Wave-Pinning

Here we consider a set of RD eqns with kinetics of the form

$$\tilde{f}(u,v) = \eta\left(k_0 + \frac{\gamma u^2}{m^2+u^2}\right)v - \eta u \qquad (1)$$

and $D_u \ll D_v$ $\qquad [\eta] = \text{time}^{-1}$

It is assumed that the RD eqs have been scaled by defining

$$x = L x^*, \quad t = \frac{L}{\sqrt{\eta D_u}} t^*, \quad u = m u^*$$

Let $\quad \varepsilon^2 = \dfrac{D_u}{\eta L^2}, \quad D = \dfrac{D_v}{\eta L^2}$

We assume that $D \sim O(1) \Rightarrow \sqrt{\dfrac{D_v}{\eta}} \approx L$

$\Rightarrow$ v can diffuse across the domain during the timescale of the reaction kinetics.

$D_u \ll D_v \Rightarrow \varepsilon$ is a small parameter.

By scaling, we arrive at the dimensionless system*

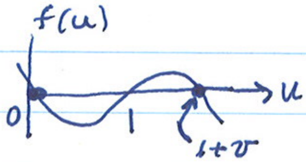
$$\varepsilon \frac{\partial u}{\partial t} = \varepsilon^2 u_{xx} + f(u,v) \qquad 2(a)$$

$$\varepsilon \frac{\partial v}{\partial t} = D\, v_{xx} - f(u,v) \qquad 2(b)$$

with $u_x = 0$, $v_x = 0$ at $x = 0, 1$ and $\quad f = \left(k_0 + \dfrac{\gamma u^2}{1+u^2}\right)v - u \qquad 2(c)$

(The above is in terms of *'d variables, but we have dropped the *s)

It is convenient to consider the caricature



$$f(u,v) = u(1-u)(u-(1+v))$$

where $u_- = 0$ and $u_+ = 1+v$ are the stable HSS, $u = 1$ is unstable.

Exploit small parameter (ε) to gain insight via asymptotic analysis.

(I) <u>Short time scale</u> (~ 1 sec)

Define $t_s = t/\varepsilon$ (← "zoom in at the events occuring very early")

Expand u, v as asymptotic series in ε

$$u(x,t_s) = u_0(x,t_s) + \varepsilon u_1(x,t_s) + \dots$$
$$v(x,t_s) = v_0(x,t_s) + \varepsilon v_1(x,t_s) + \dots$$

} higher order terms

substitute into eqs 2, keeping lowest order terms: get

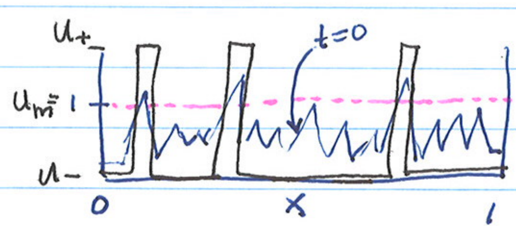
$$\frac{\partial u_0}{\partial t_s} = f(u_0, v_0) \qquad 3(a)$$

$$\frac{\partial v_0}{\partial t_s} = D\frac{\partial^2 v_0}{\partial x^2} - f(u_0, v_0) \qquad 3(b)$$

3(a) $\Rightarrow$ u_0 evolves to u_+ or u_- depending on its initial value relative to $u_m = 1$ (see diagram)

3(b) $\Rightarrow$ v_0 diffuses rapidly, attains some uniform profile

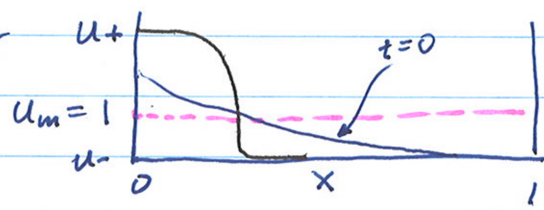
e.g.:



— = initial u_0 value

— = u_0 after short time

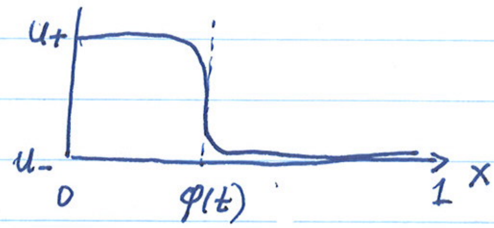
or



← We will be interested in this case

Intermediate time $t \sim 30$ sec

We ask what happens to the front that was formed on the short time scale. Let $\varphi(t)$ denote the interface position.



We break problem up into the dynamics far from the front ("outer soln") and those very close to the front ("inner soln") and use matched asymptotic analysis to ascertain how this soln. behaves.

(1) OUTER SOLN Let
$$u(x,t) = u_0(x,t) + \varepsilon u_1(x,t) + \ldots$$
$$v(x,t) = v_0(x,t) + \varepsilon v_1(x,t) + \ldots$$

substitute into eqs 2 and keep only lowest order terms

Then obtain
$$0 = f(u_0, v_0) \qquad 4(a)$$
$$0 = D\frac{\partial^2 v_0}{\partial x^2} - f(u_0, v_0) \qquad 4(b)$$

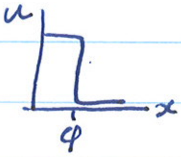
4 (a+b) $\Rightarrow$ $D\dfrac{\partial^2 v_0}{\partial x^2} = 0$

and BC's $v_x = 0$ at $x = 0, 1$

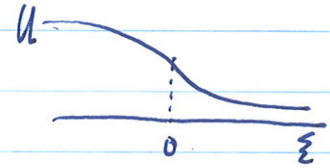
$\Rightarrow$ v is constant to left of, and to right of front
($v_>$) ($v_<$)

$\Rightarrow$ 4(a) $\Rightarrow$ u_0 takes on values u_+, u_- to left of and right of front
(these values may depend on v_0. In particular, $u_+ = 1 + v_0$).

(2) INNER SOLN (close to the sharp interface)



Zoom in at the front, by defining a stretched spatial coordinate:



(5) $$\xi = \frac{(x - \varphi(t))}{\varepsilon}$$

Let $U(\xi(x,t),t) = u(x,t)$ $\quad V(\xi(x,t),t) = v(x,t)$

Then $U(\xi,t)$, $V(\xi,t)$ are the variables transformed to this stretched coord. system

Let
$$U(\xi,t) = U_0(\xi,t) + \varepsilon U_1(\xi,t) + \dots \qquad (6a)$$
$$V(\xi,t) = V_0(\xi,t) + \varepsilon V_1(\xi,t) + \dots \qquad (6b)$$

Also note that

Multivariate (chain Rule)

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial U}{\partial \xi}\dfrac{\partial \xi}{\partial x} = \dfrac{1}{\varepsilon}\dfrac{\partial U}{\partial \xi} \\[2ex] \dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{\varepsilon}\dfrac{\partial}{\partial \xi}\left(\dfrac{\partial U}{\partial \xi}\right)\dfrac{\partial \xi}{\partial x} = \dfrac{1}{\varepsilon^2}\dfrac{\partial^2 U}{\partial \xi^2} \\[2ex] \dfrac{\partial u}{\partial t} = \dfrac{\partial U}{\partial \xi}\dfrac{\partial \xi}{\partial t} + \dfrac{\partial U}{\partial t} = \dfrac{-\varphi'(t)}{\varepsilon}\cdot\dfrac{\partial U}{\partial \xi} + \dfrac{\partial U}{\partial t} \end{cases}$$

velocity of front appears here

With these connections, substituting into eqs 2 leads to

$$\varepsilon \frac{\partial U}{\partial t} - \varphi'(t)\frac{\partial U}{\partial \xi} = F(U,V) + \frac{\partial^2 U}{\partial \xi^2} \qquad (7a)$$

similarly for V:

$$\varepsilon \frac{\partial V}{\partial t} - \varphi'(t)\frac{\partial V}{\partial \xi} = -F(U,V) + \frac{D}{\varepsilon^2}\frac{\partial^2 V}{\partial \xi^2} \qquad (7b)$$

multiply by ε^2:

$$\Rightarrow \varepsilon^3\frac{\partial V}{\partial t} - \varepsilon^2\varphi'(t)\frac{\partial V}{\partial \xi} = -\varepsilon^2 F(U,V) + D\frac{\partial^2 V}{\partial \xi^2}$$

(7b')

substitute in the asymptotic expansions, and keep only lowest order terms:

$$\varphi'(t)\frac{\partial U_0}{\partial \xi} = F(U_0, V_0) + \frac{\partial^2 U_0}{\partial \xi^2} \qquad (8a)$$

$$0 = D\frac{\partial^2 V_0}{\partial \xi^2} \quad \leftarrow \text{this can be integrated} \qquad (8b)$$

$$(8b) \Rightarrow \quad V_0 = \alpha + \beta\xi$$

But this has to match the outer solns for v as $\xi \to \pm\infty$ (which, previously, we found, were constants so

$$V_< = V_> = \alpha = V_0 = \text{constant}$$
$$(\text{and } \beta = 0)$$

$\Rightarrow v$ (and V) are spatially uniform on the entire domain

(8a) is now reduced to $\quad \varphi'(t)\dfrac{\partial U_0}{\partial \xi} = F(U_0, \alpha) + \dfrac{\partial^2 U_0}{\partial \xi^2}$

some parameter

(where V_0 was set to its constant value)

This is the same form as the scalar bistable scalar RD eqn in moving coordinates $\quad z = x - \varphi(t)$)

$$\frac{\partial u}{\partial t} = F(u, \alpha) + \frac{\partial^2 u}{\partial x^2}$$

where F looks like →

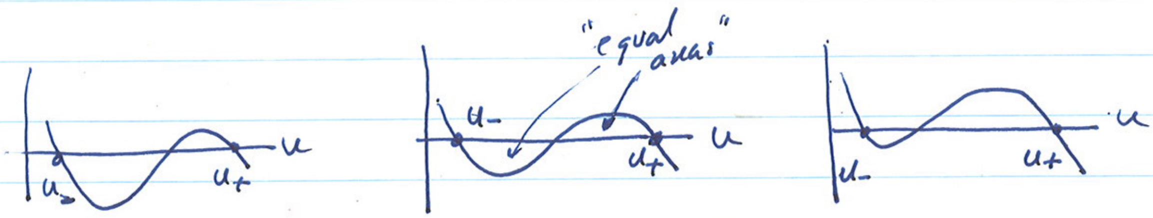
We already saw that the speed is of the form

$$\frac{d\varphi_0}{dt} = c = \frac{\int_{u_+}^{u_-} F(s,\alpha)\,ds}{\int_{-\infty}^{\infty} \left(\frac{\partial U_0}{\partial \xi}\right)^2 d\xi}$$

while we can't solve for c, we can tell when $c = 0$ (which implies a stopped wave).

Assume that there is a value of $V_0 = \alpha$ such that the integral

$$I(\alpha) = \int_{u_-}^{u_+} F(s, \alpha)\, ds = 0$$



$I < 0$ $\qquad$ $I(\alpha) = 0$ $\qquad$ $I > 0$

Then at $V_0 = \alpha$, the front's speed changes sign.

(The level of the uniform (inactive) protein v controls a parameter that determines the direction and speed of the wave).

Summary so far:

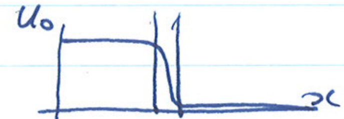
Near the sharp interface (i.e. far away from the boundaries) the system behaves like a bistable scalar RD eqn, and supports a travelling front. The speed of the front depends on the fast-diffusing variable v which is uniform in space (to first order) but which will depend on time.

Position of the (stopped) front

System eqns conserve total amt. So, returning to the outer soln, we have

$$v_0(t) + \int_0^1 u_0(x,t)\,dx = K \approx \text{constant}$$

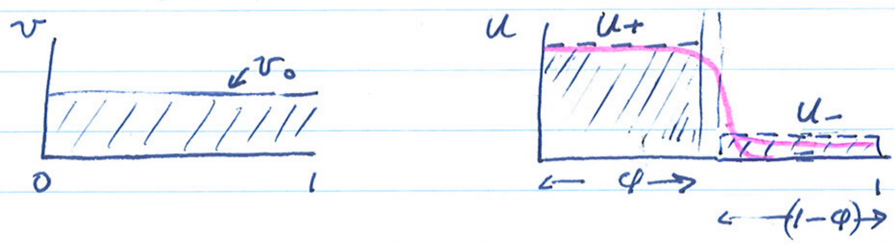
here we actually mean $\int_0^{\varphi-\varepsilon} + \int_{\varphi+\varepsilon}^{1}$ ← where outer soln is valid



but $u_0(x,t) \approx \begin{cases} u_+ & \text{for } x < \varphi(t) - \varepsilon \\ u_- & \text{for } x > \varphi(t) + \varepsilon \end{cases}$

$$v_0(t) + u_+\varphi(t) + u_-(1-\varphi(t)) + \underbrace{\cdots}_{\substack{\text{small terms} \\ O(\varepsilon)}} = K$$

(i.e. u is nearly piecewise constant except for small transition zone near the front)



u_+, u_- ← depend on v_0 in general, and thus change with time

$$\frac{du_+}{dt} = \frac{du_+}{dv_0}\frac{dv_0}{dt} = u_+'\frac{dv_0}{dt}, \qquad \frac{du_-}{dt} = \frac{du_-}{dv_0}\frac{dv_0}{dt} = u_-'\frac{dv_0}{dt}$$

Differentiate w.r.t. time

$$\frac{dv_0}{dt} + \frac{du_+}{dt}\varphi + u_+\frac{d\varphi}{dt} + \frac{du_-}{dt}(1-\varphi) - u_-\frac{d\varphi}{dt} = 0$$

$$\frac{d\varphi}{dt}\underbrace{(u_+ - u_-)} = -\frac{dv_0}{dt} - u_+'\frac{dv_0}{dt}\varphi - u_-'\frac{dv_0}{dt}(1-\varphi)$$

$$= -\frac{dv_0}{dt}\left(\underbrace{1 + \varphi u_+' + (1-\varphi)u_-'}\right)$$

$u_+ > u_-$
so this
term is +ve

$0 \le \varphi \le 1$
so this term is +ve

$\Rightarrow$ The signs of $\frac{d\varphi}{dt}$ and $\frac{dv_0}{dt}$ are opposite:

if φ increases (i.e. front moves right) then v_0 decreases (i.e. inactive protein is depleted) and vice versa.