

Chapter 8

Reaction-diffusion systems and pattern formation

8.1 Stability of a single chemical species

Suppose we have a single chemical substance that diffuses and participates in some chemical reaction. We restrict attention to a single spatial dimension and study an equation that describes this process:

$$\frac{\partial A}{\partial t} = f(A) + D \frac{\partial^2 A}{\partial x^2}, \quad (8.1)$$

About the function $f(A)$ (which represents the chemical kinetics), we will assume that there is a steady state, A_s , i.e. a value of A such that $f(A_s) = 0$. Furthermore, for the system without diffusion (the “well-mixed” system), we assume that $f'(A_s) < 0$. This latter inequality guarantees that the steady state is stable when diffusion is absent.

We now ask whether it is possible to *destabilize* the chemical by letting it diffuse. We will see that the answer is no, but the process of seeing why this is so is illuminating.

Eqn. (8.1) comes with boundary conditions. If our domain is $0 \leq x \leq L$, we could consider either Dirichlet (prescribed chemical levels) conditions or Neumann boundary conditions (no flux through boundaries). Let us arbitrarily consider Dirichlet conditions $A = 0$ at $x = 0, L$.

By the above discussion, it is clear that the constant solution $A(x, t) = A_s$ satisfies Eqn (8.1). We next consider what happens when this *homogeneous* (i.e. spatially uniform) solution is perturbed by a small deviation, i.e.

$$A(x, t) = A_s + a(x, t)$$

where a has a small amplitude. To see the result, we linearize the problem by expanding f in a Taylor series about $A = A_s$. The idea is as follows:

Exercise 8.1.1 (Linearization)

Substitute the solution $A(x, t) = A_s + a(x, t)$ into Eqn. (8.1). Use a Taylor series expansion for the function f and the fact that $f(A_s) = 0$, $dA_s/dt = 0$ (because A_s is a steady state) to deduce that the small perturbation satisfies a linear equation of the form

$$\frac{\partial a}{\partial t} = ra + D \frac{\partial^2 a}{\partial x^2}, \quad (8.2)$$

where $r = f'(A_s)$.

Now it is easy to check that one type of solution to this linear diffusion-reaction equation is

$$a(x, t) = \alpha e^{\sigma t} \sin(qx) \quad (8.3)$$

where α is initial amplitude, σ a growth rate, and q a spatial wavenumber. In fact, in order to also satisfy the Dirichlet BC at $x = L$, not any q will do, only frequencies such that $q = n\pi/L$ have the property that $\sin(qL) = 0$, satisfying $A(L, t) = 0$ (the right boundary condition). Such a sine function is a so-called “eigenfunction” of the diffusion-reaction operator, with the growthrate (eigenvalue) σ . We discuss this in more detail further on.

We will here formally ask whether this type of perturbation would grow or decay with time, i.e. whether σ could ever be made positive (signifying growth) when diffusion is introduced. Plugging the perturbation Eqn:1chemperturb into the linearized equation Eq:SingleChemLIN leads to

$$\frac{\partial(\alpha e^{\sigma t} \sin(qx))}{\partial t} = r(\alpha e^{\sigma t} \sin(qx)) + D \frac{\partial^2(\alpha e^{\sigma t} \sin(qx))}{\partial x^2},$$

Performing the required derivatives leads to

$$\sigma(\alpha e^{\sigma t} \sin(qx)) = r(\alpha e^{\sigma t} \sin(qx)) - q^2 D(\alpha e^{\sigma t} \sin(qx)),$$

Canceling out (nonzero) common factors leaves:

$$\sigma = r - q^2 D.$$

Let us examine this result. By the above discussion, since we are close to a A_s , the coefficient $r = f'(A_s) < 0$. Furthermore, since $D > 0, q^2 > 0$ the second term is negative. We see that *in this single chemical reaction* diffusion is always a *stabilizing* influence, i.e. it can only make the growth rate more negative - killing the perturbations. We can conclude that in a single chemical reaction-diffusion system we will never see an instability caused by the addition of diffusion.

Next, we look at a system of two interacting chemicals, where conditions for instability caused by diffusion can occur.

8.2 Two interacting chemicals

A generic reaction-diffusion system with two components consists of the PDE's

$$\frac{\partial A}{\partial t} = f(A, B) + D_A \frac{\partial^2 A}{\partial x^2}, \quad (8.4a)$$

$$\frac{\partial B}{\partial t} = g(A, B) + D_B \frac{\partial^2 B}{\partial x^2}, \quad (8.4b)$$

where f, g are terms for the reaction kinetics, generally nonlinear functions. Such PDE's require two boundary conditions each to be well-posed, by virtue of their second partial derivatives and parabolic character. To represent no leakage of material out of the domain the appropriate conditions are the Neumann boundary conditions.

$$\left. \frac{\partial A}{\partial x} \right|_{x=0,L} = 0, \quad \left. \frac{\partial B}{\partial x} \right|_{x=0,L} = 0, \quad (8.5)$$

About Eqs. (8.4), we will assume that the kinetics f, g are such that a spatially uniform steady state (A_s, B_s) with $f(A_s, B_s) = g(A_s, B_s) = 0$ exists, and that, moreover, in the well-mixed system (i.e. with $D_A = D_B = 0$), that steady state is stable. Stability implies that the Jacobian of the well-mixed system,

$$J = \begin{pmatrix} f_A & f_B \\ g_A & g_B \end{pmatrix} \quad (8.6)$$

has the following properties: (1) $\text{Trace}(J) = f_A + g_B < 0$ and $\text{Det}(J) = f_A g_B - g_A f_B > 0$.

We will ask whether this steady state can be destabilized by diffusion in a spatially distributed system. To do so, we consider a small perturbation of such a state,

$$A(x, t) = A_s + a(x, t), \quad B(x, t) = B_s + b(x, t)$$

where a, b are small. In this case, we can linearize the equations using a Taylor series approximation for the functions f, g as follows:

$$\frac{\partial a}{\partial t} = c_{11}a + c_{12}b + D_A \frac{\partial^2 a}{\partial x^2}, \quad (8.7a)$$

$$\frac{\partial b}{\partial t} = c_{21}a + c_{22}b + D_B \frac{\partial^2 b}{\partial x^2}, \quad (8.7b)$$

where

$$c_{11} = \frac{\partial f}{\partial A}, \quad c_{12} = \frac{\partial f}{\partial B}, \quad c_{21} = \frac{\partial g}{\partial A}, \quad c_{22} = \frac{\partial g}{\partial B}$$

We restrict attention to perturbations that have been decomposed into a superimposed set of sines and cosines of various spatial frequencies since such functions satisfy the linear diffusion equation. Here we must be careful to select

the appropriate sines or cosines that will satisfy the boundary conditions (BCs). In the case of Neumann BCs, sines are inappropriate, having nonzero slope at ($x = 0$), but cosines work provided we stick to integer multiples of 1/2-cycle of such functions, e.g. $\cos(qx)$ where $q = n\pi/L$. This way, we are assured that the derivative of the function is zero at the endpoint $x = L$, satisfying the second BC.

We refer to such functions as eigenfunctions and these act as basic building blocks out of which the general solution of a linear system will be formed. (Presently, we are considering the eigenfunctions of the diffusion operator in 1D). Note that if the RD system has Dirichlet BCs, we must use sine and not cosine functions (again restricted to multiples of 1/2-cycles). Thus the eigenfunctions will depend on the type of PDE, and on the imposed boundary conditions. In higher dimensions, the shape and geometry of the domain also influence which functions will be "allowable" solutions that act as eigenfunctions.

We thus restrict attention to perturbations that have the form

$$a(x, t) = ae^{\sigma t} \cos(qx), \quad b(x, t) = be^{\sigma t} \cos(qx).$$

Often, the problem can be studied by taking a more general approach,

$$a(x, t) = ae^{\sigma t} e^{iqx}, \quad b(x, t) = be^{\sigma t} e^{iqx}$$

which is a sum of sines and cosines.

As before, substituting these into the linearized RD system leads to the following set of algebraic equations:

$$\sigma ae^{\sigma t} \cos(qx) = c_{11}ae^{\sigma t} \cos(qx) + c_{12}be^{\sigma t} \cos(qx) - D_Aaq^2e^{\sigma t} \cos(qx), \quad (8.8a)$$

$$\sigma be^{\sigma t} \cos(qx) = c_{21}ae^{\sigma t} \cos(qx) + c_{22}be^{\sigma t} \cos(qx) - D_Bbq^2e^{\sigma t} \cos(qx), \quad (8.8b)$$

Canceling out a common factor of $e^{\sigma t} \cos(qx)$ leaves the system

$$\sigma a = c_{11}a + c_{12}b - D_Aaq^2, \quad (8.9a)$$

$$\sigma b = c_{21}a + c_{22}b - D_Bbq^2. \quad (8.9b)$$

These are a system of two algebraic equations in the coefficients a, b

$$a(\sigma + D_Aq^2 - c_{11}) + b(-c_{12}) = 0, \quad (8.10a)$$

$$a(-c_{21}) + b(\sigma + D_Bq^2 - c_{22}) = 0. \quad (8.10b)$$

We recognize that this system of homogeneous linear equations has a unique trivial solution $a = b = 0$ (which is not very interesting, as it means that there is no perturbation at all). To have a nontrivial solution, we need to assure that the determinant of the new matrix is zero, i.e.

$$\det(J_D) = \det \begin{pmatrix} \sigma + D_Aq^2 - c_{11} & -c_{12} \\ -c_{21} & \sigma + D_Bq^2 - c_{22} \end{pmatrix} = 0 \quad (8.11)$$

Some of the steps are part of the following exercise

Exercise 8.2.2 (Characteristic Equation)

- (a) Show that the above condition reduces to a quadratic characteristic equation of the form

$$\sigma^2 - \beta\sigma + \gamma = 0$$

(i.e., find the coefficients β, γ). You will find that both β and γ depend on q^2 as well as on the linearization constants c_{ij} and rates of diffusion.

- (b) Show that if $\beta < 0, \gamma > 0$ then $\sigma < 0$. Note that in that case, the perturbations decay and the system returns to its steady state.
- (c) Show that if the system is stable when well mixed then the addition of diffusion cannot violate the condition $\beta < 0$. (Conclude that the only possibility for growing perturbations is to violate the condition $\gamma > 0$.)
- (d) Let $z = q^2$ for simplicity, and define $H(z) = \gamma(q^2)$, i.e. think of the expression for γ that you got in part (c) as a function of $z = q^2$. We will refer to that function as H . Show that H has a minimum at

$$z = z_{min} = \frac{1}{2} \left(\frac{c_{11}}{D_A} + \frac{c_{22}}{D_B} \right)$$

setting $dH/dz = 0$ and solving for z_{min} . (How can we tell that this point corresponds to the minimum and not the maximum for H ?)

- (e) To make $\gamma = H$ be negative anywhere, it should surely be negative at its minimum. Find $H(z_{min})$. Then for “diffusion-driven” instability we need $H(z_{min}) < 0$. Write down this condition and show that it implies a specific relationship between the rates of diffusion and the coefficients c_{ij} .

8.3 Meinhardt-Gierer Substrate depletion

Consider the reaction-diffusion system due to Meinhardt and Gierer (1972,1974), given by Eqs. (8.4) with: where

$$f(A, B) = p_0 p + c p A^2 B^2 - \mu A, \quad (8.12a)$$

$$g(A, B) = c_0 - c p A^2 B^2 - \eta B. \quad (8.12b)$$

This type of system is often called a substrate-depletion model. The substrate, whose concentration is $S = B$, is used up in a multi-molecular interaction with some activator chemical A . That reaction produces the activator, so terms of the form $\pm c p A^2 B^2$ show up in both equations. (Production of A causes depletion

of S.) The substances are also produced at some background rates p_0p and c_0 , and decay by first-order kinetics with rates μ, η .

This system has been used to produce many interesting patterns in papers by Meinhardt and Gierer, dating back over three decades, and online simulations of some of the patterns in a biological context are available on Meinhardt's webpage. Because of the specific nonlinearity assumed here, it is slightly cumbersome to analyse the behaviour of this system. For example, it is difficult to solve for the homogeneous, space-independent steady state of this system of equations. However, a close cousin of this model is more readily analysed.

8.4 The Schnakenberg system

The Schnakenberg model is related to Meinhardt-Gierer's substrate depletion model, but with a simpler nonlinearity. Here, the two functions are

$$f(A, B) = k_1 - k_2A + k_3A^2B, \quad (8.13a)$$

$$g(A, B) = k_4 - k_3A^2B. \quad (8.13b)$$

It can be shown (Exercise 8.4.3) that, after appropriate rescaling, the dimensionless form of these equations is

$$\frac{\partial u}{\partial t} = \gamma(a - u + u^2v) + \frac{\partial^2 u}{\partial x^2}, \quad (8.14a)$$

$$\frac{\partial v}{\partial t} = \gamma(b - u^2v) + d\frac{\partial^2 v}{\partial x^2}, \quad (8.14b)$$

Exercise 8.4.3 (Nondimensionalization)

Show that the system (8.4) with the Schnakenberg functions given by (8.13) can be rescaled to the system (8.14). What scaling has been used, and how do the new parameters relate to the original set?

The Schnakenberg system is particularly convenient since it is easy to solve for its unique steady state, and thus determine all stability conditions analytically. We find that spatially homogeneous steady state solutions satisfying

$$0 = f(u, v) \equiv \gamma(a - u + u^2v), \quad (8.15a)$$

$$0 = g(u, v) \equiv \gamma(b - u^2v), \quad (8.15b)$$

are

$$u = a + b, \quad v = \frac{b}{(a + b)^2} \quad (8.16)$$

This steady state is at the intersection point of the two nullclines shown in Fig. 8.1.

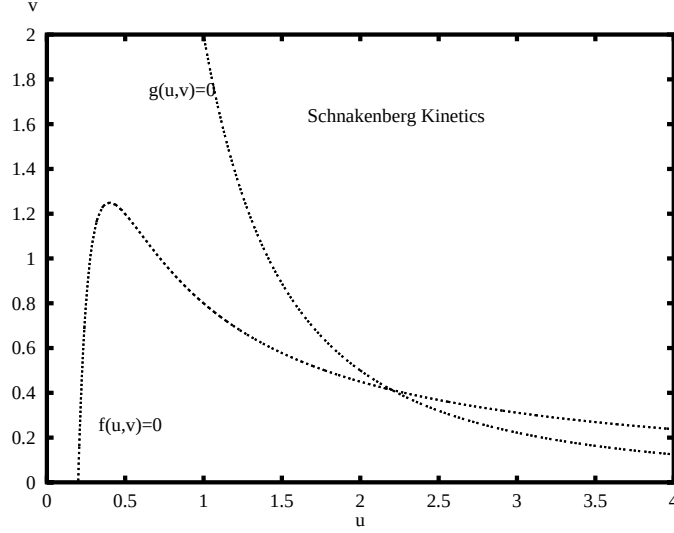


Figure 8.1: The spatially homogeneous version of the equations (8.14) has a phase-plane diagram shown above. There is a unique steady state. Parameter values chosen here, $a = 0.2, b = 2.0, \gamma = 0.2$ lead to a stable steady state at $u = 2.2, v = 0.41322$. Eigenvalues for this homogenous steady state found by XPP are $\lambda_{1,2} = -0.402091 \pm 0.17867i$, indicating a stable spiral at the HSS. The ode file that produced this plot, SchnakenODEs.ode, is given in the Appendix.

The stability of the homogeneous steady state (8.16) can be determined from its Jacobian matrix,

$$J = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} = \begin{pmatrix} \gamma(-1 + 2uv) & \gamma u^2 \\ -2\gamma uv & -\gamma u^2 \end{pmatrix} \quad (8.17)$$

The trace and determinant of J are

$$\det(J) = \gamma^2 u^2, \quad \text{Tr}(J) = \gamma(-1 + 2uv - u^2),$$

and at the homogeneous steady states these are

$$\det(J) = \gamma^2(a+b)^2, \quad \text{Tr}(J) = \left(\frac{b-a}{a+b} - (a+b)^2 \right).$$

It is evident that the determinant is always positive, and hence, stability of the homogeneous steady state to homogeneous perturbations will occur whenever $\text{Tr}(J) < 0$, i.e. when

$$b - a < (a + b)^3.$$

8.4.1 Diffusive instability

We consider perturbations that are not spatially uniform. Then in general, we may assume the special form of perturbations

$$\begin{pmatrix} u(x, t) \\ v(x, t) \end{pmatrix} = \begin{pmatrix} u_{HSS} \\ v_{HSS} \end{pmatrix} + \begin{pmatrix} A_u \\ A_v \end{pmatrix} e^{\sigma t} e^{iqx}, \quad (8.18)$$

where A_u, A_v are the (small) perturbation amplitudes. Here σ is a temporal growth rate and q , the wave number, represents the spatial frequency of the perturbation. The question we ask is whether there are any values of q which grow with time, i.e. for which $\sigma > 0$. To investigate this question, we must look at the behaviour of eigenvalues of the matrix

$$J = \begin{pmatrix} \gamma(-1 + 2uv) - q^2 & \gamma u^2 \\ -2\gamma uv & -\gamma u^2 - dq^2 \end{pmatrix} \quad (8.19)$$

It is clear from (8.19) that the trace of the Jacobian only becomes more negative with the addition of diffusion and spatial influences. Hence, the only way to destabilize this system would be via the determinant, namely we look for conditions for which

$$\det(J) < 0$$

Exercise 8.4.4 (Condition for diffusive instability)

- (a) Compute the determinant of the above matrix at the steady state (8.16). The determinant will be seen to be a function of the quantity q^2 .
- (b) For convenience, let $z = q^2$ then define $H(z) = \det(J)$ expressed as a function of z .
- (c) We ask if H can ever be negative (i.e. the determinant negative, so the Steady state is UNstable to the spacial perturbations.) First show that H has a minimum value at some $z = z_{min}$.
- (d) Next, find $H(z_{min})$. Find conditions for which H is negative at this minimum. You will hence have derived the conditions for diffusive instability of the steady state, i.e. the conditions for a Turing bifurcation.

8.5 Simulating the reaction-diffusion system

The final step in our program is to simulate the set of equations and examine its behaviour.

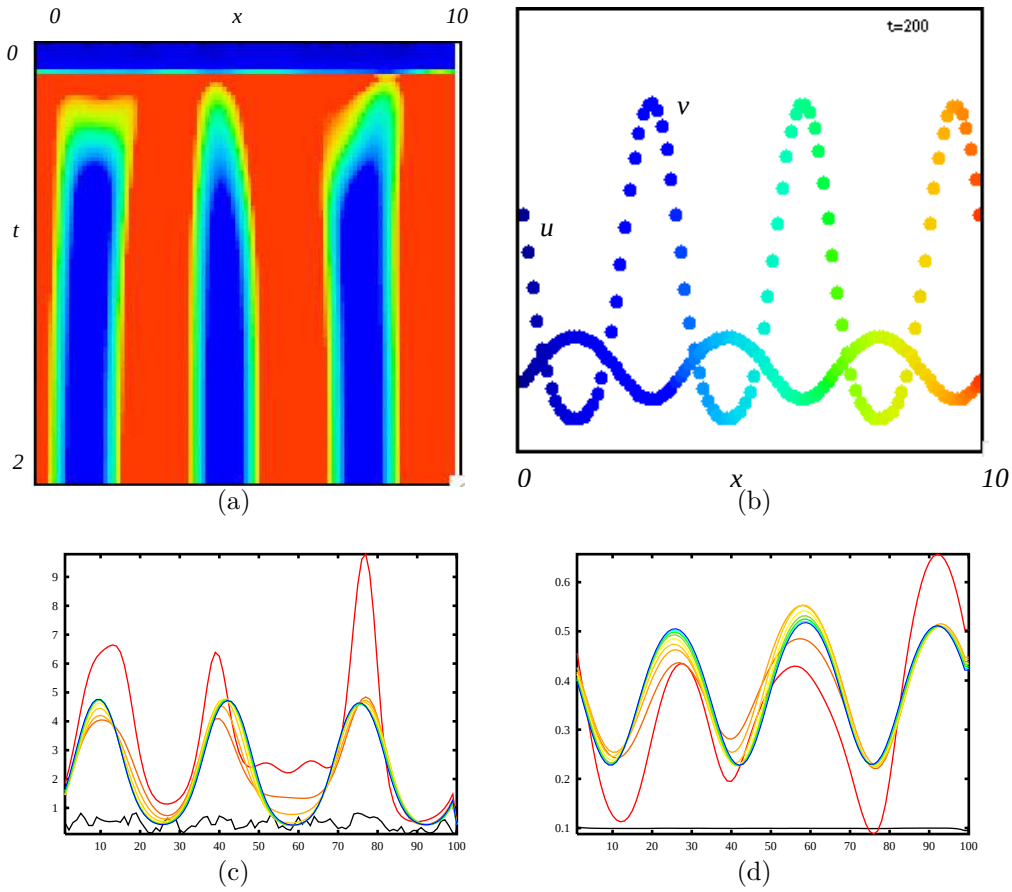


Figure 8.2: Output of the XPP file SchnakenRD.ode. (a) Using the array plot and (b) the animation window. The time and space-dependent profile of the variable u is shown in (c) and for v in (d). To plot parts (c,d) The transpose option of Xpp was used as follows: For Figure (c) *Column 1:u1, Ncols:100, ColSkip:2, Row1:1, NRows:12, RowSkip: 50. For Figure (d) *Column 1:v1, and all other options as before. The plots (c,d) have concentration on the vertical scale and number of grid point on the horizontal scale. The spacing between grid points is $\Delta x = h = 0.1$, so the domain is actually 10 units long. There is a transient in which v builds up to a high level, before dropping down to an eventual lower amplitude pattern solution.

8.6 Appendix

8.6.1 Schnakenberg ODE system

The code for the simple homogeneous (reaction only) version of the Schnakenberg system is as follows:

```
# SchnakenODEs.ode
# Reaction kinetics for Schnakenberg system
f(u,v)=gamma*(a-u+u^2*v)
g(u,v)=gamma*(b-u^2*v)

u'=f(u,v)
v'=g(u,v)

par a=.2,b=2.0,gamma=0.2
done
```

8.6.2 Full reaction-diffusion system: Schnakenberg

Here is the XPP code used to simulate the Schnakenberg system (8.14) with periodic boundary conditions.

```
# SchnakenRD.ode
# Reaction diffusion system

f(u,v)=gamma*(a-u+u^2*v)
g(u,v)=gamma*(b-u^2*v)

***** Periodic Boundaries *****
u0=u99
v0=v99

%[1..99]
u[j] '=f(u[j],v[j])+du*(u[j+1]+u[j-1]-2*u[j])/h^2
v[j] '=g(u[j],v[j])+dv*(v[j+1]+v[j-1]-2*v[j])/h^2
%

***** Periodic Boundaries *****
u100=u1
v100=v1

par a=.2,b=2.0,gamma=0.2,du=0.01,dv=1,h=.1
@ meth=cvode,bandup=2,bandlo=2,total=100
done
```

The initial conditions for this system may be input directly to an XPP window by typing I(nitilconds) (form)U(la), and then responding u[1..100] and ran(1)

when prompted by the prompt “variable”. This sets the initial conditions to be a random number whose value is uniformly distributed in the range $0 \leq R \leq 1$. Figure ??(a) was produced using the array option with the following arrayplot options: *Column1:u1, NCols:200, Row 1:0, NRows=100, RowSkip:40, Zmin:0, Zmax:2, Autoplot(0/1):0, ColSkip:2. Figure ??(b) was produced with the animation file cable100a.ani, as follows:

```
# animation for the array
# cable100a.ani
vtext .8;.95;t=;t
fcircle [1..100]/100;(u[j])/6;.02;[j]/100
fcircle [1..100]/100;(v[j])/2;.02;[j]/100
end
```

Both figures were captured as screen-shots, imported into XFIG, and saved in the .eps format.

Exercise 8.6.5 (Neumann Boundaries)

Modify the above XPP code to incorporate Neumann, instead of periodic boundary conditions.

Solution to 8.6.5: To change the code to have Neumann (no flux) boundary conditions, you might use the following

```
# ***** Neumann BC's
u1'=f(u1,v1)+du*(u2-u1)/h^2
v1'=g(u1,v1)+dv*(v2-v1)/h^2

# ***** Neumann BC's
u100'=f(u100,v100)+du*(u99-u100)/h^2
v100'=g(u100,v100)+dv*(v99-v100)/h^2
```

8.7 Exploring Meinhardt-Gierer

Zigzag waves in an RD system

Consider the reaction-diffusion system due to Meinhardt and Gierer (1974) where

$$f(u, v) = p_0 p + c p u^2 v^2 - \mu u, \quad g(u, v) = c_0 - c p u^2 v^2 - \eta v$$

Explore this system numerically with parameters as follows:

$$c = 0.1, \mu = 1.2, p_0 = 0.03, c_0 = 1, \eta = 0., p = 1.$$

Find the homogeneous steady state of the kinetic model. Then use the same software as for the Schnakenberg system to explore its pattern-formation attributes. For this set of the parameters, you will find zig-zag waves set up when a single peak is initialized in the domain. The dynamics are very sensitive to the parameters. Explore what happens for $0.07 < \eta < 0.13$.

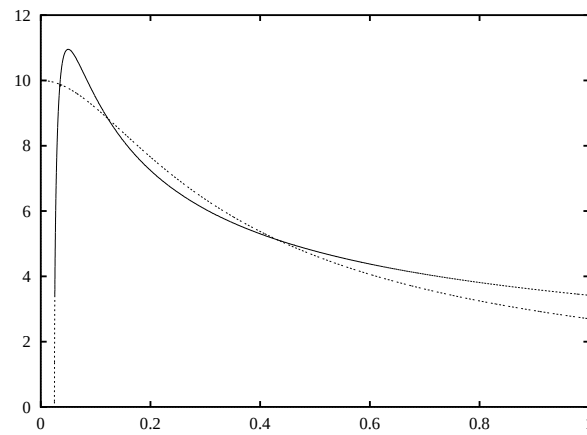
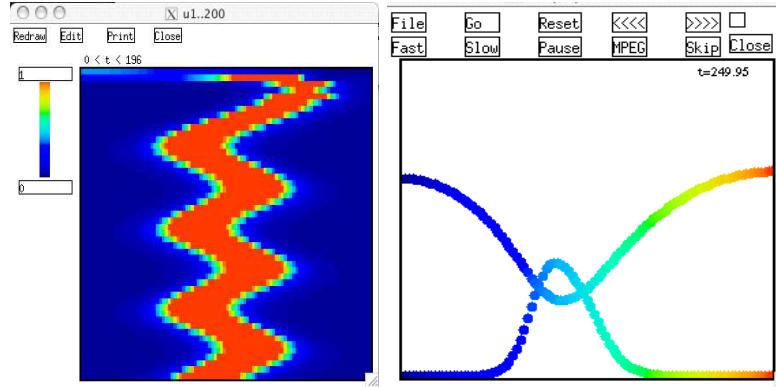


Figure 8.3: The Meinhardt-Gierer substrate depletion model produces a curious zig-zag wave dynamics. Left: u vs (x, t) for the first 200 time units. Right: profiles of the activator u and the depleted substance v at one instant of the dynamics $t = 250$. Phaseplane (lower panel) shows the configuration of the nullclines corresponding to the Meinhardt-Gierer System with parameter values indicated in the example.

8.7.1 Meinhardt-Gierer Substrate-depletion .ode file

```
# ASDepletion.ode
```

```
# This is a substrate-depletion model
#(Meinhardt & Gierer 74)
#2 SS, one unstable, one saddle, one stable
#Has a weird heteroclinic connection

a'=p0*p+c*p*a^2*s^2-mu*a
s'=c0-c*p*a^2*s^2-eta*s

par c=0.1,mu=1.2,p0=0.03,c0=1,eta=0.1
par p=1

# For this parameter set, the stable SS is at:
#A=0.0349, S=9.8809

done
```

This system can be explored with the same `rd2.ode` format, but replacing the reaction kinetics and initial conditions. For the parameters $c = 0.1, \mu = 1.2, p_0 = 0.03, c_0 = 1, \eta = 0.1, p = 1$ there are interesting zig-zag waves set up when a single peak is initialized in the domain. The dynamics are very sensitive to the parameters. For example, for $\eta = 0.14$, a large perturbation returns to the homogeneous steady state. For $\eta = 0.13$, a single peak stabilizes in the middle of the domain after some small amplitude fluctuation. For $0.07 < \eta < 0.1$ the zigzags become wider, i.e. the peak of the pattern swings more widely from one end of the cell to the other.

```
#Reaction Diffusion simulation XPP
#part of the file ASDepletRD.ode
#The rest is identical to rd2.ode.

# Kinetic functions for Meinhardt-Gierer
f(u,v)=p0*p+c*p*u^2*v^2-mu*u
g(u,v)=c0-c*p*u^2*v^2-eta*v
par c=0.1,mu=1.2,p0=0.03,c0=1,eta=0.1
par p=1
#For these params, the HSS is u=0.0349, v=9.8809

par du=.1,dv=1,h=.1

# Initialization of the chemical distribution
# Use the following formulae:
# u[1..100]=0.03+0.03*(1+cos([j]*pi/100))
# v[1..100]=9.88
```

8.7.2 Maple Jacobian

```
>with(linalg):
```

```
> f:=gamma*(a-u+u^2*v);  
> g:=gamma*(b-u^2*v);  
m:=jacobian([f,g],[u,v]);  
det(m);  
trace(m);
```