Mathematics 265 Section: Full Name: ____________________

(circle one) 101 103 Student Number: ____________________

Midterm Test Oct 6, 2010

Instructions: There are **6 pages** in this test (including this cover page).

1. **Caution: There may (or may not) be more than one version of this test paper.**
2. Ensure that your *full* name and student number appears on this page. Circle your section number.
3. No calculators, books, notes, or electronic devices of any kind are permitted.
4. Show all your work. Answers not supported by calculations or reasoning will not receive credit. Messy work will not be graded.
5. Five minutes before the end of the test period you will be given a verbal notice. After that time, you must remain seated until all test papers have been collected.
6. When the test period is over, you will be instructed to put away writing implements. Put away all pens and pencils at this point. Continuing to write past this instruction will be considered dishonest behaviour.
7. Please remain seated and pass your test paper down the row to the nearest indicated aisle. Once all the test papers have been collected, you are free to leave.
8. Exposing your test paper, copying from another student's paper, or sharing information about this test constitutes academic dishonesty. Such behaviour may jeopardize your grade on this test, in this course, and your standing at this university.

Question	Grade	Value
1		24
2		16
3		18
4		22
Total		80

I have read and understood the instructions and agree to abide by them.

Signed: ____________________

Problem 1 In each case, solve the ODE for $y(t)$:

(a) $\frac{dy}{dt} = -y^{1/2}$, and $y(0) = 1$

(b) $\frac{dy}{dt} = a - \frac{1}{t}y$ and $y(1) = 1$ (where $a > 0$ is a constant).

(c) $y'' - 5y' - 6y = 0$, and $y(0) = -1, y'(0) = 1$

Solution to Problem 1 V1:

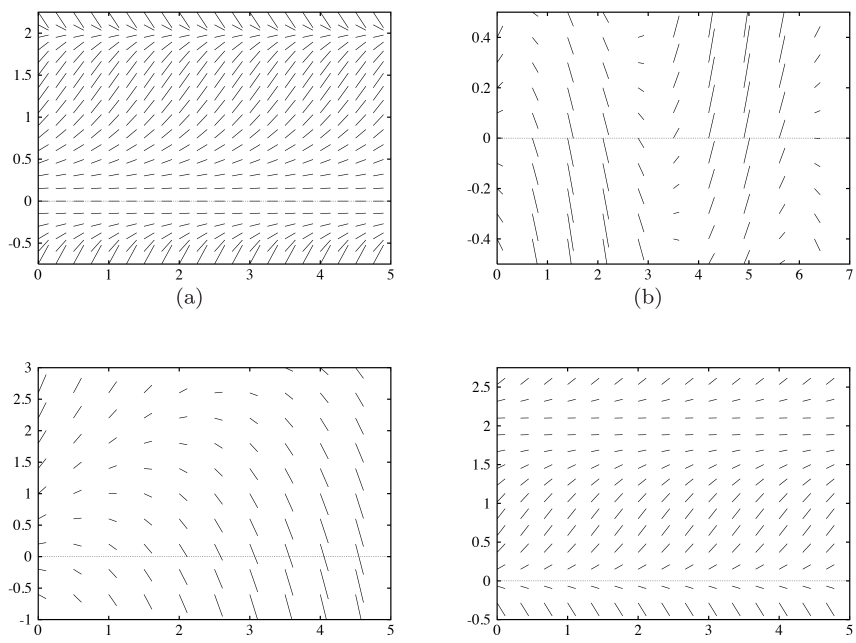
(a) This equation is nonlinear. We solve it using separation of variables $\frac{dy}{y^{1/2}} = -dt \Rightarrow \int y^{-1/2}dy = -\int dt + C \Rightarrow \frac{y^{1/2}}{1/2} = -t + C \Rightarrow y^{1/2} = -(1/2)t + C'$. We can use the initial condition now to deduce that $C' = 1$ so that $y^{1/2} = -(1/2)t + 1$ and finally $y(t) = [1 - (t/2)]^2$.

(b) We will use an integrating factor. Put in standard form $\frac{dy}{dt} + \frac{y}{t} = a$. Then the integrating factor is $\mu(t) = \exp[\int 1/tdt] = \exp[\ln(t)] = t$. So multiply both sides by t to get $t[\frac{dy}{dt} + \frac{y}{t}] = \frac{d(ty)}{dt} = at$. Integrate to get $ty = \int atdt + C = \frac{1}{2}at^2 + C$. So then $y(t) = \frac{1}{2}at + C\frac{1}{t}$. Use the initial condition to get the constant $1 = y(1) = \frac{1}{2}a + C$ $C = 1 - \frac{1}{2}a$. The solution is thus $y(t) = \frac{1}{2}at + (1 - \frac{1}{2}a)\frac{1}{t}$.

(c) The characteristic equation is $r^2 - 5r - 6 = (r - 6)(r + 1) = 0$. This has the roots $r = -1, 6$ so the general solution is $y(t) = C_1e^{-t} + C_2e^{6t}$. Then we use the initial conditions: $-1 = y(0) = C_1 + C_2$ and $1 = (-C_1 + 6C_2)$. Solving for the constants we find that $C_2 = 0, C_1 = -1$ so the solution is $y(t) = -e^{-t}$.

Problem 2 Match the direction fields in Fig 1 with the differential equations by circling a, b, c, or d in each case. Some of these differential equations do not match any of the direction fields - for those cases circle "None".

$y' = y(2-y)^2$	a	b	c	d	None
$y' = y^2(2-y)$	a	b	c	d	None
$y' = y(2-y)$	a	b	c	d	None
$y' = y - t$	a	b	c	d	None
$y' = y - \sin(t)$	a	b	c	d	None
$y' = y + t$	a	b	c	d	None



(a) (b) (c) (d)

Figure 1: Direction fields

Solution to Problem 2 V1:

$y' = y^2(2-y)$ 1(a),
$y' = y - \sin(t)$ 1(b) ,
$y' = y - t$ 1(c) ,
$y' = y(2-y)^2$ 1(d) ,

Problem 3 Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of the environment. The ODE for temperature $T(t)$ can be written as

$$\frac{dT}{dt} = k(E - T),$$

where E is the temperature of the environment, and k is a constant.

(a) Is the constant k positive or negative? Explain why.

(b) Find the temperature $T(t)$ given the initial condition $T(0) = 0$ in the case that E is constant. (Your answer will be in terms of E and k.)

(c) Now suppose that $E(t) = E_0 e^{-t}$ and $T(0) = 0$. Find $T(t)$ for $t > 0$. (**You may assume that $k \neq 1$ in this question.**)

Solution to Problem 3

(a) $k > 0$: a hot object in a cool environment should lose heat (cool down) so if $T > E$ we expect that $dT/dt < 0$, which is true if $k > 0$.

(b)

$$\frac{dT}{dt} = k(E - T), \quad \Rightarrow \frac{dT}{dt} + kT = kE$$

Integrating factor $\mu(t) = \exp[kt] = e^{kt}$ leads to

$$e^{kt}(\frac{dT}{dt} + kT) = \frac{d}{dt}[e^{kt}T] = kEe^{kt}$$

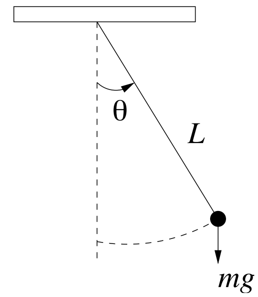
Integrate to get $e^{kt}T = \int kEe^{kt}dt + C = \frac{kE}{k}e^{kt} + C = Ee^{kt} + C$ so then $T(t) = E + Ce^{-kt}$. Now use the initial condition: $T(0) = 0$ to get $0 = T(0) = E + C$ so $C = -E$ and the solution is $T(t) = E - Ee^{-kt} = E(1 - e^{-kt})$.

(c) $E(t) = E_0 e^{-t}$ and $T(0) = 0$. Then we have a similar integrating factor and get to the equation $\frac{d}{dt}[e^{kt}T] = kE_0 e^{-t}e^{kt} = kE_0 e^{(k-1)t}$. Integrating leads to
$[e^{kt}T] = kE_0 \int e^{(k-1)t}dt + C$
$[e^{kt}T] = \frac{kE_0}{k-1}e^{(k-1)t} + C$ so $T(t) = \frac{kE_0}{(k-1)}e^{-t} + Ce^{-kt}$. Using the initial condition $T(0) = 0$ leads to $0 = T(0) = \frac{kE_0}{(k-1)} + C$ so $C = -\frac{kE_0}{(k-1)}$. Thus the solution is $T(t) = \frac{kE_0}{(k-1)}e^{-t} - \frac{kE_0}{(k-1)}e^{-kt} = \frac{kE_0}{(k-1)}\left(e^{-t} - e^{-kt}\right)$

Problem 4:

The angle that an oscillating pendulum of length L makes with the vertical direction (see Figure) approximately satisfies the equation

$$L\frac{d^2\theta}{dt^2} + g\theta = 0, \qquad (PEND)$$



where g is the acceleration due to gravity. Assume $L > 0$, and $g > 0$ are constant. Answer the following questions:

(a) Describe the motion of this pendulum starting from rest at a small deflection (You are asked to find the solution to $(PEND)$ for $\theta(0) = \theta_0, \theta'(0) = 0$.)

(b) When the pendulum becomes rusty, the motion is damped so that the ODE describing the motion becomes

$$L\frac{d^2\theta}{dt^2} + D\frac{d\theta}{dt} + g\theta = 0, \qquad (DAMP)$$

for what value of D would the oscillation stop? (**You are only asked to find the value of D for critical damping. You do not need to solve the equation.**)

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(c) The *undamped* pendulum is connected to a motor that applies a periodic force to it. The new equation describing its deflection is

$$L\frac{d^2\theta}{dt^2} + g\theta = \sin(\omega_0 t), \qquad (FORCED)$$

Write down the form of the solution to equation $(FORCED)$ and explain how that form depends on the value of ω_0. (**You do not need to solve for any constants in your solution.**)

(d) The actual (unapproximated) ODE for the motion of a pendulum is

$$L\frac{d^2\theta}{dt^2} + g\sin(\theta) = 0, \qquad (TRUE)$$

Briefly explain (1 sentence) under what conditions the differential equation $(TRUE)$ is approximated well by $(PEND)$. (**You do not need to solve this equation.**)

Solution to Problem 4

(a) We solve

$$L\frac{d^2\theta}{dt^2} + g\theta = 0, \quad \theta(0) = \theta_0, \theta'(0) = 0 \qquad (PEND)$$

This is a linear second order ODE with constant coefficients. The characteristic equation is $Lr^2 + g = 0$, so the roots are pure imaginary, $r = \pm i\sqrt{g/L}$ where $i = \sqrt{-1}$. Thus the solution is $\theta(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$ where $\omega = \sqrt{g/L}$. To find the constants, use the initial conditions. First note that $\theta'(t) = -c_1\sin(\omega t) + c_2\cos(\omega t)$. Use $\theta(0) = \theta_0, \theta'(0) = 0$ to get:

$\theta_0 = \theta(0) = c_1\cos(0) + c_2\sin(0) = c_1$

$0 = \theta'(0) = -c_1\sin(0) + c_2\cos(0) = c_2$.

So $c_1 = \theta_0$ and $c_2 = 0$. and so the solution is $\theta(t) = \theta_0\cos(\omega t) = \theta_0\cos\left(\sqrt{\frac{g}{L}}t\right)$

(b) The equation $(DAMP)$ is $L\dfrac{d^2\theta}{dt^2} + D\dfrac{d\theta}{dt} + g\theta = 0$. The characteristic equation is $Lr^2 + Dr + g = 0$ and the roots are $r = \dfrac{-D \pm \sqrt{D^2 - 4Lg}}{2L}$. Critical damping occurs when the roots are real and repeated. This is when $D^2 = 4Lg$ or $D = 2\sqrt{Lg}$.

(c) The equation of the forced undamped pendulum ($FORCED$) is $L\frac{d^2\theta}{dt^2} + g\theta = \sin(\omega_0 t)$. The solution is $\theta(t) = c_1\theta_1(t) + c_2\theta_2(t) + \theta_p(t)$ where $\theta_1(t), \theta_2(t)$ are solutions to the homogeneous ODE ($PEND$) and $\theta_p(t)$ is a particular solution to ($FORCED$).

If $\omega_0 \neq \sqrt{g/L}$ then the form of the solution is $\theta(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + A\cos(\omega_0 t) + B\sin(\omega_0 t)$

If $\omega_0 = \sqrt{g/L}$ then the form of the solution is $\theta(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + t[A\cos(\omega_0 t) + B\sin(\omega_0 t)]$

(d) The actual (unapproximated) ODE, ($TRUE$) is $L\frac{d^2\theta}{dt^2} + g\sin(\theta) = 0$, whereas ($PEND$) is $L\frac{d^2\theta}{dt^2} + g\theta = 0$. The approximation made is $\sin(\theta) \approx \theta$ which holds for small values of θ, i.e. for small deflections of the pendulum.