Nov 9, 2010

Note: Some of my notes / HW Solns had an error in

$\mathcal{L}\{y''(t)\}$. !

The correct formula is

$$\boxed{\mathcal{L}\{y''(t)\} = s^2 F(s) - s\,y(0) - y'(0)}$$

I apologize for this confusing error, and I hope I've corrected all instances at this point!

LEK

Solns to HW6

Problem 1:

Find Laplace Transform:

1(a) $\sinh(at) = \frac{1}{2}(e^{at} - e^{-at})$

$$\mathcal{L}\{\sinh(at)\} = \frac{1}{2}\left(\mathcal{L}\{e^{at}\} - \mathcal{L}\{e^{-at}\}\right) = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right)$$

$$= \frac{1}{2}\,\frac{(s+a)-(s-a)}{(s-a)(s+a)} = \frac{1}{2}\cdot\frac{2a}{s^2-a^2} = \frac{a}{s^2-a^2}$$

1(b) t^2e^{at}

$$\mathcal{L}\{t^2e^{at}\} = F(s-a), \quad \text{where} \quad F(s) = \mathcal{L}\{t^2\} = \frac{2}{s^3}$$

$$= \frac{2}{s^3}\Big|_{s\to s-a} = \frac{2}{(s-a)^3}$$

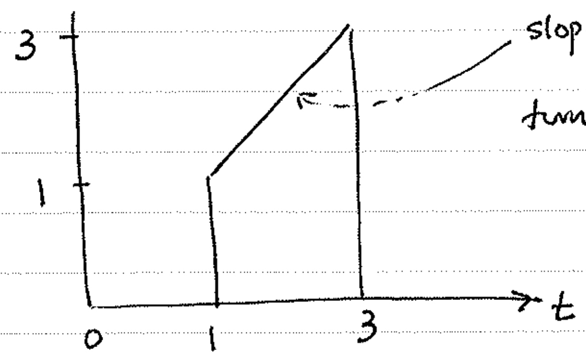
(c) $e^{at}\left(H(t-1) - H(t-2)\right)$

$$\mathcal{L}\{e^{at}(H(t-1) - H(t-2))\} = F(s-a) \quad \text{where } F(s) = \mathcal{L}\{H(t-1) - H(t-2)\} = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

$$= \frac{e^{-(s-a)}}{(s-a)} - \frac{e^{-2(s-a)}}{(s-a)}$$

2) Laplace transform:

(a)



slope 1, poin(1,1) $\Rightarrow$ $y = t$

turns on at $t=1$, off at $t=3$

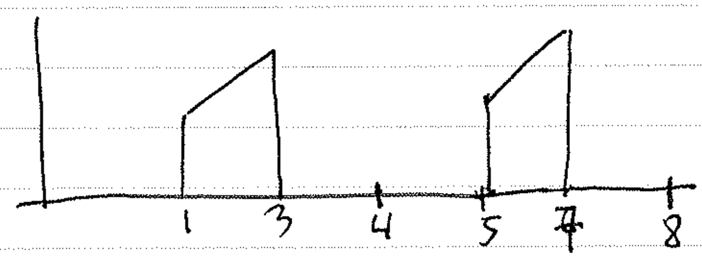
$$f(t) = t\,(H(t-1) - H(t-3))$$

rewrite as

$$f(t) = (t-1+1)\,H(t-1) - (t-3+3)\,H(t-3)$$
$$= (t-1)\,H(t-1) - (t-3)\,H(t-3) + H(t-1) - 3\,H(t-3)$$

$$F(s) = \mathcal{L}\{f(t)\} = \frac{e^{-s}}{s^2} - \frac{e^{-3s}}{s^2} + \frac{e^{-s}}{s} - \frac{3e^{-3s}}{s}$$
$$= -e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right) + e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

(b) If fn has period 4: Here is a sketch



(c) Its Lapl. trans. is $\frac{1}{1-e^{-4s}} \cdot F(s)$ where $F(s)$ is as in part (a)

3) Find the inverse Laplace transform

ⓐ Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3} + \frac{1}{s^2+2s-8}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\,\frac{2!}{(s-1)^3} + \frac{1}{(s^2+2s+1)-9}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{2}\,\frac{2!}{s^3}\Big|_{s\to s-1} + \frac{1}{(s+1)^2-3^2}\right\}$

$= \frac{1}{2}e^t\,t^2 + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2-3^2}\Big|_{s\to s+1}\right\}$

$= \frac{1}{2}e^t t^2 + \frac{1}{3}e^{-t}\cdot\sinh(3t)$ $\leftarrow$ uses $\mathcal{L}\{\sinh(3t)\}$

ⓑ Evaluate $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s/2}}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s/2}}{3}\cdot F(s)\right\}$ $\leftarrow$ where $F(s)=\frac{3}{s^2+9}$

$= \frac{1}{3}\,H\left(t-\frac{\pi}{2}\right)\cdot\sin\left(3\left(t-\frac{\pi}{2}\right)\right) = \frac{1}{2}\,H\left(t-\frac{\pi}{2}\right)\cos(3t)$

(two forms of the soln) equivalent.

ⓒ $F(s) = \frac{6}{s^3-9s} = 6\cdot\frac{1}{s}\cdot\frac{1}{s^2-9} = -\frac{2}{3s} + \frac{1}{3(s-3)} + \frac{1}{3(s+3)}$

(partial fractions)

$\mathcal{L}^{-1}\{F(s)\} = -\frac{2}{3} + \frac{1}{3}e^{3t} + \frac{1}{3}e^{-3t}$

Problem 4

(a) $y'' + y = H(t-3\pi) \qquad y(0) = 1, \quad y'(0) = 0$

$$\left(s^2F(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{1}\right) + F(s) = \frac{e^{-3\pi s}}{s}$$

$$F(s)(s^2+1) - s = \frac{e^{-3\pi s}}{s}$$

$$F(s) = \frac{s}{(s^2+1)} + \frac{e^{-3\pi s}}{s(s^2+1)} = \frac{s}{s^2+1} + \left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-3\pi s}$$

$$\mathcal{L}^{-1}\{F(s)\} = \cos(t) + (1 - \cos t)\Big|_{\substack{t \\ \text{shifted to } t-3\pi \\ \text{and stepped up there}}}$$

$$= \cos(t) + (1 - \cos(t-3\pi))\, H(t-3\pi)$$

$$= \cos(t) + (1 + \cos t)\, H(t-3\pi)$$

Problem 4 cont'd

(b) $y'' - 2y' + y = e^t \qquad y(0) = 0 \qquad y'(0) = 1$

$$\boxed{[s^2F(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{1}]} - 2[sF(s) - \underbrace{y(0)}_{0}] + F(s) = \mathcal{L}\{e^t\} = \frac{1}{(s-1)}$$

$$F(s)[s^2 - 2s + 1] - \boxed{1} = \frac{1}{(s-1)}$$

$$F(s) = \frac{\boxed{1}}{(s^2-2s+1)} + \frac{1}{(s^2-2s+1)(s-1)} = \frac{1}{(s-1)^2} + \frac{1}{(s-1)^3}$$

$$y(t) = \mathcal{L}^{-1}\{F(s)\} = te^t + \frac{1}{2}t^2e^t$$

(c) $y'' + 4y' + 5y = \delta(t - 2\pi) \qquad y(0) = 0 \qquad y'(0) = 0$

$$(s^2F(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0}) + 4(sF(s) - \underbrace{y(0)}_{0}) + 5F(s) = e^{-2\pi s}$$

$$s^2F(s) + 4sF(s) + 5F(s) = e^{-2\pi s}$$

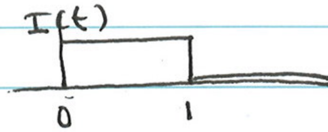
$$F(s)(s^2 + 4s + 5) = e^{-2\pi s}$$

$$F(s) = \frac{1}{(s^2+4s+5)}e^{-2\pi s} = \frac{e^{-2\pi s}}{(s^2+4s+4)+1} = \frac{e^{-2\pi s}}{(s+2)^2+1}$$

$$y(t) = \mathcal{L}^{-1}\left\{e^{-2\pi s}\left(\frac{1}{s^2+1}\right)_{s\to s+2}\right\} = u_{2\pi}(t)\sin(t-2\pi)e^{2(t-2\pi)} = H(t-2\pi)\sin(t)e^{2(t-2\pi)}$$

Problem 6

Use the Laplace Transform to solve the drug-infusion problem: Let $c(t)$ be conc. of drug in the blood given an infusion rate $I(t)$ shown below with $c(0)=0$ and with decay rate r.



infusion turned on at $t=0$, off at $t=1$.

The ODE and initial conds are:

$$\frac{dc}{dt} = I - rc \qquad c(0)=0 \qquad I = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}$$

$$I = 1 - H(t-1) \quad \text{or}$$

$$I = H(t) - H(t-1)$$

(since we do not care what happens before $t=0$ here).

$$\frac{dc}{dt} + rc = I(t) = 1 - H(t-1)$$

$$\mathcal{L}\left\{\frac{dc}{dt}\right\} + r\,\mathcal{L}\{c\} = \mathcal{L}\{1\} - \mathcal{L}\{H(t-1)\}$$

$$sF(s) - \underbrace{c(0)}_{=0} + r\,F(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$(s+r)\,F(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$F(s) = \frac{1}{s(s+r)} - \frac{1}{s(s+r)}e^{-s}$$

Partial fraction

$$= \frac{A}{s} + \frac{B}{s+r} - \left(\frac{A}{s} + \frac{B}{s+r}\right)e^{-s}$$

because the fractions are identical, we need only one pair of constants

$$\Rightarrow \quad 1 = A(s+r) + B(s)$$

$$s \to 0 \qquad 1 = Ar \qquad A = 1/r$$

$$s \to -r \qquad 1 = B(-r) \qquad B = -1/r$$

uses $\mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}F(s)$

$$F(s) = \frac{1}{r}\left(\frac{1}{s} - \frac{1}{s+r}\right) - \frac{1}{r}\left(\frac{1}{s} - \frac{1}{s+r}\right)\boxed{e^{-s}}$$

results in shift and step

Inverse transform

$$\mathcal{L}^{-1}(F(s)) = \frac{1}{r}\left(1 - e^{-rt}\right) - \frac{1}{r}\cdot H(t-1)\left(1 - e^{-rt}\right)\Big|_{t \to t-1}$$

$$= \frac{1}{r}\left(1 - e^{-rt}\right) - \frac{1}{r}\,H(t-1)\left(1 - e^{-r(t-1)}\right)$$