Detailed calculations for HW 4.

Particular soln to:

$$my'' + \gamma y'(t) + ky(t) = F_0 \cos(\omega t)$$

plug in

$$Y_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

or can use

$$Y_p(t) = R\cos(\omega t - \delta)$$

$$Y_p'(t) = -R\omega \sin(\omega t - \delta)$$

$$Y_p''(t) = -R\omega^2 \cos(\omega t - \delta)$$

We want to find R, δ using Method of undetermined coeffs.

$$m\left[-R\omega^2\cos(\omega t-\delta)\right] + \gamma\left[-R\omega\sin(\omega t-\delta)\right] + k\left[R\cos(\omega t-\delta)\right] = F\,\boxed{\cos(\omega t)}$$

before matching "like terms" we have to express all trig functions in terms of same argument $(\omega t - \delta)$. See Trig Identities showing that

$$\boxed{\cos(\omega t) = \cos\delta\cos(\omega t-\delta) - \sin\delta\sin(\omega t-\delta)}$$

Now we can match terms on both sides of the above algebraic eqn

terms multiplying $\cos(\omega t-\delta)$: $\left(-mR\omega^2 + kR\right) = F\cos\delta$ (1)

terms multiplying $\sin(\omega t-\delta)$: $-R\omega\gamma = -F\sin\delta$ (2)

$$(1) \Rightarrow \quad R(k - m\omega^2) = F\cos\delta$$

$$(2) \Rightarrow \quad \sin\delta = \frac{R\omega\gamma}{F}$$

$\left\{\begin{array}{l}(1)^2 + (2)^2 \Rightarrow \\ \sin^2\delta + \cos^2\delta = 1\end{array}\right.$ $\quad R^2\omega^2\gamma^2 + R^2(k - m\omega^2)^2 = F^2 \Rightarrow R^2 = \dfrac{F^2}{(\omega^2\gamma^2 + (k-m\omega^2)^2)}$

$$\Downarrow$$

$$R = \frac{F}{\Delta} \Leftarrow R = \frac{F}{\sqrt{\omega^2\gamma^2 + (k-m\omega^2)^2}}$$

where $\Delta = \sqrt{\omega^2\gamma^2 + m^2\left(\frac{k}{m} - \omega^2\right)^2}$

$$= \sqrt{\omega^2\gamma^2 + m^2\left(\omega_0^2 - \omega^2\right)^2}$$

natural freq. $\left(\omega_0^2 = \frac{k}{m}\right)$ forcing freq.

Full solution to damped forced spring-mass sys.

$$y(t) = e^{\sigma t}\left(c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)\right) + R\cos(\omega t - \delta)$$

$\omega_0 = \frac{1}{2m}\sqrt{|\gamma^2 - 2mk|}$

$\sigma = -\frac{b}{2a} < 0$ (in general)

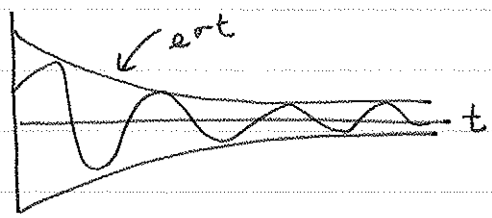
$= -\frac{\gamma}{2m} < 0$ (for spring-mass)

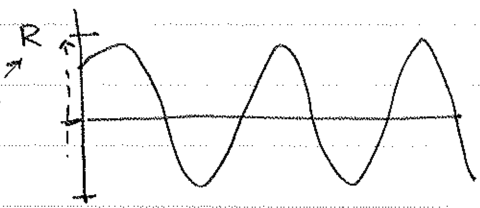
amplitude R
frequency ω
shift δ

decaying oscillations of unforced system

"transient solution"

forced system response





After some time, this transient will go away and we'll be left with this

Q: How does the amplitude R of the forced oscillation depend on the forcing frequency?

Answer:

$$\frac{R}{F} = \frac{1}{\sqrt{\omega^2\gamma^2 + m^2(\omega_0^2 - \omega^2)^2}} \Rightarrow \frac{Rk}{F} = \frac{R}{F(1/k)} = \frac{R}{F(\sqrt{1/k^2})} = \frac{1}{\sqrt{\frac{\omega^2\gamma^2}{k^2} + \frac{m^2}{k^2}(\omega_0^2 - \omega^2)^2}}$$

$$\frac{Rk}{F} = \frac{1}{\sqrt{\frac{\omega^2\gamma^2}{(k/m)\cdot mk} + \frac{1}{\omega_0^4}(\omega_0^2 - \omega^2)^2}} = \frac{1}{\sqrt{\frac{\omega^2}{\omega_0^2}\Gamma + \left(1 - \frac{\omega^2}{\omega_0^2}\right)^2}}$$

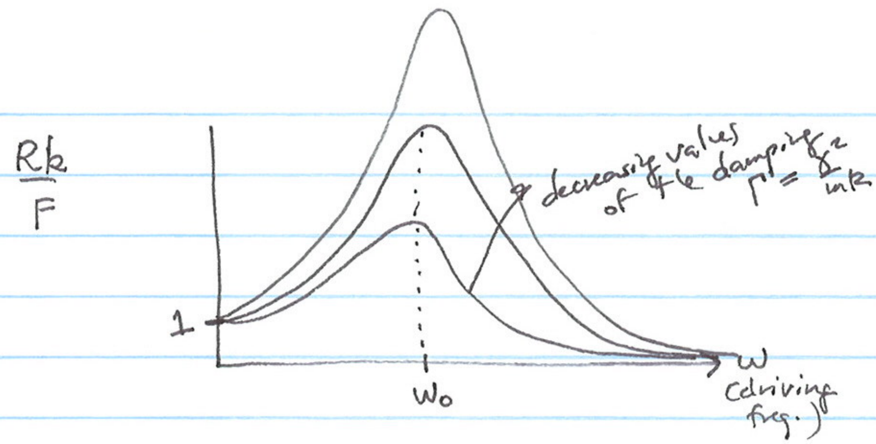
Now we consider this as a fnc. of ω, the driving force

where $\Gamma = \gamma^2/mk$.

when $\omega = 0$ $\quad \frac{Rk}{F} = \frac{1}{\sqrt{0 + 1^2}} = 1$; when $\omega \to \infty$, $\frac{Rk}{F} \to 0$

$\frac{Rk}{F}$ largest for $\omega = \omega_0$ when the term $(\;)^2 = 0$

(Then, for $\omega = \omega_0$ we have $\frac{Rk}{F} = \frac{1}{\sqrt{\Gamma}} = \frac{\sqrt{mk}}{\gamma} \Rightarrow R \approx \frac{F}{\gamma}\sqrt{\frac{m}{k}} = \frac{F}{\gamma\omega_0}$



$$\boxed{\frac{R}{(F/k)} = \frac{1}{\sqrt{\frac{\omega^2}{\omega_0^2}\Gamma + \left(1-\frac{\omega^2}{\omega_0^2}\right)^2}} \qquad \Gamma = \frac{\gamma^2}{mk}}$$

if $\omega = 0$ then $\frac{R}{(F/k)} = \frac{1}{\sqrt{0 \cdot \Gamma + (1-0)^2}} = 1$

if $\omega = \omega_0$ then $\frac{R}{(F/k)} = \frac{1}{\sqrt{\Gamma + 0}} = \frac{1}{\sqrt{\Gamma}}$ ← as Γ gets small this value gets very large

if $\omega \gg \omega_0$ then $\frac{R}{(F/k)} \to 0$

Beats : Driving frequency close to (but not same as) natural frequency

$$my'' + ky = F\cos(\omega t) \qquad y(0)=0 \quad y'(0)=0$$

forcing ("driving") frequency ω.

We showed (Oct 4) that solution to this is

$$y(t) = \frac{F}{k-\omega^2 m}\left(\cos(\omega t) - \cos(\omega_0 t)\right) \qquad \text{where } \omega_0 = \sqrt{\frac{k}{m}}$$

This part comes from particular solution

This part comes from soln to corresponding homog. eqn.

natural frequency of spring-mass

(This is after a lot of algebra and using $Y_p = A\cos(\omega t) + B\sin(\omega t)$, method of undetermined coeffs to find A, B, and initial conditions to find other constants.)

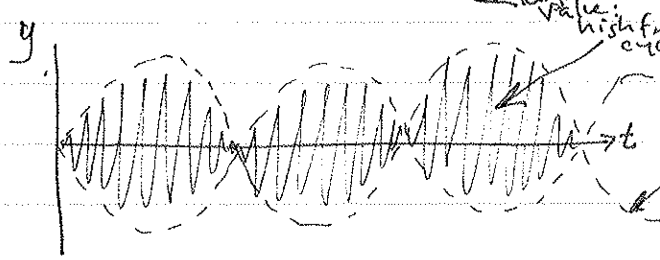
Q: What does this solution look like? How does it depend on the driving frequency ω?

Answ: Using Trig identity, I4 can show that

$$y(t) = \frac{2F}{m(\omega_0^2 - \omega^2)} \sin\left(\left[\frac{\omega+\omega_0}{2}\right]t\right) \sin\left(\left[\frac{\omega-\omega_0}{2}\right]t\right)$$

large value high freq cycles

(small value) for $\omega \approx \omega_0$ provides freq. of "envellope" of oscillations (dashed lines)



this is called "beats"