{σ_i} for $i = 1, \dots, k$ inductively, so that $K(\sigma_i)$ intersects the line segments l_{ij} transversely. Suppose we have already made choices of $v_{\sigma_1}, \dots, v_{\sigma_{i-1}}$. For $1 \leq j < i \leq k$, let $\mathcal{Q}_{ij} \subseteq l_{ij} \subseteq \sigma_i$ be the finite set of points that belong to the intersection $l_{ij} \cap K(\sigma_i)$. Indeed, \mathcal{Q}_{ij} is a finite set of points, since $K(\sigma_i)$ is a codimension one subspace of σ_i , and each l_{ij} is a line segment intersecting $K(\sigma_i)$ transversely. Let $\mathcal{P}_{ij} \subseteq l_{ij} \subseteq \sigma_i$ be the finite set of points such that $f(\mathcal{P}_{ij}) = f(\mathcal{Q}_{ji})$, i.e. for each point $p \in \mathcal{P}_{ij}$ its image $f(p)$ belongs to the intersection $f(\sigma_i) \cap f(K(\sigma_j))$. In other words, $\mathcal{P}_{ij} = f|_{\sigma_i}^{-1}(f(\mathcal{Q}_{ji}))$. See Figure 3.

Let $\mathcal{P}_i = \bigcup_{j < i} \mathcal{P}_{ij}$. By Lemma 3.4 there exists a choice of v_{σ_i} such that $K(\sigma_i) \cap \mathcal{P}_i = \emptyset$ and $K(\sigma_i)$ intersects each line of \mathcal{L}_i transversely. Thus we have chosen v_{σ_i} for $i = 1, \dots, k$, such that f restricted to $K(X)$ is an embedding.

It remains to show that f restricted to the regular ϵ -neighborhood of $K(X)$ is still an embedding, for a sufficiently small $\epsilon > 0$.

Choose $\delta > 0$ so that $\delta < d_{\mathbb{R}^n}(f(K(\sigma_i)), f(K(\sigma_j)))$ for any d -simplices σ_i, σ_j which intersect in at most one vertex. Since f is uniformly continuous, there is $\epsilon > 0$ so that for any $x, y \in X$ $d_X(x, y) < \epsilon$ implies $d_{\mathbb{R}^n}(f(x), f(y)) < \delta/2$.

Let $x_1 \neq x_2 \in N_\epsilon$ such that for $i = 1, 2$, $x_i \in \sigma_i$ for some d -simplex σ_i . If σ_1, σ_2 have a common face of dimension > 0 , then by Lemma 3.1, f is an embedding on $\sigma_1 \cup \sigma_2$ as $d + d - d_3 \geq 2d - 1$. In particular, $f(x_1) \neq f(x_2)$.

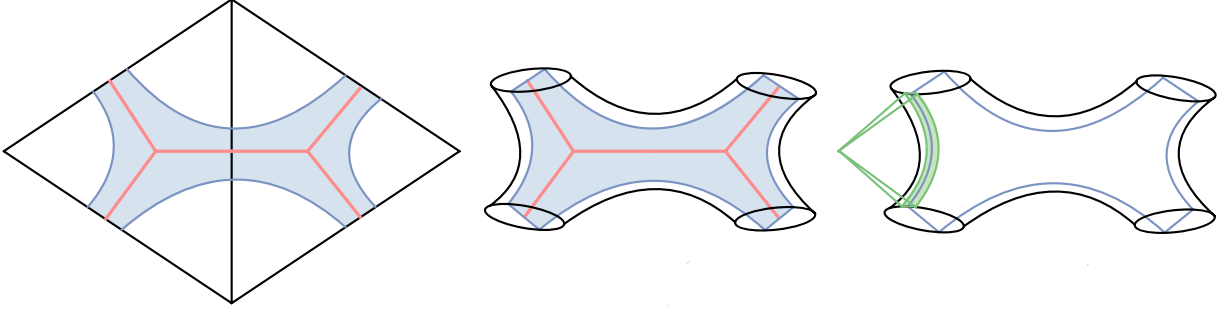


FIGURE 4. Construction of $P(X)$: (1) choice of a regular neighborhood of the spine in the complex X , (2) a regular neighborhood M of the spine in \mathbb{R}^n , (3) coning-off of submanifolds of the boundary of M .

Now suppose σ_1, σ_2 have at most one common vertex. Since the distance from $f(x_i)$ to $f(K(\sigma_i))$ is at most $\delta/2$ we have

$$\begin{aligned} d_{\mathbb{R}^n}(f(x_1), f(x_2)) &\geq \\ d_{\mathbb{R}^n}(f(K(\sigma_i)), f(K(\sigma_j))) - d_{\mathbb{R}^n}(f(x_1), f(K(\sigma_1))) - d_{\mathbb{R}^n}(f(x_2), f(K(\sigma_2))) &> \\ \delta - \delta/2 - \delta/2 &= 0. \end{aligned}$$

In particular, $f(x_1) \neq f(x_2)$. □

3.3. Embedding a simplicial complex in a pseudomanifold with boundary.

Theorem 1.1. For $d \geq 2$, every finite d -dimensional simplicial complex X is a deformation retract of an orientable $(2d - 1)$ -dimensional pseudomanifold with boundary $P(X)$ with isolated singularities.

Proof.

(1) *Construction of $P(X)$:* Let X be a d -dimensional simplicial complex. Let $f : X \rightarrow \mathbb{R}^{2d-1}$ be any general position map. By Proposition 3.6, f restricts to an embedding of N_ϵ of $K(X)$ in X for some $\epsilon > 0$ and appropriate choice of $K(X)$ inside of X . Thus we can view N_ϵ as a simplicial complex in \mathbb{R}^{2d-1} (to simplify the notation we do not write $f(N_\epsilon)$). This step is illustrated by the first picture of Figure 4. By Lemma 2.3 some subdivision \hat{N} of N_ϵ is a full subcomplex of some triangulation R of \mathbb{R}^{2d-1} (i.e. $|R| = \mathbb{R}^{2d-1}$). We denote the induced subdivision of $K(X)$ by \hat{K} . In particular, \hat{K} is a subcomplex of \hat{N} .

Let $\epsilon' > 0$ be sufficiently small so that the regular ϵ' -neighborhood M of \hat{K} in R satisfies $|M \cap \hat{N}| \subseteq |\hat{N}^\circ|$. By Theorem 2.8 M is a combinatorial $(2d - 1)$ -manifold with boundary, and $|\dot{M}| = \partial M$. See the second picture of Figure 4.

Since \hat{N} is a subcomplex of R , the regular ϵ' -neighborhood L of \hat{K} in \hat{N} is naturally a subcomplex of M , and \dot{L} is an embedded subcomplex of ∂M . By Lemma 2.7 and Lemma 2.6, \dot{L} is a disjoint union $\bigsqcup_{v \in X^{(0)}} L_v$ where each L_v is PL-homeomorphic to $\text{Lk}_X(v)$. A regular neighborhood N_v of L_v in ∂M is a codimension zero combinatorial submanifold of ∂M . See the last picture of Figure 4.

We define $P(X)$ as $M \cup \bigsqcup_{v \in X^{(0)}} \text{Cone}(w_v, N_v)$, i.e. M with each N_v coned off. Clearly $P(X)$ is a simplicial complex (in \mathbb{R}^n for some n depending on the number of cone points).

We refer to the new vertices w_v as *cone vertices* of $P(X)$.

(2) $P(X)$ is a $(2d - 1)$ -pseudomanifold with isolated singularities:

Let us analyze the links of vertices in $P(X)$. If $u = w_v$ is a cone vertex, then $\text{Lk}_{P(X)}(u)$ is the regular neighborhood N_v , which is a combinatorial $(2d - 2)$ -manifold with boundary. If u is not in the link of a cone vertex, then $u \in M$ and $\text{Lk}_{P(X)}(u)$ is equal to $\text{Lk}_M(u)$. Finally, if u is in the link of a cone vertex w_v , then $u \in \partial M$ and $\text{Lk}_{P(X)}(u)$ is obtained from $\text{Lk}_M(u)$ by coning off $\text{Lk}_{N_v}(u)$. The latter is either PL-homeomorphic to the boundary of a $(2d - 2)$ -simplex (if $u \in N_v - \partial N_v$), or PL-homeomorphic to a $(2d - 3)$ -simplex (if $u \in \partial N_v$). In the first case, $\text{Lk}_{P(X)}(u)$ is a combinatorial $(2d - 2)$ -sphere, and in the second, it is a $(2d - 2)$ -disc. This proves that $P(X)$ is a $(2d - 1)$ -pseudomanifold with isolated singularities.

(3) $P(X)$ is orientable: Since M is a codimension-zero manifold with boundary in \mathbb{R}^{2d-1} , each of its simplices can be oriented consistently with some fixed orientation of \mathbb{R}^{2d-1} . Then for any $(2d - 2)$ -dimensional simplex τ of M which is not in ∂M , the two orientations on τ induced by the orientations of the two $(2d - 1)$ -simplices containing τ as a face must be reversed. Each N_v is a $(2d - 2)$ -dimensional manifold with boundary, and the orientation of simplices of M induces an orientation on each $(2d - 2)$ -simplex τ of N_v . Let the orientation of τ be given by ordering (v_0, \dots, v_{2d-2}) . We orient each simplex $v * \tau$ of $\text{Cone}_{w_v}(N_v)$ as $-(w_v, v_0, \dots, v_{2d-2})$. Then the orientation induced on τ by the orientation of $v * \tau$ is $-(v_0, \dots, v_{2d-2})$, as required. It remains to check that for any $(2d - 2)$ -simplex containing w_v and which is contained in two $(2d - 1)$ -simplices, the two orientations induced by the $(2d - 1)$ -simplices are opposite. Let δ be a $(2d - 3)$ -simplex in $N_v - \partial N_v$ spanned by vertices v_1, \dots, v_{2d-2} . Since N_v is an orientable combinatorial manifold with boundary, δ belongs to two $(2d - 2)$ -simplices τ, τ' , and their orientations can be represented by the orderings of their vertices (v_0, \dots, v_{2d-2}) and $-(v'_0, \dots, v_{2d-2})$ as they must induce the opposite orientations on δ . Thus $w_v * \tau$ and $w_v * \tau'$ also induce opposite orientations on $w_v * \delta$.

(4) X is a strong deformation retract of $P(X)$: Any regular neighborhood M admits a strong deformation retraction onto $K(X)$, which by Lemma 2.1 can be extended to the strong deformation retraction of $P(X)$ onto X . \square

Remark 3.7. The assumption that $d \geq 2$ is clearly necessary, as 1-dimensional pseudomanifolds are 1-dimensional manifolds. If we try to follow the steps of our construction of $P(X)$ in the case of $d = 1$, we get a finite discrete set of intervals as M , and disconnected subsets N_v . In that case, coning off sets N_v does not yield a pseudomanifold (but in fact the original complex X).

4. REFLECTION GROUP TRICK FOR PSEUDOMANIFOLDS

We now use a version of Davis's reflection group trick to embed a pseudomanifold with isolated singularities D into a closed pseudomanifold P with isolated singularities. The "usual" reflection group trick turns an aspherical manifold N with boundary into a closed aspherical manifold M which retracts onto N . The general version of the trick we use is described in [Dav08, Section 11.5].

Remark 4.1. Davis also considers a version of the reflection group trick for pseudomanifolds with boundary in [Dav08, Section 13.4]. However, Davis additionally requires that

every point in ∂P has a neighborhood homeomorphic to $U \times [0, \epsilon)$ where U is some open neighborhood of the point in ∂P . This condition fails in general for the cone points in $P(X)$ as constructed in the proof of Theorem 1.1. Thus we cannot directly quote this version.

Definition 4.2. A *mirror structure* on a CW-complex Y consists of an index set S and a family of subcomplexes $\{Y_s\}_{s \in S}$. The subcomplexes Y_s are the *mirrors* of Y . We say Y is a mirrored space over S .

Let Y be a mirrored space over S , and let W be a right-angled Coxeter group with vertex set S . Given $y \in Y$, let $S(y) := \{s \in S \mid y \in Y_s\}$, and $W_{S(y)}$ be the Coxeter subgroup generated by $S(y)$. Given any Coxeter group generated by S , let $\mathcal{U}(W, Y)$ be the *basic construction* associated to this data:

$$\mathcal{U}(W, Y) := (W \times Y) / \sim,$$

where $(w, y) \sim (w', y')$ if and only if $y = y'$ and $wW_{S(y)} = w'W_{S(y)}$.

Let Y be a simplicial complex, and L a flag subcomplex. This produces a mirror structure on the barycentric subdivision Y' , where $S = L^{(0)}$. For $s \in S$, the mirror Y_s is the star of the vertex s in the barycentric subdivision L' of L , i.e. Y_s is the geometric realization of the poset of simplices of L containing s .

Let W_L be the right-angled Coxeter group with respect to L . We can then form the basic construction $\mathcal{U}(W_L, Y')$. Topologically, there is no difference between $\mathcal{U}(W_L, Y')$ and $\mathcal{U}(W_L, Y)$, but later we will want a succinct description of the links of vertices, and Y' helps. If Γ is a torsion-free finite index subgroup of W_L , then the output of the reflection group trick is the space $\mathcal{U}(W_L, Y')/\Gamma$.

The universal cover of $\mathcal{U}(W_L, Y')/\Gamma$ is itself a basic construction $\mathcal{U}(W_{\tilde{L}}, \tilde{Y}')$, where \tilde{Y}' is the universal cover of Y' and $W_{\tilde{L}}$ is the right-angled Coxeter group based on the lifted flag triangulation \tilde{L} [Dav08, pg. 168]. In particular, if Y is aspherical then $\mathcal{U}(W_{\tilde{L}}, \tilde{Y}')$ is contractible, so $\mathcal{U}(W_L, Y')/\Gamma$ is aspherical. The orbit map $\mathcal{U}(W_L, Y') \rightarrow Y'$ induces a retraction from $\mathcal{U}(W_L, Y')/\Gamma \rightarrow Y'$.

Proposition 4.3. Suppose that P is a pseudomanifold with isolated singularities obtained from a compact combinatorial d -manifold M with boundary by attaching cones to some number of disjoint codimension zero PL-submanifolds with boundary $(N_i, \partial N_i)$ of ∂M . Suppose that the induced triangulation L on ∂P is flag.

Then for any torsion-free finite index subgroup $\Gamma \subseteq W_L$, the space $Q = \mathcal{U}(W_L, P')/\Gamma$ for any is a closed pseudomanifold with isolated singularities. Furthermore, if P is aspherical, then Q is aspherical.

Proof. The triangulation L induces a mirror structure on P' with mirrors $\text{St}_{L'}(s)$ for $s \in S = L^{(0)}$, as above. We prove that $\mathcal{U}(W_L, P')$ is a pseudomanifold with isolated singularities, which implies the result as this is preserved by the Γ -action. Since W_L acts on $\mathcal{U}(W_L, P')$ with strict fundamental domain P' , it suffices to consider vertices in the chamber P' . If $v \notin L = \partial P'$, then P' is the unique chamber containing v , and $\text{Lk}_{P'}(v)$ is a combinatorial $(2d - 1)$ -sphere. Then the link of v in $\mathcal{U}(W_L, P')$ is the same.

Now let v be a cone vertex. Then $\text{Lk}_{P'}(v)$ is PL-homeomorphic to N_i for some i . The vertex v is contained in a unique mirror of P , namely $\text{St}(v, L')$, which is a cone on ∂N_i . Thus the link of v in $\mathcal{U}(W_L, X)$ is the double of N_i over ∂N_i , which is a closed combinatorial manifold.

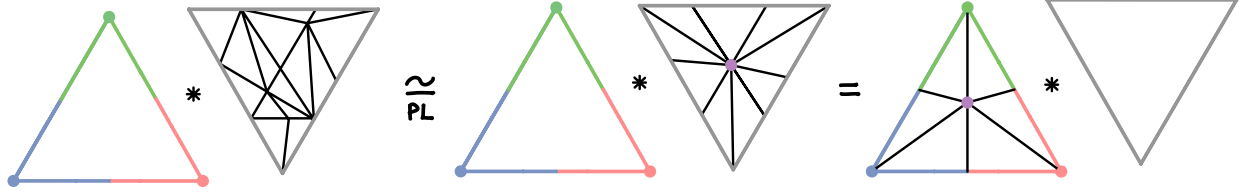


FIGURE 5. The link $\text{Lk}_{P'}(v_\sigma)$ with the induced mirror structure. Each vertex $s \in S(v_\sigma)$ is distinguished in the first picture, and the associated mirror is the join of its star in the left cycle with the boundary of the right disc. The PL-homeomorphism takes the disc $\text{Lk}_{P'}(\sigma)$ to a cone over a cycle, whose cone vertex is distinguished in the picture.

Finally suppose v is a non-cone vertex of L' , i.e. $v = v_\sigma$ is the barycenter of some simplex $\sigma \in L$. The link $\text{Lk}_{P'}(v_\sigma)$ decomposes as a join $(\partial\sigma)' * \text{Lk}_P(\sigma)'$. Since P' is a pseudomanifold with isolated singularities, $\text{Lk}_P(\sigma)'$ is PL-homeomorphic to a simplex Δ (with the boundary $\partial\Delta$ contained in L), which is also PL-homeomorphic to the cone $\text{Cone}(w, \partial\Delta)$. See Figure 5. Let $S(v_\sigma) = \{s \in S \mid v_\sigma \in X_s\}$; note that the elements of $S(v_\sigma)$ correspond to the vertices of σ . There is an induced mirror structure on $\text{Lk}_{P'}(v_\sigma)$ where the s -mirror is the star of the vertex s in $(\partial\sigma)' * \partial\text{Lk}_P(\sigma)'$. The link of v_σ in $\mathcal{U}(W_L, P')$ is the corresponding basic construction $\mathcal{U}(W_{S(v_\sigma)}, \text{Lk}_{P'}(v_\sigma))$.

The PL-homeomorphism between $\text{Lk}_P(\sigma)'$ and $\text{Cone}(w, \partial\Delta)$ induces a PL-homeomorphism $\text{Lk}_{P'}(v_\sigma) \rightarrow (\partial\sigma)' * \Delta$ which takes the each s -mirror to $\text{St}_s(\partial\sigma)' * \partial\Delta$. We note that $(\partial\sigma)' * \Delta = \sigma' * \partial\Delta$, which can be seen by “pushing” the vertex w from $\text{Cone}(w, \partial\Delta)$ to the center of $\partial\sigma$, see Figure 5.

Reflecting σ using the mirror structure $\text{St}_s(\partial\sigma)'$ and the right-angled Coxeter group $W_{S(v_\sigma)} \cong (\mathbb{Z}/2)^{|S(v_\sigma)|}$ yields a combinatorial sphere of dimension $\dim \sigma$. Thus, the link of v_σ in $\mathcal{U}(W_L, P')$ is a join of two combinatorial spheres which is a combinatorial sphere. \square

Combining Proposition 4.3 with Theorem 1.1 gives the following:

Theorem 1.3. Every finite d -dimensional simplicial complex X embeds as a retract into a closed $(2d - 1)$ -dimensional pseudomanifold $Q(X)$ with isolated singularities. If X is aspherical, then $Q(X)$ can be chosen to be aspherical.

REFERENCES

- [Bjö95] A. Björner. Topological methods. In *Handbook of combinatorics, Vol. 1, 2*, pages 1819–1872. Elsevier Sci. B. V., Amsterdam, 1995.
- [BKK02] Mladen Bestvina, Michael Kapovich, and Bruce Kleiner. Van Kampen’s embedding obstruction for discrete groups. *Invent. Math.*, 150(2):219–235, 2002.
- [Bry02] J. L. Bryant. Piecewise linear topology. In *Handbook of geometric topology*, pages 219–259. North-Holland, Amsterdam, 2002.
- [CM99] Donald J. Collins and Charles F. Miller, III. The word problem in groups of cohomological dimension 2. In *Groups St. Andrews 1997 in Bath, I*, volume 260 of *London Math. Soc. Lecture Note Ser.*, pages 211–218. Cambridge Univ. Press, Cambridge, 1999.
- [Dav08] M.W. Davis. *The geometry and topology of Coxeter groups*, volume 32 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2008.
- [DHW23] P. Dani, M. Haulmark, and G. Walsh. Right-angled Coxeter groups with non-planar boundary. *Groups, Geometry, and Dynamics*, 17(1):127–155, 2023.

- [DO01] M. Davis and B. Okun. Vanishing theorems and conjectures for the ℓ^2 -homology of right-angled Coxeter groups. *Geom. Topol.*, 5:7–74, 2001.
- [DR93] A. N. Dranišnikov and D. Repovš. Embeddings up to homotopy type in Euclidean space. *Bull. Austral. Math. Soc.*, 47(1):145–148, 1993.
- [HAM93] Cynthia Hog-Angeloni and Wolfgang Metzler. Geometric aspects of two-dimensional complexes. In *Two-dimensional homotopy and combinatorial group theory*, volume 197 of *London Math. Soc. Lecture Note Ser.*, pages 1–50. Cambridge Univ. Press, Cambridge, 1993.
- [Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.
- [MR26] J. Manning and L. Ruffoni. Incubable hyperbolic 3-pseudomanifold groups. 2026.
- [Sta65] J.R. Stallings. *The Embedding of Homotopy Types Into Manifolds*. Princeton University Press, 1965.
- [vK33] E. R. van Kampen. Komplexe in euklidischen raumen. *Abh. Math. Sem. Univ. Hamburg*, 9(1):72–78, 1933.
- [Zee63] E.C. Zeeman. *Seminar on Combinatorial Topology*. Institut Des Hautes Études Scientifiques, 1963.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, BC, CANADA
V6T 1Z2

Email address: `kasiamath@math.ubc.ca`

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA 70806

Email address: `kschrevelsu.edu`