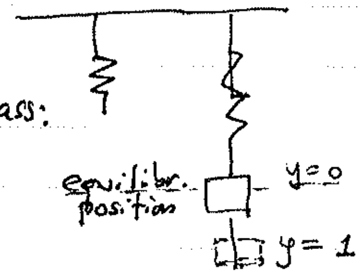
Nov 24, 2010 Math 265

Applications of Lin. 1st order ODE systems to:

→ spring mass system viewed as a sys. of 1st order ODEs.

Recall we had derived the following ODE for displacem. of mass:



(*) $$\boxed{m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0} \qquad \underbrace{y(0)=1 \quad y'(0)=0}_{\text{released from a stretched position with } v=0}$$

Now we derive the corresponding system of 1st order ODES. To do so, let $v \equiv \frac{dy}{dt}$ = veloc. of the mass at time t

↑ defn of v then $\frac{d^2y}{dt^2} = \frac{dv}{dt}$ use this in (*)

(*) $$\Rightarrow \quad m\frac{dv}{dt} + cv + ky = 0 \quad \Rightarrow \quad \frac{dv}{dt} = -\frac{c}{m}v - \frac{k}{m}y$$

So the single (2nd order) ODE for spring-mass system can be "traded in" for a system of 1st order ODEs. in the variables $y(t)$, $v(t)$

$$\left.\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\frac{k}{m}y - \frac{c}{m}v \end{aligned}\right\} \quad \text{or } \vec{x}(t) = M\vec{x} \text{ when } \vec{x}(t) = \begin{pmatrix} y(t) \\ v(t) \end{pmatrix}$$

and $M = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix}$

The initial conditions are $\vec{x}(0) = \begin{pmatrix} y(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

→ Let us study the behavior of this system for $m = 1$ kg $k = 1$ kg/s² and understand the effect of variable damping, i.e. $c \geq 0$.

The system is then $\frac{d\vec{x}}{dt} = M\cdot\vec{x}$ $\quad M = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix}$

Eigenvalues: satisfy $\det(M - rI) = 0$ $\quad \det\begin{pmatrix} -r & 1 \\ -1 & -(c+r) \end{pmatrix} = r(c+r)+1 = 0$

charac. eqn is $r^2 + cr + 1 = 0$

eigenvalues: $r_{12} = \frac{-c \pm \sqrt{c^2-4}}{2}$

remark:
$\beta = \mathrm{Tr}\, M = -c$
$\gamma = \det M = 1$
So char. eqn is
$r^2 - \beta r + \gamma = 0$
$r^2 + cr + 1 = 0$

Let us consider some specific examples of what can happen

Cases:

① $c = 4$ (large damping) : $r_{1,2} = \dfrac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$ ← both roots are real and negative (since $\sqrt{3} < 2$)

eigenvectors: $(M - rI) \cdot \vec{v} = 0$

$$\begin{pmatrix} 0-r & 1 \\ -1 & -(c+r) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -r v_1 + v_2 \\ -v_1 - (c+r) v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Use first of these eqns (as they are redundant)
then, take $v_1 = 1$ so $v_2 = r$ $\quad \vec{V} = \begin{pmatrix} 1 \\ r \end{pmatrix}$

so eigenvalue $r_1 = -2 + \sqrt{3} \approx -0.3$ has corresponding eigenvector $\vec{V}_1 = \begin{pmatrix} 1 \\ -2+\sqrt{3} \end{pmatrix} \approx \begin{pmatrix} 1 \\ -0.3 \end{pmatrix}$

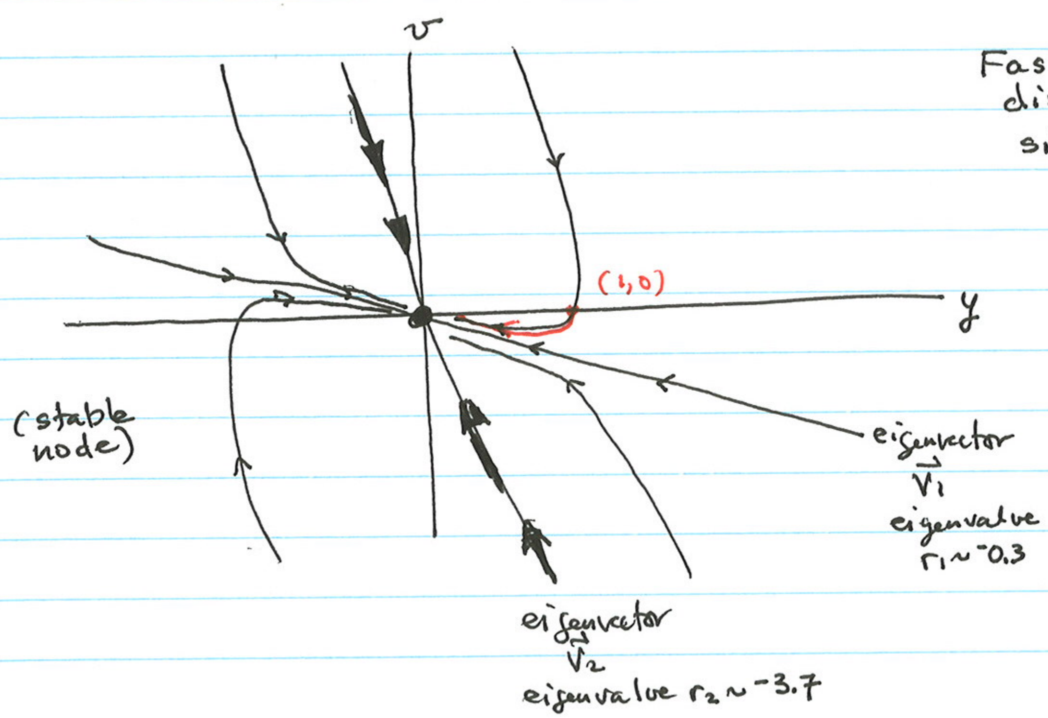
eigenvalue $r_2 = -2 - \sqrt{3} \approx -3.7$ .. $\vec{V}_2 = \begin{pmatrix} 1 \\ -2-\sqrt{3} \end{pmatrix} \approx \begin{pmatrix} 1 \\ -3.7 \end{pmatrix}$

General soln:

$$\vec{x}(t) = c_1 \vec{V}_1 e^{r_1 t} + c_2 \vec{V}_2 e^{r_2 t} = C_1 \begin{pmatrix} 1 \\ -2+\sqrt{3} \end{pmatrix} e^{(-2+\sqrt{3})t} + C_2 \begin{pmatrix} 1 \\ -2-\sqrt{3} \end{pmatrix} e^{(-2-\sqrt{3})t}$$

(woops! ran outta space)

Let us sketch the general soln in the yv plane:



Fast flow in the direction of $\vec{V}_2$ since $|r_2| \gg |r_1|$

Initial conds $\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow$ solve for constants c_1, c_2:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -2+\sqrt{3} \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2-\sqrt{3} \end{pmatrix}$$

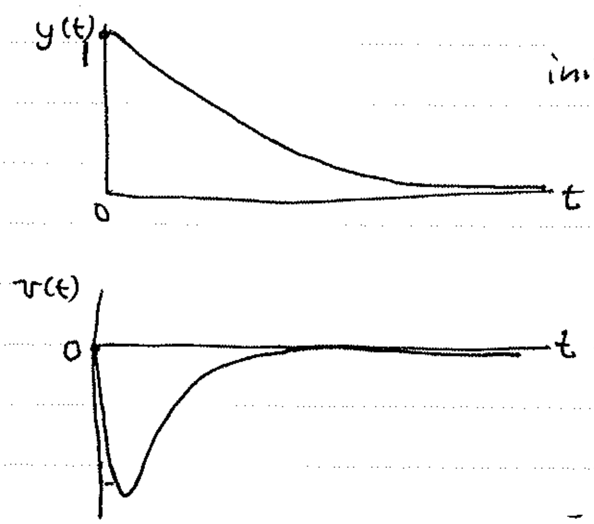
$$\Rightarrow \left. \begin{aligned} 1 &= c_1 + c_2 \\ 0 &= (-2+\sqrt{3})\, c_1 + (-2-\sqrt{3})\, c_2 \end{aligned} \right\} \Rightarrow \quad c_1 = \frac{1}{2} + \frac{\sqrt{3}}{3} \qquad c_2 = \frac{1}{2} - \frac{\sqrt{3}}{3}$$

So Soln is

$$\vec{x}(t) = \begin{pmatrix} y(t) \\ v(t) \end{pmatrix} = \left(\frac{1}{2} + \frac{\sqrt{3}}{3}\right) \begin{pmatrix} 1 \\ -2+\sqrt{3} \end{pmatrix} e^{(-2+\sqrt{3})t} + \left(\frac{1}{2} - \frac{\sqrt{3}}{3}\right) \begin{pmatrix} 1 \\ -2-\sqrt{3} \end{pmatrix} e^{(-2-\sqrt{3})t}$$

(can be simplified, but let's skip that for now.)

Can plot:



initially $y = 1$
$v = 0$

These plots correspond to the red trajectory in the yv plane (previous page)

Notice that for this case since $c^2 > 1$, there is no oscillation: the spring creeps back to its rest position.

This is the case

Case ② Critical damping $c^2-4=0$ $\leftarrow \beta^2-4\gamma=0$

i.e, $c=2$ eigenvalues: $r_{1,2} = -\frac{2}{2} = -1$ (repeated)

Eigenvector(s): $(M-rI)\cdot\vec{v}=0$

$$\begin{pmatrix}1 & 1\\ -1 & -(2-1)\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \qquad \left.\begin{aligned}v_1+v_2&=0\\ -v_1-v_2&=0\end{aligned}\right\}\Rightarrow v_1=-v_2$$

only one eigenvector $\begin{pmatrix}1\\-1\end{pmatrix}$

Solns: $\vec{x}_1(t) = \vec{v}_1 e^{rt} = \begin{pmatrix}1\\-1\end{pmatrix}e^{-t}$

Second soln: $\vec{x}_2(t) = \begin{pmatrix}1\\-1\end{pmatrix}te^{-t} + \underbrace{\begin{pmatrix}q_1\\q_2\end{pmatrix}}_{\vec{Q}}e^{-t}$

find this vector so that $\vec{x}_2$ is a soln.

From last lecture, we know that

$$\vec{v}_1 = (M-rI)\cdot\vec{Q}$$

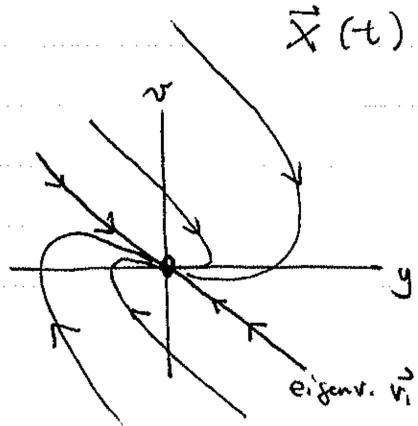
$$\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\begin{pmatrix}q_1\\q_2\end{pmatrix} \qquad \left.\begin{aligned}q_1+q_2&=1\\ -q_1-q_2&=-1\end{aligned}\right\} q_1=1-q_2$$

e.g. pick $q_1=1$, $q_2=0$

so $\vec{x}_2(t) = \begin{pmatrix}1\\-1\end{pmatrix}te^{-t} + \begin{pmatrix}1\\0\end{pmatrix}e^{-t}$

Gen'l Soln:

$$\vec{x}(t) = c_1\underbrace{\begin{pmatrix}1\\-1\end{pmatrix}e^{-t}}_{\vec{x}_1(t)} + c_2\underbrace{\left[\begin{pmatrix}1\\-1\end{pmatrix}te^{-t} + \begin{pmatrix}1\\0\end{pmatrix}e^{-t}\right]}_{\vec{x}_2(t)}$$



← The phase portrait looks like this

Case (3) small damping.

$c = 1 \qquad c^2 - 4 = -3$

eigenvalues $\quad r_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i \quad$ (complex conjugates) $\quad r = \sigma \pm \mu i$

$\sigma = -\frac{1}{2} \quad \mu = \frac{\sqrt{3}}{2}$

quasi freq.

eigenvectors $\quad \vec{v}_1 = \begin{pmatrix} 1 \\ r_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} + \frac{\sqrt{3}}{2} i \end{pmatrix}$

complex conj

$\vec{v}_2 = \begin{pmatrix} 1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} - \frac{\sqrt{3}}{2} i \end{pmatrix}$

$\vec{v}_{1,2} = \vec{a} \pm \vec{b} i \quad$ where $\quad \vec{a} = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ \frac{\sqrt{3}}{2} \end{pmatrix}$

Solns: $\quad \vec{x}_1 = \vec{v}_1 e^{r_1 t} = (\vec{a} + \vec{b} i)\, e^{(\sigma + i\mu) t}$

$\vec{x}_2 =$ complex conjug. of the above

Find two <u>real</u> valued solns by writing out real and imag parts:

$$\vec{x}_1 = (\vec{a} + \vec{b} i)\, e^{\sigma t} \left(\cos(\mu t) + i \sin(\mu t) \right)$$
$$= e^{\sigma t} \left([\vec{a} \cos(\mu t) - \vec{b} \sin(\mu t)] + i\, [\vec{b} \cos(\mu t) + \vec{a} \sin(\mu t)] \right)$$

let

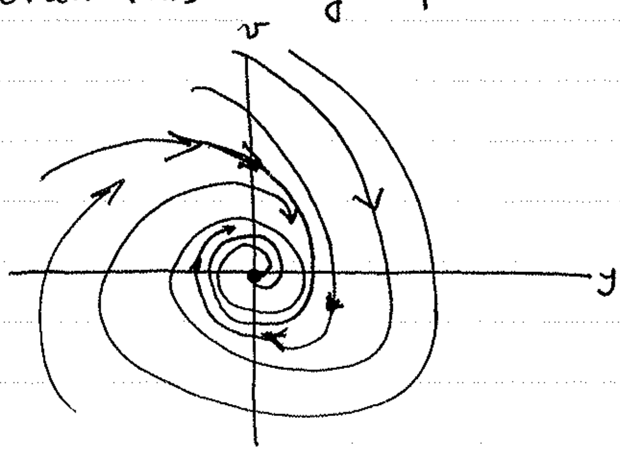
$$\vec{u}(t) = e^{\sigma t} [\vec{a} \cos(\mu t) - \vec{b} \sin(\mu t)] = e^{-\frac{t}{2}} \left(\begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \cos \tfrac{\sqrt{3}}{2} t - \begin{pmatrix} 0 \\ \frac{\sqrt{3}}{2} \end{pmatrix} \sin \tfrac{\sqrt{3}}{2} t \right)$$
$$\vec{v}(t) = e^{\sigma t} [\vec{b} \cos(\mu t) + \vec{a} \sin(\mu t)] = e^{-\frac{t}{2}} \left(\begin{pmatrix} 0 \\ \frac{\sqrt{3}}{2} \end{pmatrix} \cos(\tfrac{\sqrt{3}}{2} t) - \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \sin \tfrac{\sqrt{3}}{2} t \right)$$

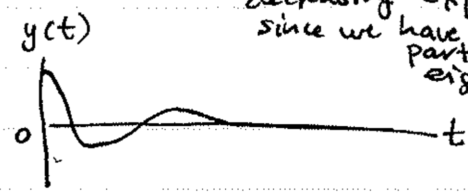
genl' soln $\quad \vec{x}(t) = c_1 \vec{u}(t) + c_2 \vec{v}(t)$

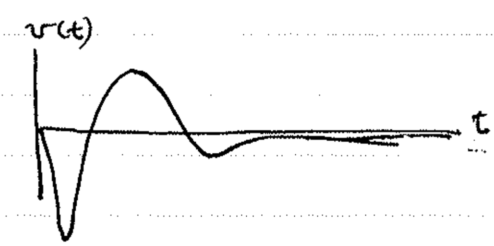
Note: ampl. of osc. is decreasing exponentially since we have −ve real part of eigenvalue.

Let us draw this in yv plane:

stable spiral







$\boxed{\sigma = 0,\ \omega = 1}$

Case 4: No damping

$c = 0$ eigenvalues $r = \pm \frac{\sqrt{-4}}{2} = \pm i$ pure imaginary

eigenvectors: $\vec{v}_1 = \begin{pmatrix} 1 \\ r_1 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} i$

$\vec{v}_2 = \begin{pmatrix} 1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} i$

$\vec{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

as before, real valued solns are

$\vec{u}(t) = e^{\sigma t}(\vec{a}\cos(\omega t) - \vec{b}\sin(\omega t))$ but $\sigma = 0$ so $e^{\sigma t} = 1$

$\vec{v}(t) = e^{\sigma t}(\vec{b}\cos(\omega t) + \vec{a}\sin(\omega t))$

Thus $\vec{u}(t) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t\right)$

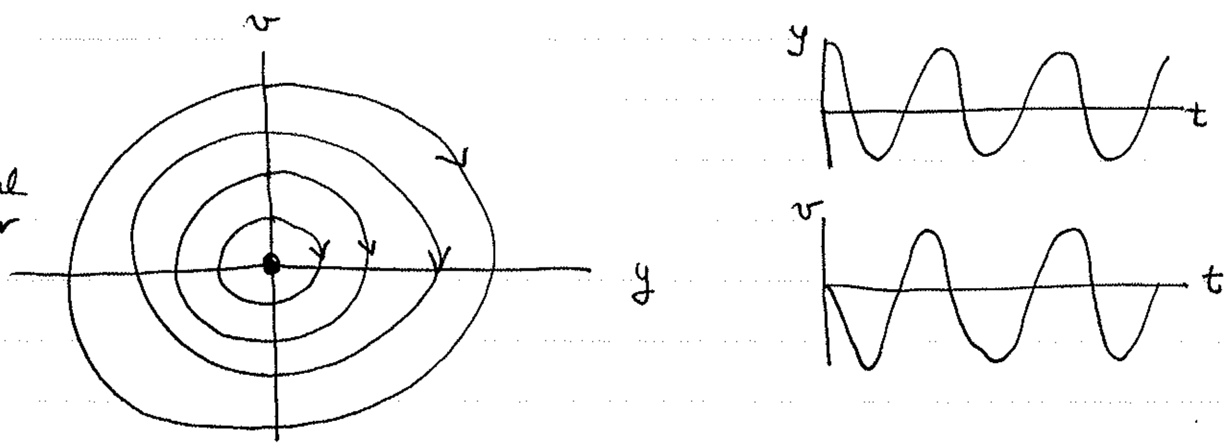
$\vec{v}(t) = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin t\right)$

real valued solns (fundam. set) as before

$$\vec{x}(t) = c_1 \vec{u}(t) + c_2 \vec{v}(t) = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

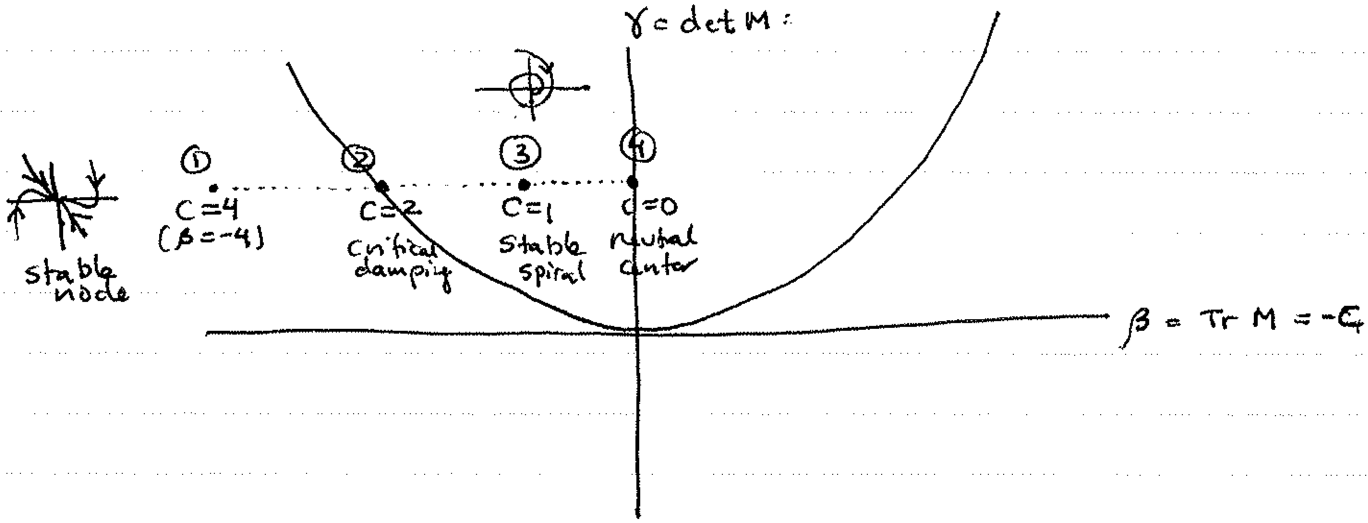
Sketch:

neutral center



We have seen that just varying one parameter (in this case, the damping) can change the behaviour of the spring-mass system

In fact, we can summarize the whole set of behaviors as follows:



Cases 1, 2, 3, 4 correspond to the above four sets of parameter values. Note that we kept the mass $m=1$ and spring constant $k=1$ so that $\gamma = \det M = 1$ was the same in all cases. Varying c corresponds to varying $\beta = \mathrm{Tr}\, M$.

We have encountered similar cases in our previous study of the 2nd order ODE for spring-mass dynamics. Here we just looked at it from the point of view of a system of ODEs.