Travelling waves and Wave-Pinning in a set of PDEs.

Goals:
- To explore travelling wave speed in a single Reaction-Diffusion (RD) equation
- To examine a pair of RD eqns with mass conservation and "cubic" type kinetics
- Based on vastly different rates of diffusion, to arrive at a scaled, dimensionless model with a small parameter (ε)
- To exploit the small parameter in asymptotic expansions both near an interface of the wave ("inner soln") and far away ("outer soln")
- To arrive at an ODE that characterizes the motion of that interface, and to show that it can stall ($\equiv$ wavepinning) under the appropriate conditions.

Preliminaries : Speed of travelling wave in a single RD eqn.

Let $u(x,t)$ be variable of interest on $-\infty \leq x \leq \infty$

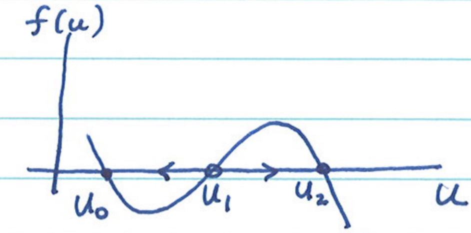
Suppose u satisfies

$$u_t = f(u) + D\, u_{xx}$$

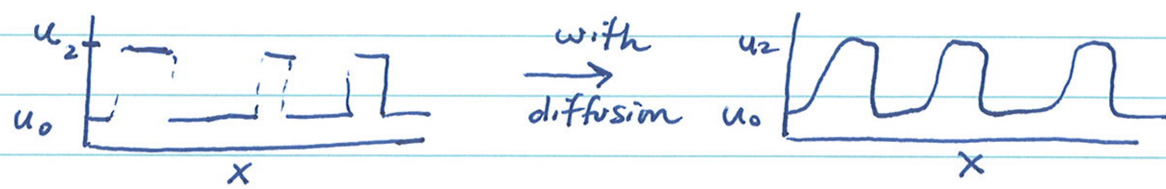
(reaction) (diffusion)

with no-flux boundary conditions (BC's) : $u_x = 0$ at $x = \pm\infty$

For example :
"cubic kinetics"

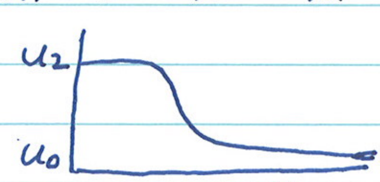


In the well-mixed system, such kinetics lead to one of two stable steady states, u_0 or u_2 (depending on initial conditions.) We would expect that spatially, we could obtain a solution to the pde that "mixes" these steady states:



Here, we have shown that diffusion smooths out the sharp transitions.

We will be interested here in single interfaces:

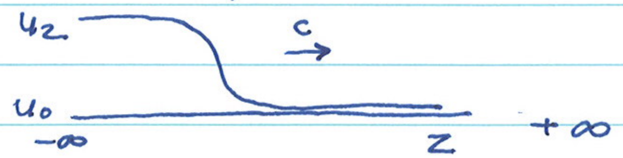


we ask whether such solns exist, and, in particular whether they are stationary or travelling waves. We are, moreover, interested in speed and direction of motion.

<u>Travelling waves</u>: To look for such soln's, we transform to a moving coordinate system (moving at speed c)

Let $z = x - ct$ $\qquad U(z(x,t)) = u(x,t)$

In this coordinate system, the wave will look like:



Use chain rule to write $u_x = \frac{\partial u}{\partial x} = \frac{dU}{dZ}\frac{\partial z}{\partial x} = \frac{dU}{dZ}\cdot 1$

$$u_t = \frac{\partial u}{\partial t} = \frac{dU}{dZ}\frac{\partial z}{\partial t} = \frac{dU}{dZ}(-c)$$

Transforming to the new variables leads to

$$-c\frac{dU}{dZ} = f(U) + D\frac{d^2U}{dZ^2} \qquad \text{and } \frac{dU}{dZ} = 0 \text{ for } Z = \pm\infty$$

<u>Wave speed</u>: First note that the following small calculation is useful:
$$\boxed{\frac{d}{dz}\left(\left(\frac{du}{dz}\right)^2\right) = 2\frac{du}{dz}\cdot\frac{d}{dz}\left(\frac{du}{dz}\right) = 2\frac{du}{dz}\frac{d^2u}{dz^2}}$$

Our eqn, in the moving coordinate frame is

$$-c\frac{du}{dz} = f(u) + D\frac{d^2u}{dz^2}$$

Multiply by $\frac{du}{dz}$:

$$-c\left(\frac{du}{dz}\right)^2 = f(u)\frac{du}{dz} + D\frac{d^2u}{dz^2}\frac{du}{dz}$$

$$= f(u)\frac{du}{dz} + \frac{D}{2}\frac{d}{dz}\left(\left(\frac{du}{dz}\right)^2\right) \quad \text{(see above)}$$

Integrate both sides:

$$-c\int_{-\infty}^{\infty}\left(\frac{du}{dz}\right)^2 dz = \int_{-\infty}^{\infty} f(u)\frac{du}{dz}\,dz + \frac{D}{2}\int_{-\infty}^{\infty}\frac{d}{dz}\left(\left(\frac{du}{dz}\right)^2\right)dz$$

$$= \int_{u(-\infty)}^{u(\infty)} f(u)\,du + \underbrace{\frac{D}{2}\left(\frac{du}{dz}\right)^2\Bigg|_{-\infty}^{\infty}}$$

BCs $\Rightarrow \frac{du}{dz} = 0$ at $z = \pm\infty$

so this term vanishes.

Since we look for a moving front as shown on previous page, $u(-\infty) = u_2$, $u(+\infty) = u_0$

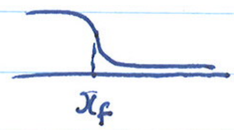
so

$$c = -\int_{u_2}^{u_0} f(u)\,du \Bigg/ \int_{-\infty}^{\infty}\left(\frac{du}{dz}\right)^2 dz$$

Remarks:

- Same calculation works equally well for non-constant front speed, e.g. set

$$z = x - x_f(t)$$



where $x_f(t)$ is the "position" of the moving front (e.g. some coordinate on that interface, such as inflection pt. or point corresponding to arbitrary u value).

Then $\quad z = Z(x,t) \qquad \frac{\partial z}{\partial x} = 1 \qquad \frac{\partial z}{\partial t} = -\frac{dx_f}{dt} = -v(t)$

where $v(t)$ is the time-dependent wave speed.

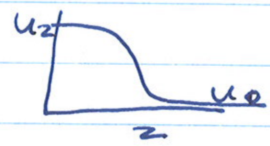
- The above calculation on previous page may not provide an explicit value of the speed c, but we can extract from it the direction of motion ($c>0$ or $c<0$) and whether the wave can ever be stationary ($c=0$). Note that the denominator is always >0.

Also: $\quad c = 0 \iff$ stationary wave $\iff$

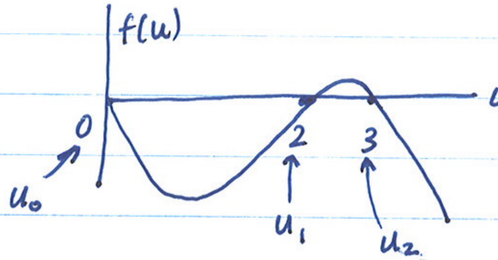
$$\int_{u_2}^{u_0} f(u)\,du = 0$$

Exercise: (1) Suppose $f(u) = (u-u_0)(u_1-u)(u-u_2)$

If $u_0 = 0$, $u_1 = 2$ and $u_2 = 3$, which way would a front solution of the form move?



Soln: sketch $f(u)$; a cubic. Even a rough sketch will convince us that

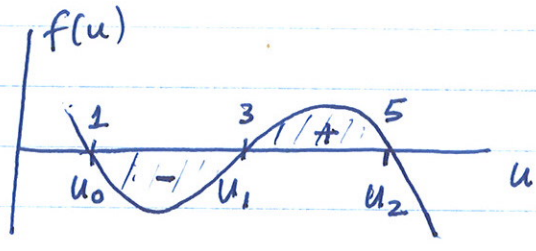


$$c = -\int_3^0 f(u)\,du = -\left[\begin{array}{c}\text{positive}\\ \text{quantity}\end{array}\right]$$

$c < 0$ so move to left.

(2) For the same form of f, given that $u_0 = 1$ and $u_1 = 3$, for what value of u_2 would we expect a stationary wave?

Soln: To answer this, we look for u_2 such that the plot of $f(u)$ has 0 integral over the range (u_0, u_2)



By symmetry of the cubic, picking $u_2 = 5$ will do it.

since have "equal and opposite" "areas" (shown shaded)

Thus
$$\int_{u_2}^{u_0} f(u)\,du = \int_{u_2}^{u_1} + \int_{u_1}^{u_0} = 0$$