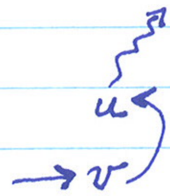
Meinhardt (1983)

$$u_t = e u^2 v - \mu u + D_u \Delta u$$
$$v_t = e_0 - e u^2 v + D_v \Delta v$$




SS $\mu u = e u^2 v \Rightarrow u = 0$ or else $\mu = e u v$

$$e_0 = e u^2 v$$

$$\frac{e_0}{\mu} = \frac{e u^2 v}{e u v} = v \Rightarrow u = \frac{\mu}{e v} = \frac{\mu^2}{e e_0}$$

HSS

$$J = \begin{bmatrix} -\mu + 2euv & eu^2 \\ -2euv & -eu^2 \end{bmatrix} \qquad J_{ss} = \begin{bmatrix} \mu & eu_0^2 \\ -2\mu & -eu_0^2 \end{bmatrix}$$

$$\beta = \mathrm{Tr}\, J_{ss} = \mu - e u_0^2$$
$$\gamma = \det J_{ss} = -\mu e u_0^2 + 2\mu e u_0^2 = \mu e u_0^2 > 0$$

HSS stable iff $\beta < 0$ : $e u_0^2 > \mu$

$$e\left(\frac{\mu^2}{e e_0}\right)^2 > \mu$$
$$\frac{e \mu^4}{e^2 e_0^2} > \mu$$
$$\frac{\mu^3}{e e_0^2} > 1$$

spatial part.

$$J_\Delta = \begin{bmatrix} \mu - k^2 D_u & e u_0^2 \\ -2\mu & -e u_0^2 - k^2 D_v \end{bmatrix}$$

$$\gamma = -(\mu - k^2 D_u)(e u_0^2 + k^2 D_v) + 2\mu e u_0^2$$
$$= -\left[\mu e u_0^2 + (k^2)(-D_u e u_0^2 + D_v \mu) - (k^2)^2 D_u D_v\right] + 2\mu e u_0^2$$
$$= \mu e u_0^2 + k^2(-D_u e u_0^2 + D_v \mu) + (k^2)^2 D_u D_v$$

Meinhardt's system

$$u_t = e u^2 v - \mu u + D_u u_{xx}$$
$$v_t = e_0 - e u^2 v + D_v v_{xx}$$

Dimensional analysis : define scales for

time : $\tau$

space: $\bar{x}$

conc: $\bar{u}, \bar{v}$

Let:

$t = t^* \tau$
$x = x^* \bar{x}$
$u = u^* \bar{u}$
$v = v^* \bar{v}$

$$\left\{ \begin{aligned} \frac{\bar{u}}{\tau} \frac{\partial u^*}{\partial t^*} &= (e \bar{u}^2 \bar{v}) u^{*2} v^* - (\mu \bar{u}) u^* + \frac{D_u \bar{u}}{\bar{x}^2} \frac{\partial^2 u^*}{\partial x^{*2}} \\ \frac{\bar{v}}{\tau} \frac{\partial v^*}{\partial t^*} &= e_0 - (e \bar{u}^2 \bar{v}) u^{*2} v^* + \frac{D_v \bar{v}}{\bar{x}^2} \frac{\partial^2 v^*}{\partial x^{*2}} \end{aligned} \right.$$

$$\left\{ \begin{aligned} \frac{\partial u^*}{\partial t^*} &= (e \tau \bar{u} \bar{v}) u^{*2} v^* - (\mu \tau) u^* + \left(\frac{D_u \tau}{\bar{x}^2}\right) \frac{\partial^2 u^*}{\partial x^{*2}} \\ \frac{\partial v^*}{\partial t^*} &= \left(\frac{e_0 \tau}{\bar{v}}\right) - (e \bar{u}^2 \tau) u^{*2} v^* + \left(\frac{D_v \tau}{\bar{x}^2}\right) \frac{\partial^2 v^*}{\partial x^{*2}} \end{aligned} \right.$$

chose $\tau = 1/\mu$ $\quad \bar{v} = e_0 \tau = e_0/\mu$

$\bar{u}^2 = \frac{1}{e\tau} = \mu/e \qquad \bar{u} = \left(\frac{\mu}{e}\right)^{1/2}$

$\bar{x}^2 = D_u \tau \qquad \bar{x} = (D_u/\mu)^{1/2}$

Drop $*$s. Get:

$$\frac{\partial u}{\partial t} = \alpha u^2 v - u + u_{xx}$$

$$\frac{\partial v}{\partial t} = 1 - u^2 v + d v_{xx}$$

$$\alpha = e \tau \bar{u} \bar{v} = \frac{e}{\mu} \left(\frac{\mu}{e}\right)^{1/2} \frac{e_0}{\mu} = \frac{e^{1/2} e_0}{\mu^{3/2}}$$

$$d = D_v / D_u$$

HSS: $\alpha u^2 v - u = 0 \Rightarrow u = 0$ or $\alpha u v - 1 = 0 \Rightarrow \alpha u v = 1$

$1 - u^2 v = 0 \Rightarrow u^2 v = 1$

$$\Rightarrow \frac{u^2 v}{\alpha u v} = 1 \qquad \boxed{u = \alpha, \quad v = 1/\alpha^2}$$

$uv = 1/\alpha$

$\alpha u v = 1$

$u^2 = \alpha^2$

Stability of HSS

$$J = \begin{bmatrix} 2\alpha u v - 1 & \alpha u^2 \\ -2uv & -u^2 \end{bmatrix}\Bigg|_{ss} = \begin{bmatrix} 1 & \alpha^3 \\ -2/\alpha & -\alpha^2 \end{bmatrix}$$

$\beta = \operatorname{Tr} J = 1 - \alpha^2$

$\gamma = \det J = -\alpha^2 + 2\alpha^3/\alpha = -\alpha^2 + 2\alpha^2 = \alpha^2 > 0$

For stability, need $1 - \alpha^2 < 0$ $\quad \alpha^2 > 1 \Rightarrow \alpha > 1$ or $\underbrace{\alpha < -1}_{\text{we'll discount this}}$

Diffusive instability to perturbations $e^{\sigma t} e^{iqx}$ $\quad q =$ wavenumber

Calculation reduces to the Jacobian

$$J_\Delta = \begin{bmatrix} 1 - q^2 & \alpha^3 \\ -2/\alpha & -\alpha^2 - dq^2 \end{bmatrix}$$

$\beta_\Delta = \operatorname{Tr}(J_\Delta) = 1 - \alpha^2 - q^2(1+d) < 0 \leftarrow$ provided $\beta < 0$ as per above

$\gamma_\Delta = \det(J_\Delta) = -(1-q^2)(\alpha^2 + dq^2) + 2\alpha^2$

$= -[\alpha^2 + q^2(d - \alpha^2) - dq^4] + 2\alpha^2$

$= \alpha^2 + q^2(\alpha^2 - d) + d(q^2)^2$

Ask if $\exists\, q^2$ such that $\gamma_\Delta < 0$

$\gamma_{min}$ attained when $0 = \dfrac{d\gamma_\Delta}{dq^2} = (\alpha^2 - d) + 2dq^2 \Rightarrow q^2 = -\left(\dfrac{\alpha^2 - d}{2d}\right)$

has to be positive for this $q$ to exist

$$q^2 = \frac{d-\alpha^2}{2d} = \frac{1}{2} - \frac{\alpha^2}{2d} > 0 \qquad \Rightarrow \frac{\alpha^2}{d} < 1 \Rightarrow \alpha^2 < d$$

Wavenumber that first goes unstable is

$$q_{min}^2 = \left(\frac{1}{2} - \frac{\alpha^2}{2d}\right)$$

$$\gamma_\Delta = \alpha^2 + q^2(\alpha^2-d) + d(q^2)^2 = A\left(q^2 - q_{min}^2\right)^2 + B < 0$$

$$\gamma_\Delta(q_{min}^2) = \alpha^2 + \left(\frac{1}{2} - \frac{\alpha^2}{2d}\right)(\alpha^2-d) + d\left(\frac{1}{2} - \frac{\alpha^2}{2d}\right)^2$$

$$= \alpha^2 - \frac{1}{2d}(\alpha^2-d)(\alpha^2-d) + d\cdot\left(\frac{1}{2d}\right)^2(d-\alpha^2)^2$$

$$= \alpha^2 - \frac{1}{2d}(\alpha^2-d)^2 + \frac{1}{4d}(\alpha^2-d)^2$$

$$= \alpha^2 - \frac{1}{4d}(\alpha^2-d)^2 < 0$$

Boundary of stability region: $\gamma_\Delta = 0 \Leftrightarrow \alpha^2 = \frac{1}{4d}(\alpha^2-d)^2$

$$4\alpha^2 = \frac{(\alpha^2-d)^2}{d} \Leftrightarrow 2\alpha = \frac{\alpha^2-d}{\sqrt{d}}$$

$$4\alpha^2 d = \alpha^4 - 2\alpha^2 d + d^2$$

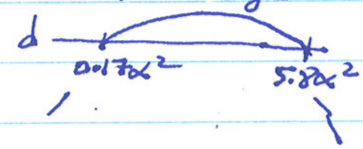
$$d^2 - (6\alpha^2)d + \alpha^4 = 0$$

$$d^2 = \frac{6\alpha^2 \pm \sqrt{36\alpha^4 - 4\alpha^4}}{2} = \frac{3\alpha^2 \pm \sqrt{32\alpha^4}}{2}$$

$$= 3\alpha^2 \pm 2\sqrt{2}\alpha^2 = \alpha^2(3 \pm 2\sqrt{2}) \doteq \alpha^2(0.17, 5.8)$$

When $d = \alpha^2$ then $\gamma_\Delta(q_{min}^2) = \alpha^2 > 0$ $\Rightarrow$ no instabil. inside this range

thus need $d > 5.8\alpha^2$ for instabil.



Eigenvalues of well mixed system:

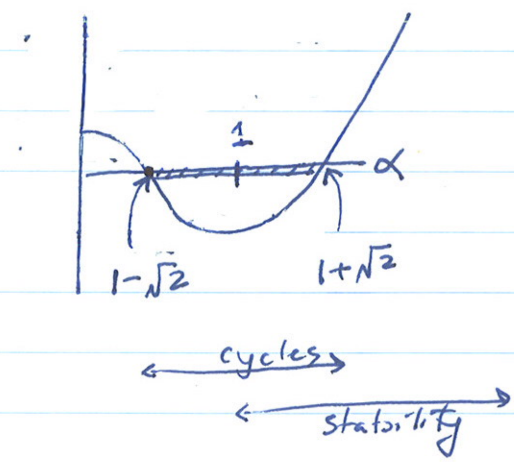
$$\det\left(J_{\substack{\text{well}\\\text{mixed}}} - \lambda I\right) = \det \begin{bmatrix} 1-\lambda & \alpha^3 \\ -2/\alpha & -\alpha^2-\lambda \end{bmatrix}$$

$$\lambda^2 - \beta\lambda + \gamma = 0 \qquad \beta = \mathrm{Tr}(J) = 1-\alpha^2$$
$$\gamma = \det J = \alpha^2$$

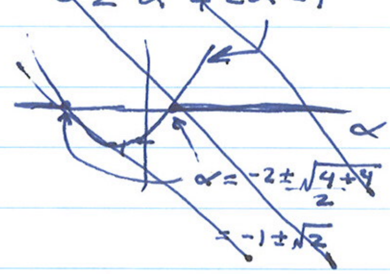
$$\lambda = \frac{\beta \pm \sqrt{\beta^2 - 4\gamma}}{2} = \frac{1-\alpha^2 \pm \sqrt{(1-\alpha^2)^2 - 4\alpha^2}}{2}$$
$$= \frac{1}{2}\left(1-\alpha^2 \pm \sqrt{1-6\alpha^2+\alpha^4}\right)$$

stability whenever $1-\alpha^2 < 0 \Rightarrow \alpha^2 > 1$

cycles if $4\alpha^2 > (1-\alpha^2)^2$
$$\Rightarrow \alpha^4 - 6\alpha^2 + 1 < 0$$




$0 < \alpha^2 + 2\alpha - 1$



## LPA system for same model

$$\frac{\partial u}{\partial t} = \underbrace{\alpha u^2 v - u}_{f(u,v)} + u_{xx} \qquad \leftarrow \text{`slow'}$$

$$\frac{\partial v}{\partial t} = \underbrace{1 - u^2 v}_{g(u,v)} + d\, v_{xx} \qquad \leftarrow \text{`fast'}$$

HSS $u = \alpha \quad v = 1/\alpha^2$

LPA: $u$ replaced by $u^l$ and $u^g$

$v$ " " $v^g$

$$\frac{du^g}{dt} = \underbrace{\alpha (u^g)^2 v^g - u^g} \leftarrow f(u_g, v_g)$$

$$\frac{dv^g}{dt} = \underbrace{1 - (u^g)^2 v^g} \leftarrow g(u_g, v_g)$$

$$\frac{du^l}{dt} = \alpha (u^l)^2 v^g - u^l \qquad \tilde{f}(u^l, v_g)$$

SS
$u^g = \alpha$, $v^g = 1/\alpha^2$
$u^l = \alpha$

Jacobian:
$$J = \begin{bmatrix} f_{u_g} & f_{v_g} & f_{u^l} \\ g_{u_g} & g_{v_g} & g_{u^l} \\ \tilde{f}_{u_g} & \tilde{f}_{v_g} & \tilde{f}_{u^l} \end{bmatrix} = \begin{bmatrix} 1 & \alpha^3 & 0 \\ -2/\alpha & -\alpha^2 & 0 \\ 0 & \alpha(u^l)^2 & 2\alpha u^l v_g - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \alpha^3 & 0 \\ -2/\alpha & -\alpha^2 & 0 \\ 0 & \alpha^3 & 1 \end{bmatrix}$$

Eigenvalues of $J$ are those of this block plus one more here on diagonal which is positive, $\lambda_l = 1$ $\Rightarrow$ pulse grows

$$\left[\begin{array}{c:c} \text{Well mixed sys Jacob} & 0 \\ \hdashline \text{stuff} & 1 \end{array}\right]$$

Schnakenberg

$$u_t = a - u + u^2 v + D_u \Delta u \qquad f(u,v) = a - u + u^2 v$$

$$v_t = b - u^2 v + D_v \Delta v \qquad g(u,v) = b - u^2 v$$

S.S.

$$u^2 v = b$$

$$a - u + u^2 v = 0 \Rightarrow a - u + b = 0 \Rightarrow u = a + b$$

$$\Rightarrow v = \frac{b}{u^2} = \frac{b}{(a+b)^2}$$

[Note: at ss $uv = \frac{b}{a+b}$ and $u^2 = (a+b)^2$ (We use these below).

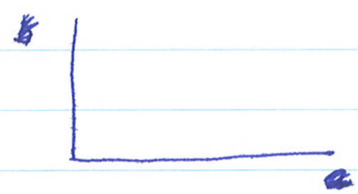
$$J = \begin{bmatrix} -1 + 2uv & u^2 \\ -2uv & -u^2 \end{bmatrix} \qquad J_{ss} = \begin{bmatrix} -1 + 2\frac{b}{a+b} & (a+b)^2 \\ -2\frac{b}{(a+b)} & -(a+b) \end{bmatrix}$$

$$\beta = \operatorname{Tr} J = -1 - (a+b) + \frac{2b}{a+b} = \frac{-(a+b)[1 + a + b] + 2b}{a+b}$$

$$\gamma = \det J = (a+b) - 2b + 2b(a+b)$$
$$= a - b + 2b(a+b)$$

For stability of HSS need $\beta < 0 \Rightarrow (a+b)(1+a+b) > 2b$

$$a + b + a^2 + 2ab + b^2 > 2b$$
$$a^2 + a(2b+1) + b^2 - b > 0$$

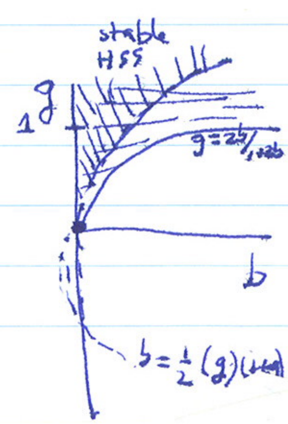


Let $g \equiv a + b$ then

$$\beta = -1 - g + \frac{2b}{g}$$
$$\gamma = g - 2b + 2bg$$
$$= g + 2b(g-1) = g(1+2b) - 2b$$

$$\gamma > 0 \Rightarrow g(1+2b) > 2b \qquad g > \frac{2b}{1+2b}$$

$$\beta < 0 \Rightarrow 1 + g > \frac{2b}{g} \qquad g(1+g) > 2b$$



Schakenberg LPA

$$u = u^g$$
$$v = v^g$$
and also $u^l$

global $\begin{cases} u_t = a - u + u^2 v \\ v_t = b - u^2 v \end{cases}$

local $\quad u^l_t = a - u^l + (u^l)^2 v$

$$\cancel{J = \begin{bmatrix} -1- & 2uv & 0 \\ -2 & & \\ & & \end{bmatrix}}$$

ss $\quad v = \dfrac{b}{u^2} = \dfrac{b}{a+b} \qquad u = a+b$

$$\frac{b}{a+b}(u^l)^2 - u^l + a = 0$$

$$(u^l)^2 - \frac{(a+b)}{b}u^l + \frac{a(a+b)}{b} = 0$$

(two solns)

$$J = \begin{bmatrix} -1+2uv & u^2 & 0 \\ -2uv & -u^2 & 0 \\ 0 & -1+2u^l v & (u^l)^2 \end{bmatrix}$$

eigenval

$$\det \begin{bmatrix} -1+2uv-\lambda & u^2 & 0 \\ -2uv & -u^2-\lambda & 0 \\ 0 & -1+2u^l v & (u^l)^2-\lambda \end{bmatrix} = 0$$

$$\left[(u^l)^2 - \lambda\right] \det\left[\underbrace{2\times 2 \text{ sys}}_{\text{orig. eigenval}}\right] = 0$$

$\lambda = (u^l)^2$ ← additional eigenval.