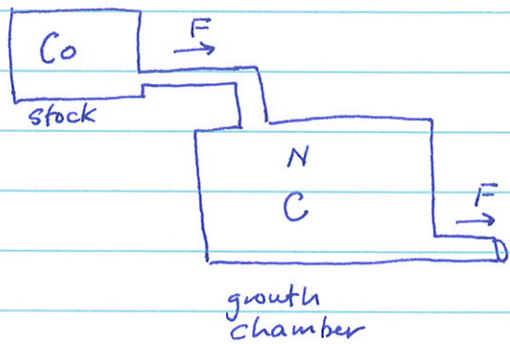
Unit on Dimensional Analysis Chemostat LEK p121-129



Device for sustained growth of bacteria

C_0 = conc of stock solution of nutrient

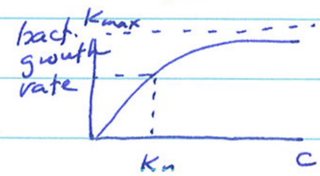
$C(t)$ = conc nutrient in growth chamber

$N(t)$ = " bacteria

adjustable parameters: C_0, also F = flow rate
V = volume of growth chamber

$$\begin{cases} \dfrac{dN}{dt} = K(C)\,N - \dfrac{FN}{V} \\[2ex] \dfrac{dC}{dt} = -\alpha\, K(C)\,N - \dfrac{FC}{V} + \dfrac{FC_0}{V} \end{cases}$$

$$K(C) = \frac{K_{max}\,C}{K_n + C}$$



Dimension carrying eqns:

$$\begin{cases} \dfrac{dN}{dt} = \dfrac{K_{max}\,C}{K_n + C}\,N - \dfrac{FN}{V} \\[2ex] \dfrac{dC}{dt} = -\alpha\,\dfrac{K_{max}\,C}{K_n + C}\,N - \dfrac{FC}{V} + \dfrac{FC_0}{V} \end{cases}$$

Units	N :	mass/vol	gm/cm^3
	C :	mass/vol	gm/cm^3
	F :	vol/time	cm^3/s
	V :	vol	cm^3
	K_{max} :	1/time	$1/s$
	K_n :	mass/vol	gm/cm^3
	α :	—	— (dimensionless)

Transform system to Dimensionless form:

Let
$$t = \bar{\tau}\,\hat{t}$$
$$N = \bar{N}\,\hat{N}$$
$$c = \bar{c}\,\hat{c}$$

dimension-carrying scales to be chosen

dimensionless variables

original variables

Notation: in LEK's book, *s were used for the dimensionless var, and ^ for choices of scales. Avoid confusing!

Rewrite eqns in terms of ^ (dimensionless) variables.

$$\begin{cases} \dfrac{d(\bar{N}\hat{N})}{d(\tau\hat{t})} = K_{max}\dfrac{\bar{c}\hat{c}}{K_n + \bar{c}\hat{c}}(\bar{N}\hat{N}) - \dfrac{F}{V}(\bar{N}\hat{N}) \\[2ex] \dfrac{d(\bar{c}\hat{c})}{d(\tau\hat{t})} = -\alpha\dfrac{K_{max}(\bar{c}\hat{c})}{K_n + (\bar{c}\hat{c})}(\bar{N}\hat{N}) - \dfrac{F}{V}(\bar{c}\hat{c}) + \dfrac{FC_0}{V} \end{cases}$$

$$\begin{cases} \dfrac{\bar{N}}{\tau}\dfrac{d\hat{N}}{d\hat{t}} = K_{max}\dfrac{\hat{c}}{\left(\frac{K_n}{\bar{c}}\right) + \hat{c}}\bar{N}\hat{N} - \dfrac{F}{V}\bar{N}\hat{N} \\[2ex] \dfrac{\bar{c}}{\tau}\dfrac{d\hat{c}}{d\hat{t}} = -\alpha K_{max}\dfrac{\hat{c}}{\left(\frac{K_n}{\bar{c}}\right) + \hat{c}}\bar{N}\hat{N} - \dfrac{F}{V}\bar{c}\hat{c} + \dfrac{FC_0}{V} \end{cases}$$

$$\begin{cases} \dfrac{d\hat{N}}{d\hat{t}} = \tau K_{max}\dfrac{\hat{c}}{\left(\frac{K_n}{\bar{c}}\right) + \hat{c}}\hat{N} - \dfrac{\tau F}{V}\hat{N} \\[2ex] \dfrac{d\hat{c}}{d\hat{t}} = -\tau\alpha K_{max}\dfrac{\bar{N}}{\bar{c}}\dfrac{\hat{c}}{\left(\frac{K_n}{\bar{c}}\right) + \hat{c}}\hat{N} - \dfrac{\tau F}{V}\hat{c} + \dfrac{\tau F C_0}{V\bar{c}} \end{cases}$$

Now chose 'scales' for convenience: let $\tau = V/F$, $\bar{c} = K_n$,

$$\bar{N} = \frac{\bar{c}}{\tau\alpha K_{max}} = \frac{FK_n}{V\alpha K_{max}}$$

$$\begin{cases} \dfrac{d\hat{N}}{d\hat{t}} = \alpha_1 \dfrac{\hat{c}}{1+\hat{c}} \hat{N} - \hat{N} \\[2ex] \dfrac{d\hat{c}}{d\hat{t}} = - \dfrac{\hat{c}}{1+\hat{c}} \hat{N} - \hat{c} + \alpha_2 \end{cases}$$

$$\alpha_1 = \tau K_{max} = \frac{V K_{max}}{F}$$

$$\alpha_2 = \frac{\tau F}{V} \frac{C_0}{\bar{C}} = \frac{C_0}{K_n}$$

We could now drop the ^'s and arrive at:

$$\begin{cases} \dfrac{dN}{dt} = \alpha_1 \left(\dfrac{c}{1+c}\right) N - N \\[2ex] \dfrac{dc}{dt} = - \left(\dfrac{c}{1+c}\right) N - c + \alpha_2 \end{cases}$$

This is the dimensionless system, whose behaviour is seen to be determined by just two parameters, α_1, α_2.

Analysis:

Nullclines:

$\dfrac{dN}{dt} = 0$ at $\alpha_1 \left(\dfrac{c}{1+c}\right) N - N = 0 \Rightarrow N = 0$ or

$$\alpha_1 \frac{c}{1+c} = 1 \Rightarrow$$

$$\alpha_1 c = 1 + c$$

$$c(\alpha_1 - 1) = 1$$

$$c = \frac{1}{\alpha_1 - 1}$$

Conclusion (1) Need $\alpha_1 > 1 \Rightarrow \dfrac{V K_{max}}{F} > 1$ or $\boxed{F < V K_{max}}$

(flow rate should not be too fast)

S.S (1)

Nullclines $\dfrac{dc}{dt} = 0$ at $-\left(\dfrac{c}{1+c}\right) N - c + \alpha_2 = 0$

- if $N = 0$ then $c = \alpha_2$ $\Rightarrow$ ss $(N_{ss}, C_{ss} =) (0, \alpha_2)$
  (from above)

  This is undesirable for a chemostat where it is desired to have $N_{ss} > 0$ i.e. to keep culture going.

- otherwise $N = \dfrac{(1+c)}{c} (\alpha_2 - c)$

$$\Rightarrow N = \left(\frac{1}{c} + 1\right)(\alpha_2 - c)$$

in this case (from before $C_{ss_2} = \frac{1}{\alpha_1 - 1}$)

$$N_{ss_2} = \left(\frac{\alpha_1 - 1}{1} + 1\right)\left(\alpha_2 - \frac{1}{\alpha_1 - 1}\right) = \alpha_1\left(\alpha_2 - \frac{1}{\alpha_1 - 1}\right)$$

ss 2

N_{ss_2}, C_{ss_2}

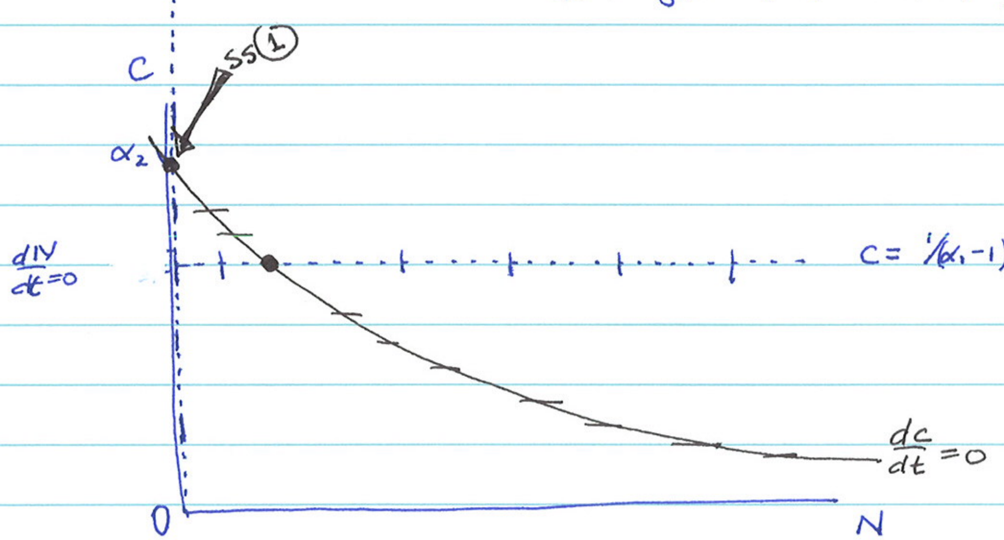
Conclusion 2: to have $N_{ss} > 0$ need $\alpha_2 > \frac{1}{\alpha_1 - 1}$

i.e. $\frac{C_0}{K_n} > \frac{1}{\frac{V K_{max}}{F} - 1}$

$$\boxed{C_0 > \frac{K_n F}{V K_{max} - F}}$$

(stock nutrient soln should be concentrated enough to have bacteria grow faster than they get washed out)

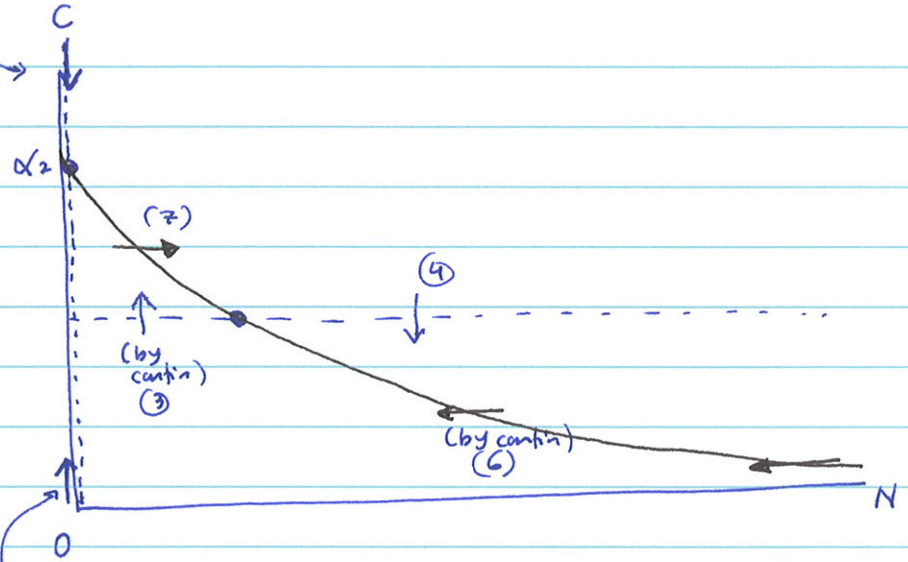
dN/dt=0



p195 LEK

proper configuration when the conditions in $\boxed{}$'s are satisfied.

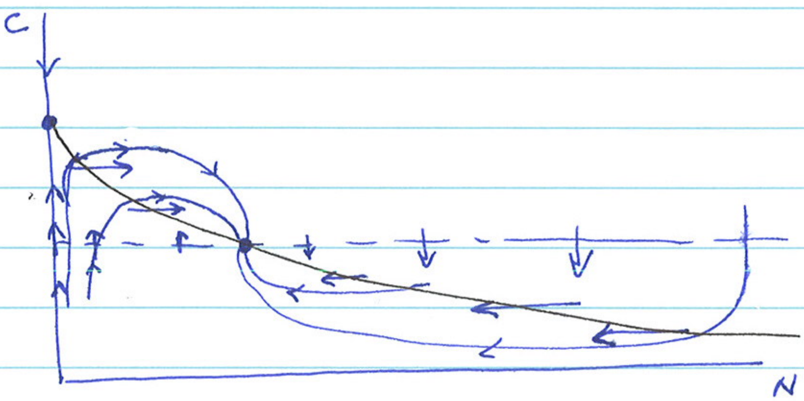
for large c, and $N=0$ $\frac{dc}{dt} \approx -c+\alpha_2$ (2)



(1)

close to (0,0) $\frac{dc}{dt} \sim \alpha_2 > 0$

(5) for c very small $\frac{dN}{dt} \sim -N$



Exercises - Chemostat

(1) Draw a phase-plane diagram for the case where the chemostat "~~wha~~ washes out" i.e. when the flow rate is too fast.

(2) Selection of problems 10 - 13 on p 202-203 in LEK. (attached)

e.g.

Show that there is always a single stable S.S. and that there will not be any oscillations in the approach to that s.s.

(3) Create an XPP ode simulation of the chemostat, showing phase plane behaviour. (Reasonable values of params will be needed).

Use XPP auto to produce a bifurcation diagram showing what happens as the flow rate F is gradually increased.