Review of Linear Stability Analysis : ODEs

Consider the system of two ODEs:

$$\begin{cases} \dfrac{du}{dt} = f(u,v) & (1a) \\[2ex] \dfrac{dv}{dt} = g(u,v) & (1b) \end{cases}$$

We will assume that f, g are smooth functions, and that there exists a steady state value u_{ss}, v_{ss} such that

$$f(u_{ss}, v_{ss}) = 0, \qquad g(u_{ss}, v_{ss}) = 0 \qquad (2)$$

Consider small perturbations of that steady state, i.e. let

$$u(t) = u_{ss} + u' \qquad v(t) = v_{ss} + v' \qquad (3)$$

Substitute these into the ODES, use the fact that $\dfrac{du_{ss}}{dt} = \dfrac{dv_{ss}}{dt} = 0$ and (2). Then use a Taylor expansion of $f(u_{ss}+u', v_{ss}+v')$ and $g(u_{ss}+u', v_{ss}+v')$ - retaining only linear terms - to arrive at

$$\begin{cases} \dfrac{du'}{dt} = \left.\dfrac{\partial f}{\partial u}\right|_{ss} u' + \left.\dfrac{\partial f}{\partial v}\right|_{ss} v' & (4a) \\[2ex] \dfrac{dv'}{dt} = \left.\dfrac{\partial g}{\partial u}\right|_{ss} u' + \left.\dfrac{\partial g}{\partial v}\right|_{ss} v' & (4b) \end{cases}$$

where $|_{ss}$ denotes "evaluate at the steady state (u_{ss}, v_{ss})". Then (4) is a linear 2×2 system of ODES of the form

$$\begin{cases} (5a) & \dfrac{du'}{dt} = au' + bv' \\[2ex] (5b) & \dfrac{dv'}{dt} = cu' + dv' \end{cases} \qquad a = \left.\frac{\partial f}{\partial u}\right|_{ss}, \; b = \left.\frac{\partial f}{\partial v}\right|_{ss}, \text{ etc.}$$

Stability of the steady state (u_{ss}, v_{ss}) hinges on whether solutions to (5) grow or decay, which, in turn, depends on eigenvalues of the matrix

$$J = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \leftarrow \text{(the Jacobian matrix)} \qquad (6)$$

The Linear Algebra approach

Using Linear Algebra, these eigenvalues are roots of

$$\det\left[J - \lambda I\right] = 0 \qquad (7)$$

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0 \quad \Rightarrow \quad (a-\lambda)(d-\lambda) - bc = 0$$

$$\Rightarrow \quad \lambda^2 - (a+d)\lambda + (ad-bc) = 0 \qquad (8)$$

Let $\beta = a+d$ $\leftarrow$ note that $\beta = \mathrm{Trace}(J)$

$\gamma = ad - bc$ $\leftarrow$ $\gamma = \det(J)$

Then the characteristic equation (8) is the quadratic

$$\lambda^2 - \beta\lambda + \gamma = 0$$

whose roots are

$$\lambda = \frac{\beta \pm \sqrt{\beta^2 - 4\gamma}}{2} \qquad (9)$$

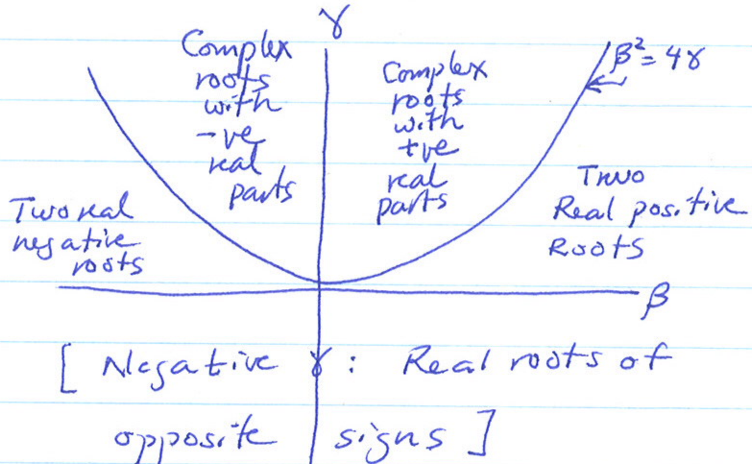
Elimination of 1ODE approach

Alternately, the same result can be obtained from (5) by differentiating 5(a) and using 5a,b to elliminate v. This leads to a second-order ODE

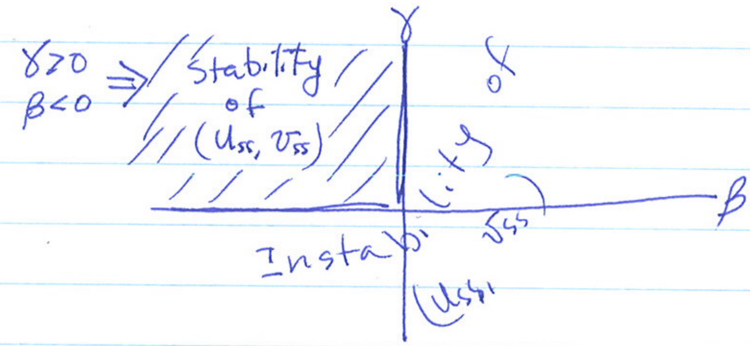
$$\frac{d^2u'}{dt^2} - (a+d)\frac{du'}{dt} + (ad-bc)u' = 0$$

Assuming solutions of the form $u'(t) = C'e^{\lambda t}$ then leads to the same characteristic equation (8), and eigenvalues (9).

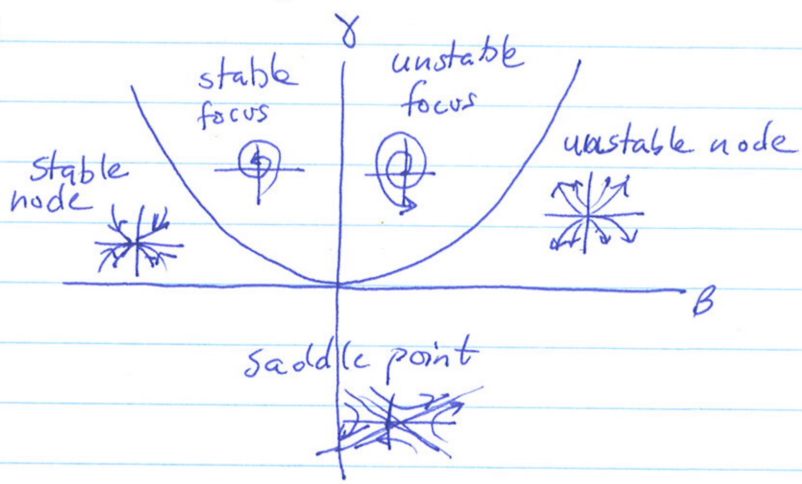
It remains to characterize conditions on β and γ that correspond to various types of eigenvalues: real or complex, positive or negative. This can be done as follows:



Using the above, we can characterize conditions for stability (both eigenvalues have negative real parts)



Moreover, we can identify the kind of behaviour close to (u_{ss}, v_{ss}) as follows:



Summary: The steady state (u_{ss}, v_{ss}) is stable provided

$$\boxed{\beta = \mathrm{Tr}\, J = a+d < 0 \quad \text{and} \quad \gamma = \det J = ad-bc > 0} \tag{10}$$

Linear Stability analysis for Reaction-Diffusion Systems (RD)

We now consider a pair of two PDEs of the RD type,

$$\begin{cases} \dfrac{\partial u}{\partial t} = f(u,v) + D_u \Delta u & \text{(D1a)} \\[2ex] \dfrac{\partial v}{\partial t} = g(u,v) + D_v \Delta v & \text{(D1b)} \end{cases}$$

where Δ refers to the Laplacian $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$ in 3 dimensions.
Here we treat the 1 dimensional version $\Delta = \frac{\partial^2}{\partial x^2}$, i.e. for $0 \le x \le L$,

$$\begin{cases} \dfrac{\partial u}{\partial t} = f(u,v) + D_u \dfrac{\partial^2 u}{\partial x^2} & \text{(D2a)} \\[2ex] \dfrac{\partial v}{\partial t} = g(u,v) + D_v \dfrac{\partial^2 v}{\partial x^2} & \text{(D2b)} \end{cases}$$

(D2c)

We take no flux (Neumann) boundary conditions $\left\{ \frac{\partial u}{\partial x} = 0 \quad \frac{\partial v}{\partial x} = 0 \quad \text{at } x = 0, L \right\}$.

We will assume that there exists a homogeneous steady state soln, (HSS)

$$(u_{ss}, v_{ss}) \quad \text{such that} \quad f(u_{ss}, v_{ss}) = 0, \quad g(u_{ss}, v_{ss}) = 0$$

(And also, of course, $\frac{\partial u_{ss}}{\partial t} = 0$ $\frac{\partial v_{ss}}{\partial t} = 0$ $\frac{\partial u_{ss}}{\partial x} = 0$, $\frac{\partial v_{ss}}{\partial x} = 0$).

We also assume that this HSS is stable in the well-mixed system (D2) where $D_u = D_v \equiv 0$. Recall stability condition (10) was expressed in terms of the partial derivatives of f and g (with respect to u, v) evaluated at the HSS.

We now consider small (spatially non-uniform) perturbations of the HSS, i.e. we let

$$\text{(D3)} \quad \begin{cases} u(x,t) = u_{ss} + u'(x,t) \\ v(x,t) = v_{ss} + v'(x,t) \end{cases} \quad \text{where } u', v' \text{ are small.}$$

Substituting these into (D2), expanding in a Taylor series, and using properties of the HSS, we obtain (after keeping only the linear terms)

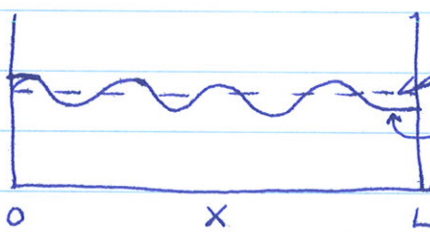
$$
(D4) \quad \begin{cases} \dfrac{\partial u'}{\partial t} = a u' + b v' + D_u \dfrac{\partial^2 u'}{\partial x^2} \\[2ex] \dfrac{\partial v'}{\partial t} = c u' + d v' + D_v \dfrac{\partial^2 v'}{\partial x^2} \end{cases}
$$

where $a = \left.\dfrac{\partial f}{\partial u}\right|_{ss}$, $b = \left.\dfrac{\partial f}{\partial v}\right|_{ss}$, $c = \left.\dfrac{\partial g}{\partial u}\right|_{ss}$, $d = \left.\dfrac{\partial g}{\partial v}\right|_{ss}$

The system (D4) is linear. Its eigenfunctions are of the form

$$\zeta \cos(qx) \qquad\qquad q = \frac{n\pi}{L} \quad n = 1, 2, \ldots$$

so that solutions are $\zeta' e^{\lambda t} \cos(qx)$ where λ, the eigenvalue determines whether this type of solution grows ($\mathrm{Re}\,\lambda > 0$) or decays ($\mathrm{Re}\,\lambda < 0$)

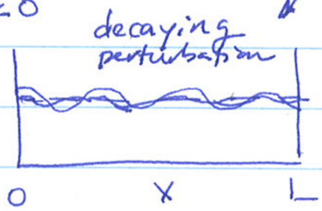


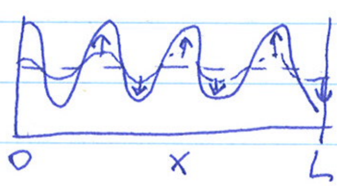
HSS

perturbation $\zeta e^{\lambda t} \cos(qx)$.
The Neumann BC (D2c) can only be satisfied if $q = \frac{n\pi}{L}$, and for eigenfunc. $\cos(qx)$, not $\sin(qx)$.

$\lambda < 0$

(HSS stable)





$\lambda > 0$
growing perturbation
(HSS unstable)

Note that this linear analysis will only predict the short time behavior of the perturbation; in the unstable case, once the perturbation grows, the nonlinear terms that were neglected in (D4) will dominate and affect the results.

Assume
$$u'(x,t) = \hat{u}\, e^{\lambda t} \cos(qx)$$
$$v'(x,t) = \hat{v}\, e^{\lambda t} \cos(qx)$$
where $\hat{u}, \hat{v}$ are constants ($\equiv$ perturb. amplitudes at $t=0$)

Substitute into (D4) and simplify algebraically to obtain

$$\lambda \hat{u} = a\hat{u} + b\hat{v} - D_u q^2 \hat{u}$$
$$\lambda \hat{v} = c\hat{u} + d\hat{v} - D_v q^2 \hat{v}$$

This is an algebraic system that can be written

$$\begin{bmatrix} a - D_u q^2 - \lambda & b \\ c & d - D_v q^2 - \lambda \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To have solutions other than the trivial $\hat{u} = \hat{v} = 0$ (which is the case of <u>no</u> perturbation) it must be true that

$$\det \begin{bmatrix} a - D_u q^2 - \lambda & b \\ c & d - D_v q^2 - \lambda \end{bmatrix} = \det \left[J_D - \lambda I \right] = 0$$

where
$$J_D = \begin{bmatrix} a - D_u q^2 & b \\ c & d - D_v q^2 \end{bmatrix}$$

is an "augmented" Jacobian. (compare with the Jacobian of the "well mixed" system where $D_u = D_v = 0$).

We thus ask whether the matrix J_D has eigenvalues with positive real parts ($\Rightarrow$ growing perturbations $\Rightarrow$ unstable HSS) or not.

$$\text{Let} \quad \beta_D = \operatorname{Trace} J_D = (a - D_u q^2 + d - D_v q^2)$$
$$\gamma_D = \det J_D = (a - D_u q^2)(d - D_v q^2) - bc.$$

By previous assumption, the well-mixed system has a stable S.S.
Recall this implies that

(wellmixed): $\beta \equiv a + d < 0$

Then this immediately implies that $\beta_D < 0$ since $\beta_D < \beta$ by inspection. Now consider γ and γ_D. By the same argument, we have that

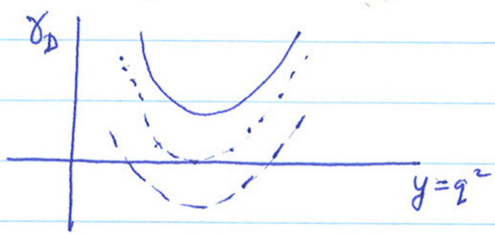
(well mixed): $\gamma = ad - bc > 0$

Note that $\gamma_D = ad - bc \underbrace{- q^2(aD_v + dD_u)} + (q^2)^2 D_u D_v$

It is possible to violate $\gamma_D > 0$ if this term is sufficiently large. Should this be the case, there will be a positive eigenval, and the HSS will be <u>unstable</u>.

We next examine when / how this could happen, i.e. for what conditions $\gamma_D < 0$ for some wavenumber q.

Let $y \equiv q^2$ and note that $\gamma_D = D_u D_v y^2 - (aD_v + dD_u)y + \gamma$ is a parabola in y as shown. Then γ_D has a minimum at $d\gamma_D/dy = 0$, i.e. $y_{min} = \frac{1}{2}\frac{(aD_v + dD_u)}{D_u D_v}$



or, rewritten, $y_{min} = \frac{1}{2}\left(\frac{a}{D_u} + \frac{d}{D_v}\right)$.
$y_{min} = q^2_{min}$

Thus, we expect that (as some parameter varies) γ_D will first go negative for the wavenumber

$$q_{min} = \left[\frac{1}{2}\left(\frac{a}{D_u} + \frac{d}{D_v}\right)\right]^{1/2}$$

This should be the first wavenumber to destabilize the HSS.

Existence of (realvalued) q_{min} $\Longleftrightarrow$ $\frac{a}{D_u} + \frac{d}{D_v} > 0$

We also require (for instability) that $\gamma_p(q^2 = q^2_{min}) < 0$.

It can be shown that this implies that

$$ad - bc < \frac{D_u D_v}{4}\left(\frac{a}{D_u} + \frac{d}{D_v}\right)^2$$

<u>Synthesis/Summary</u>:

We now group all inequalities and interpret what they mean.

Stability of well-mixed system:

$\beta = a + d < 0$ $\Rightarrow$ either a or d or both < 0

Existence of q_{min}: $\frac{a}{D_u} + \frac{d}{D_v} > 0$ $\Rightarrow$ D_u cannot be same as D_v, since then $\Rightarrow \frac{a+d}{D} > 0$ contradicts the inequal $\beta < 0$

$\Rightarrow D_u \neq D_v$

also $\Rightarrow$ not both a and d < 0 so only <u>one</u> of them is negative.

Suppose $d < 0$; then $a > 0$ and $\left|\frac{a}{D_u}\right| > \left|\frac{d}{D_v}\right|$

Remark: this can be rewritten as $\left|\frac{D_u}{a}\right|^{1/2} < \left|\frac{D_v}{d}\right|^{1/2}$ which says that the "diffusion distance" of u is <u>smaller</u> than that of v

Stability of wellmixed system:

$\gamma = ad - bc > 0$ by above, $a > 0$, $d < 0 \Rightarrow ad < 0$

so we need $-bc > 0 \Rightarrow bc < 0 \Rightarrow$ one of b or c must be negative (but not both), Sign patterns of the Jacobian

If b is negative :

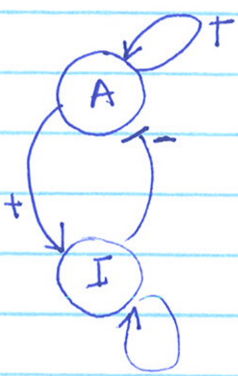
$$\begin{bmatrix} + & - \\ + & - \end{bmatrix}$$

If c is negative:

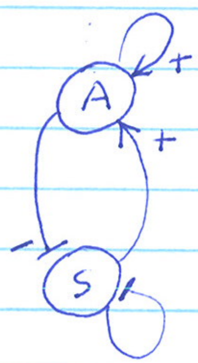
$$\begin{bmatrix} + & + \\ - & - \end{bmatrix}$$

Condition for diffusive ("Turing") instability:

$$\boxed{0 < ad - bc < \frac{D_u D_v}{4}\left(\frac{a}{D_u} + \frac{d}{D_v}\right)^2}$$



activator - inhibitor system



substrate - depletion system

$$\gamma_p < 0 \quad \Rightarrow$$

$$\gamma_p < 0 \text{ at } q^2 = q^2_{min} \quad \Rightarrow$$

$$\frac{D_u D_v}{4}\left(\frac{a}{D_u} + \frac{d}{D_v}\right)^2 - \frac{1}{2}\left(a D_v + d D_u\right)\left(\frac{a}{D_u} + \frac{d}{D_v}\right) + ad - bc < 0$$

$$\underbrace{\frac{1}{4} D_u D_v \left(\frac{a}{D_u} + \frac{d}{D_v}\right)^2 - \frac{1}{2} D_u D_v \left(\frac{a}{D_u} + \frac{d}{D_v}\right)^2} + (ad - bc) < 0$$

$$: \ -\frac{1}{4} D_u D_v \left(\frac{a}{D_u} + \frac{d}{D_v}\right)^2 + (ad - bc) < 0$$

$$\Rightarrow \qquad 0 < \frac{(ad - bc)}{D_u D_v} < \frac{1}{4}\left(\frac{a}{D_u} + \frac{d}{D_v}\right)^2$$