Chapter 15

Stability and Linearization, Systems of Differential Equations

15.1 Linear approximation and stability

Consider the differential equation

$$\frac{dx}{dt} = f(x)$$

Suppose that the steady state of this equation is $x = X_{ss}$.

(a) Use linear approximation to express $f(X_{ss} + \Delta x)$ in terms of the values of f and f' at X_{ss}.

(b) Consider a point that is close to the steady state, i.e. let $x = X_{ss} + \Delta x$. Substitute this into the differential equation and simplify your result to obtain a new differential equation for the deviation Δx from steady state.

(c) A steady state is said to be stable if small deviations from that steady state decay. Show that X_{ss} is stable if $f'(X_{ss}) < 0$. We will refer to this condition as **the stability criterion**.

Detailed Solution:

(a) By linear approximation,

$$f(X_{ss} + \Delta x) \approx f(X_{ss}) + \Delta x f'(X_{ss}).$$

(b) Plug in $x = X_{ss} + \Delta x$ to obtain

$$\frac{d(X_{ss} + \Delta x)}{dt} = f(X_{ss} + \Delta x) \approx f(X_{ss}) + \Delta x f'(X_{ss}).$$

Use the fact that $f(X_{ss}) = 0$ (Steady state) and $dX_{ss}/dt = 0$ to arrive at

$$\frac{d(\Delta x)}{dt} = \Delta x f'(X_{ss}).$$

(c) The above equation is a simple linear ODE of the form

$$\frac{dx}{dt} = ax$$

where $a = f'(X_{ss})$. The solutions (i.e. the deviations from steady state) will decay exponentially provided that $a < 0$. Thus stability is equivalent to $f'(X_{ss}) < 0$.

15.2 Stability, single ODE

Consider the differential equation

$$\frac{dx}{dt} = f(x)$$

and suppose that this equation has a stable a steady state located at $x = 1$. Which of the following statements could then be true?

(a) $f(1) = 1$, $f'(1) = 0$

(b) $f(1) = 1$, $f'(1) = -1$

(c) $f(1) = 0$, $f'(1) = 1$

(d) $f(1) = 0$, $f'(1) = -1$

(e) $f(1) = -1$, $f'(1) = -1$

Detailed Solution:

Because $x = 1$ is a steady state, it must be true that $f(1) = 0$. Stability implies that $f'(1) < 0$. The only possibility is (d).

15.3 Single ODE's cont'd

For each of the following single ODE's, find all steady states and determine stability of those steady states using the stability criterion developed above. (If the stability criterion does not apply, explain why.)

(a) $\frac{dy}{dt} = y^2 - 3y + 2$

(b) $\frac{dx}{dt} = rx(1 - x)$, for $r > 0$

(c) $\frac{dx}{dt} = 3x^2(1 - x)$

(d) $\frac{dy}{dt} = y$

Detailed Solution:

(a) For $\frac{dy}{dt} = f(y) = y^2 - 3y + 2 = (y-1)(y-2)$, the steady states are $y = 1, 2$ and the criterion involves $f'(y) = 2y - 3$. We find that $f'(1) = -1 < 0$ so $y = 1$ is stable, and $f'(2) = 4 - 3 = 1 > 0$ so $y = 2$ is an unstable steady state.

(b) For $\frac{dx}{dt} = f(x) = rx(1-x)$, for $r > 0$, the steady states are $x = 0, 1$ and $f'(x) = r - 2rx$. Thus $f(0) = r > 0$ implies that 0 is unstable. $f'(1) = r - 2r = -r < 0$ implies that $x = 1$ is stable.

(c) For the equation $\frac{dx}{dt} = f(x) = 3x^2(1-x) = 3(x^2 - x^3)$, the steady states are also $x = 0, 1$ and $f'(x) = 3(2x - 3x^2)$. Thus $f'(0) = 0$ so this steady state is neither stable nor unstable (i.e. it is a kind of neutral state). $f'(1) = -3$ so the steady state at 1 is stable.

(d) The equation $\frac{dy}{dt} = y$ is linear. Its only steady state is $y = 0$ and this is clearly unstable since all positive solutions are growing. (Alternately, $f(y) = y, f'(y) = 1 > 0$ for all y implying instability.)

15.4 Linear approximation, Two variables

In the following problems, a function $f(x, y)$ depends on two variables. Use linear approximation to express $f(X_0 + \Delta x, Y_0 + \Delta y)$ in terms of the values of f and its partial derivatives at (X_0, Y_0). Simplify as much as possible. (Your answer will be in terms of $\Delta x, \Delta y$.)

(a) $f(x, y) = x + y$ at $X_0 = 0, Y_0 = 0$

(b) $f(x, y) = x^2 + y^3$ at $X_0 = 1, Y_0 = 1$

(c) $f(x, y) = x - xy$ at $X_0 = 1, Y_0 = 1$

(d) $f(x, y) = x(1 - 2x - 3y)$ at $X_0 = 0, Y_0 = 1$

Detailed Solution:

$$f(X_0 + \Delta x, Y_0 + \Delta y) \approx f(X_0, Y_0) + \Delta x f_x(X_0, Y_0) + f_y(X_0, Y_0)\Delta y$$

(a) The function is already linear, and moreover $f_x = f_y = 1$ for all x, y thus $f(X_0 + \Delta x, Y_0 + \Delta y) = \Delta x + \Delta y$

(b) Here $f_x = 2x, f_y = 3y^2$ so $f_x(1, 1) = 2, f_y(1, 1) = 3$ $f(1 + \Delta x, 1 + \Delta y) \approx f(1, 1) + 2\Delta x + 3\Delta y = 2 + 2\Delta x + 3\Delta y$

(c) $f(x, y) = x - xy$ so $f_x = 1 - y, f_y = -x$, $f_x(1, 1) = 0, f_y(1, 1) = -1$ so $f(1 + \Delta x, 1 + \Delta y) \approx f(1, 1) - \Delta y = -\Delta y$.

(d) For $f(x, y) = x(1 - 2x - 3y) = x - 2x^2 - 3xy$ we have $f_x = 1 - 4x - 3y, f_y = -3x$. Thus $f_x(0, 1) = 1, f_y(0, 1) = 0$, so $f(\Delta x, 1 + \Delta y) \approx f(0, 1) + \Delta x = \Delta x$.

15.5 Stability, System of ODE's

Consider the system of ODE's given below

$$\frac{dx}{dt} = f(x,y) \quad (15.1)$$

$$\frac{dy}{dt} = g(x,y) \quad (15.2)$$

Suppose that (X_{ss}, Y_{ss}) is a steady state of this system. Show that close to this steady state, the system can be approximated by the linear system

$$\frac{dx}{dt} = ax + by \quad (15.3)$$

$$\frac{dy}{dt} = cx + dy \quad (15.4)$$

where the coefficients a, b, c, d are partial derivatives of f, g evaluated at (X_{ss}, Y_{ss}).

Detailed Solution:

This follows from linear approximation to f and g close to (X_{ss}, Y_{ss}). Using results of Problem 15.4,

$$f(X_{ss} + \Delta x, Y_{ss} + \Delta y) \approx f(X_{ss}, Y_{ss}) + f_x(X_{ss}, Y_{ss})\Delta x + f_y(X_{ss}, Y_{ss})\Delta y$$

and

$$g(X_{ss} + \Delta x, Y_{ss} + \Delta y) \approx g(X_{ss}, Y_{ss}) + g_x(X_{ss}, Y_{ss})\Delta x + g_y(X_{ss}, Y_{ss})\Delta y$$

We also need to use the fact that

$$f(X_{ss}, Y_{ss}) = 0, \quad g(X_{ss}, Y_{ss}) = 0$$

Plugging in

$$x = X_{ss} + \Delta x, \quad y = Y_{ss} + \Delta y$$

into the system of nonlinear equations, and using these linear approximation results will lead to the desired outcome, with

$$a = f_x(X_{ss}, Y_{ss}), b = f_y(X_{ss}, Y_{ss}), c = g_x(X_{ss}, Y_{ss}), d = g_y(X_{ss}, Y_{ss})$$

15.6 A Linear System

Consider the linear system of ODEs

$$\frac{dx}{dt} = f(x,y) = ax + by \quad (15.5)$$

$$\frac{dy}{dt} = = g(x,y) = cx + dy \quad (15.6)$$

This system has a single steady state, at $(0, 0)$. Show that the partial derivatives of f, g are precisely the linear coefficients a, b, c, d in this system. Show that the system is stable is $\beta = a + d < 0$ and $\gamma = ad - bc > 0$.

Detailed Solution:

The first part is an easy calculation. The second part follows from the fact that eigenvalues are of the form

$$\lambda_{1,2} = \frac{\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}$$

using some of the arguments about this formula from the last homework set. In particular, when $\beta = a + d < 0$ and $\gamma = ad - bc > 0$, the eigenvalues both have negative real parts, so solutions are decreasing exponentials.

15.7 Stability, Continued

Consider the system of differential equations given below,

$$\frac{dx}{dt} = x(1 - 2y - x) \tag{15.7}$$

$$\frac{dy}{dt} = y - x \tag{15.8}$$

(a) Find all steady states of this system.

(b) Determine the stability of each of the steady states by linearizing the system about that steady state and determining the behaviour of that linear system.

Detailed Solution:

(a) Steady states: $x = y$ (to satisfy $dy/dt = 0$), so $0 = x(1 - 2y - x) = x(1 - 3x)$ so the steady states are $(0, 0)$ and $(1/3, 1/3)$

(b) Let $f(x, y) = x(1 - 2y - x) = x - 2xy - x^2$. Then $f_x = 1 - 2y - 2x, f_y = -2x$. The second equation is already in linear form, and need not be processed further. Linearization about (0,0) will lead to

$$\frac{dx}{dt} = x \tag{15.9}$$

$$\frac{dy}{dt} = -x + y \tag{15.10}$$

For this linear system, $\beta = a + d = 2 > 0, \gamma = ad - bc = 1 > 0$ implies that the eigenvalues are positive so the steady state is unstable. Linearization about $(1/3, 1/3)$ will lead to

$$\frac{dx}{dt} = -(1/3)x - (2/3)y \tag{15.11}$$

$$\frac{dy}{dt} = -x + y \tag{15.12}$$

Here $\beta = a+d = 2/3, \gamma = -1/3-2/3 = -1 < 0$ implying that there is one positive eigenvalue, so this steady state is a saddle.