Math 265 HW8 Solutions L. Keshet

(the HW was to solve these exam problems)

The University of British Columbia

Final Examination - December 2009

Mathematics 265

Section 101

Closed book examination Time: 2.5 hours

Last Name: ____________ **First:** ____________ **Signature** ____________

Student Number ____________

Note: in 2010 exam we will NOT allow a formula sheet.

Special Instructions:

- Be sure that this examination has 13 pages. Write your name on top of each page.
[You are allowed to bring into the exam one $8\frac{1}{2} \times 11$ formula sheet filled on both sides. No calculators or any other aids are allowed.]
- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:
  (a) having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
  (b) speaking or communicating with other candidates; and
  (c) purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		10
2		10
3		10
4		15
5		20
6		20
7		15
Total		100

[10] 1. Solve the initial value problem:

$$(e^x \sin x)y' = -1 - (e^x \cos x)y,$$

with $y(\pi/2) = e^{-\pi/2}$.

$$(e^x \sin x)\, y' + (e^x \cos x) y = -1$$

$$y' + \frac{\cos x}{\sin x}\, y = -\frac{1}{e^x \sin x}$$

Intgr. factor

$$\mu(x) = \exp \int \frac{\cos x}{\sin x}\, dx = \exp \int \frac{1}{u}\, du = \exp \ln u$$

$u = \sin x$

$du = \cos x\, dx$

$$= u = \sin x$$

$$\underbrace{\sin x \left[y' + \frac{\cos x}{\sin x}\, y \right]}_{\frac{d}{dx}[y \sin x]} = -\frac{1}{e^x} \frac{\cancel{\sin x}}{\cancel{\sin x}}$$

$$\frac{d}{dx}\left[y \sin x \right] = -e^{-x}$$

$$y \sin x = -\int e^{-x}\, dx + C$$

$$= e^{-x} + C$$

$$y = \frac{1}{\sin x}\left[e^{-x} + C \right]$$

$$y\left(\tfrac{\pi}{2}\right) = e^{-\pi/2} = \frac{1}{\sin \pi/2}\left[e^{-\frac{\pi}{2}} + C \right]$$

$$e^{-\frac{\pi}{2}} = 1 \cdot \left[e^{-\frac{\pi}{2}} + C \right]$$

$c = 0$

$$y = \frac{e^{-x}}{\sin x}$$

[10] **2.** Find all solutions of the differential equation

$$(x+1)^3 y' + (x+1)e^{-y} = 0.$$

$$\frac{dy}{e^{-y}} = \frac{-(x+1)}{(x+1)^3}\,dx$$

$$\int e^{y}\,dy = \int -\frac{1}{(x+1)^2}\,dx + C$$

$$e^{y} = \int -u^{-2}\,du + C \qquad u = x+1$$

$$= (-1)\frac{(-1)}{u} + C$$

$$= \frac{1}{(x+1)} + C$$

Note!

$$y = \ln\left[C + \frac{1}{x+1}\right]$$

[10] **3**. Consider the differential equation

$$y'' + p(t)y' + q(t)y = 0 \qquad (*)$$

where p and q are continuous functions for all t.

(a) Can $y(t) = \sin(t^2)$ be a solution on an interval containing $t = 0$ of the differential equation $(*)$? Explain your answer.

if $y(t) = \sin(t^2)$ then $y'(t) = 2t\cos(t^2)$

$y''(t) = 2\left(\cos(t^2) - t \cdot 2t\sin(t^2)\right)$

$= 2\cos(t^2) - 4t^2\sin(t^2)$

at $t=0$ $y(t)=0$, $y'(t)=0$ $y''(t)=2$

but $(*)$ implies $y''=0$ so no, this is inconsist.

(i.e. plug back into ODE)

(b) Calculate the Wronskian of t and t^2.

$$W = \det\begin{bmatrix} t & t^2 \\ 1 & 2t \end{bmatrix} = 2t^2 - t^2 = t^2$$

(c) Can t and t^2 both be solutions of the same differential equation $(*)$? Explain clearly.

We notice that at $t=0$ Wronskian $W=0$, so this cannot be a fundam. set on any interval that includes $t=0$. But there is even more to say. suppose these are both solns of $(*)$. then

t is a soln $\Rightarrow$ $0 + p(t) + q(t)t = 0$ (1)

t^2 is a soln $\Rightarrow$ $2 + 2tp(t) + q(t)t^2 = 0$ (2)

(1) $\Rightarrow$ $p(t) = -q(t)t$ now sub into (2) to get

$2 + (-2t^2q(t)) + q(t)t^2 = 0$

$2 - 2t^2q(t) = 0$

$t^2q(t) = 1 \Rightarrow q(t) = \frac{1}{t^2}$

So on any interval includig $t=0$, these 2 solns lead to the coefficient $q(t)$ beig undefined at $t=0$. So this would be a contradiction.
(But on intervals that do <u>not</u> include $t=0$, there is no problem.)

[15] 4. Use the method of undetermined coefficients to find the general solution of

$$y'' + y = \cos t.$$

soln to hom. eqn:

$y'' + y = 0$

char eq

$r^2 + 1 = 0$

$r^2 = -1$

$r = \pm i$

$y(t) = c_1 \cos t + c_2 \sin t$

guess for partic. soln of nonhom prob.

$Y_p(t) = t(A\cos t + B\sin t)$ to avoid dupl. the soln of hom. prob.

$Y_p'(t) = t(-A\sin t + B\cos t) + 1(A\cos t + B\sin t)$

$Y_p''(t) = t(-A\cos t - B\sin t) + 1\cdot(-A\sin t + B\cos t) + (-A\sin t + B\cos t)$

$= -t(A\cos t + B\sin t) - 2A\sin t + 2B\cos t$

$Y_p''(t) + Y_p(t) = \cos t$

$\left[-t(\cancel{A\cos t + B\sin t}) - 2A\sin t + 2B\cos t\right] + t(\cancel{A\cos t + B\sin t}) = \cos t$

$-2A = 0 \qquad A = 0$

$2B = 1 \qquad B = \frac{1}{2}$

$Y_P(t) = \frac{1}{2} t \sin t$

genl soln:

$y(t) = c_1 \cos t + c_2 \sin t + \frac{1}{2} t \sin t$

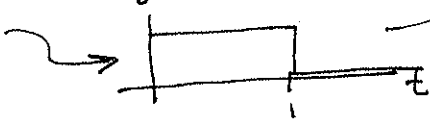
[20] 5. Consider the initial value problem

$$y'' - 3y' + 2y = g(t), \qquad y(0) = 1,\ y'(0) = 1$$

where

$$g(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{if } t \ge 1. \end{cases}$$

$g(t) = 1 - u_1(t)$



(a) Compute the Laplace transform $\mathcal{L}\{g(t)\}$.

(b) Use the method of Laplace transforms to solve the initial value problem.

$$\left(s^2F(s) - sy(0) - y'(0)\right) - 3\left(sF(s) - y(0)\right) + 2F(s) = \mathcal{L}\{g\}$$

$$F(s)\left(s^2 - 3s + 2\right) - s - 1 + 3 = \mathcal{L}\{g\}$$

$$F(s) = \frac{s-2}{(s^2-3s+2)} + \frac{\mathcal{L}\{g\}}{(s^2-3s+2)}$$

$$= \frac{\cancel{s-2}}{(\cancel{s-2})(s-1)} + \frac{1}{(s-2)(s-1)} \cdot \left(\frac{1-e^{-s}}{s}\right)$$

$$= \frac{1}{(s-1)} + \frac{1}{s(s-1)(s-2)}\left(1-e^{-s}\right)$$

See partial fractions, below

$$F(s) = \frac{1}{s-1} + \left(\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{s-1} + \frac{1}{2}\cdot\frac{1}{s-2}\right)\left(1-e^{-s}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{1}{s-2} - \left(\frac{1}{2}\frac{1}{s} - \frac{1}{s-1} + \frac{1}{2}\frac{1}{s-2}\right)e^{-s}$$

(the e^{-s} results in a time shift $t \to t-1$ and a step fn)

$$y(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2} + \frac{1}{2}e^{2t} - \left(\frac{1}{2} - e^{t} + \frac{1}{2}e^{2t}\right)\Big|_{t \to t-1} \cdot u_1(t)$$

$$= \frac{1}{2}\left(1 + e^{2t}\right) - u_1(t)\left[\frac{1}{2} - e^{t-1} + \frac{1}{2}e^{2(t-1)}\right]$$

Partial Fracts:

$$\frac{1}{(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$$

$$1 = A(s-1)(s-2) + Bs(s-2) + Cs(s-1)$$

$s \to 0$: $1 = A(-1)(-2)$ $\quad A = \frac{1}{2}$

$s \to 1$: $1 = B \cdot 1(-1)$ $\quad B = -1$

$s \to 2$: $1 = C\,2(2-1)$ $\quad C = \frac{1}{2}$

$$\therefore \text{Frac} = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{s-1} + \frac{1}{2}\frac{1}{s-2}$$

[20] **6.** Consider the following 2×2 matrix with real coefficients

$$A = \begin{pmatrix} 3 & 1 \\ 0 & a \end{pmatrix},$$

and consider the system of differential equations

$$X'(t) = AX(t), \tag{1}$$

where $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.

(a) Write the determinant $|A - \lambda I|$ in factored form and then find the eigenvalues λ_1 and λ_2 of the matrix A in terms of a.

(b) In the case where $a \neq 3$, find the eigenvectors corresponding to λ_1 and λ_2.

(c) If $a \neq 3$, find two solutions $X^{(1)}(t)$ and $X^{(2)}(t)$, in terms of a, so that $\{X^{(1)}, X^{(2)}\}$ forms a fundamental set of solutions for (1).

(d) Deduce the general solution of (1) in the case $a \neq 3$.

(e) What is/are the eigenvalue(s) of A in the case where $a = 3$? Find, in this case, fundamental solutions $X^{(1)}(t)$ and $X^{(2)}(t)$, and then the general solution of (1).

6a) eigenvalues: $\det(A - rI) = 0$

$\det\begin{pmatrix} 3-r & 1 \\ 0 & a-r \end{pmatrix} = 0 = (3-r)(a-r)$

eigenvalues are $r_1 = 3$, $r_2 = a$ ← (also called here λ_1, λ_2)

(Note: since matrix is diagonal, we could have just written this directly)

(b) Eigenvectors: $\begin{pmatrix} 3-r & 1 \\ 0 & a-r \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$(3-r)v_1 + v_2 = 0$ let $v_1 = 1$ then $v_2 = -3+r$ so $\vec{V} = \begin{pmatrix} 1 \\ -3+r \end{pmatrix}$

Corresponding to $r_1 = 3$ is $\vec{V}_1 = \begin{pmatrix} 1 \\ -3+3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$r_2 = a$ $\vec{V}_2 = \begin{pmatrix} 1 \\ -3+a \end{pmatrix}$

(c) $\vec{X}_1(t) = \vec{V}_1 e^{3t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t}$, $\vec{X}_2 = \vec{V}_2 e^{at} = \begin{pmatrix} 1 \\ -3+a \end{pmatrix} e^{at}$

will be a fundamental set of solns as long as $a \neq 3$

Extra space (if needed)

6 (d) general soln is then $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$ provided $a \neq 3$

i.e. $\vec{x}(t) = c_1\begin{pmatrix}1\\0\end{pmatrix}e^{3t} + c_2\begin{pmatrix}1\\-3+a\end{pmatrix}e^{at}$

6(e) If $a=3$, have only one soln, $\vec{x}_1(t) = \begin{pmatrix}1\\0\end{pmatrix}e^{3t}$ (case of repeated roots)

Form a 2nd soln $\vec{x}_2(t) = \begin{pmatrix}1\\0\end{pmatrix}te^{3t} + \vec{Q}e^{3t}$

we can find only one eigenvector $\begin{pmatrix}1\\0\end{pmatrix}$ here, so use the method shown (see repeated roots lecture)

then $\vec{Q}$ satisfies

$\vec{V} = (M - rI)\cdot\vec{Q}$ where $\vec{V} = \begin{pmatrix}1\\0\end{pmatrix}$ is the known eigenvector

$\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\cdot\begin{pmatrix}q_1\\q_2\end{pmatrix} \Rightarrow q_2 = 1$, q_1 arbitrary

e.g. $q_1 = 1$

$\vec{x}_2 = \begin{pmatrix}1\\0\end{pmatrix}te^{3t} + \begin{pmatrix}1\\1\end{pmatrix}e^{3t}$ so $\underbrace{\vec{Q} = \begin{pmatrix}1\\1\end{pmatrix}}$

So fundam. set is

$\vec{x}_1 = \begin{pmatrix}1\\0\end{pmatrix}e^{3t}$ $\vec{x}_2 = \begin{pmatrix}1\\1\end{pmatrix}te^{3t}$

gen'l soln

$\vec{x}(t) = c_1\begin{pmatrix}1\\0\end{pmatrix}e^{3t} + c_2\left[\begin{pmatrix}1\\0\end{pmatrix}te^{3t} + \begin{pmatrix}1\\1\end{pmatrix}e^{3t}\right]$

[15] 7. Consider the differential equation $y'' - 4y' + 13y = 0$.

(a) Transform the above equation into a system of first order differential equations, and write it in matrix form $X'(t) = AX(t)$.

(b) Find two real-valued solutions $X^{(1)}(t)$ and $X^{(2)}(t)$ that form a fundamental set of solutions to the system $X' = AX$ from part (a). What is the general solution of the system?

(c) Describe the behaviour of the solutions as $t \to \infty$.

(a) There are several ways to proceed here. Note that the 2nd order ODE has char. eqn $r^2 - 4r + 13 = 0$ and its eigenvalues are $r = 2 \pm 3i$ (easy to show). The corresponding matrix for system would have same char. eqn so $\beta = -4 = \text{Trace}(M)$ $\gamma = 13 = \det(M)$. Any such matrix would work so pick $M = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$ for example. Then

$$\frac{d\vec{x}}{dt} = M\vec{x}$$ is equivalent to the above 2nd order ODE.

or: we could define $v = y'$ and sub into ODE to get a system
$$\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -13y + 4v \end{cases}$$

(b) Eigenvalues (as before): $r = 2 \pm 3i$

Eigenvectors: $\begin{pmatrix} 2-r & -3 \\ 3 & 2-r \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$(2-r)v_1 - 3v_2 = 0$ let $v_1 = 1$ then $v_2 = \frac{2-r}{3}$

so $\vec{V} = \begin{pmatrix} 1 \\ \frac{2-r}{3} \end{pmatrix}$

$$\vec{V}_{1,2} = \begin{pmatrix} 1 \\ \frac{\mp 3i}{3} \end{pmatrix} = \begin{pmatrix} 1 \\ \mp i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \equiv \vec{a} \pm \vec{b} i$$

$$e^{rt} = e^{2t}\left(\cos(3t) \pm i \sin(3t)\right)$$

Now build up real valued solns

$$\vec{u}(t) = e^{2t}\left(\vec{a}\cos 3t - \vec{b}\sin 3t\right), \quad \vec{v}(t) = e^{2t}\left(\vec{b}\cos 3t + \vec{a}\sin 3t\right)$$

get $\vec{X}(t) = c_1\vec{u}(t) + c_2 v(t)$, etc.

(c) as $t \to \infty$ $u(t)$ and $v(t)$ are both exponentially increasing oscillations. (i.e. cycles with amplitudes that grow).