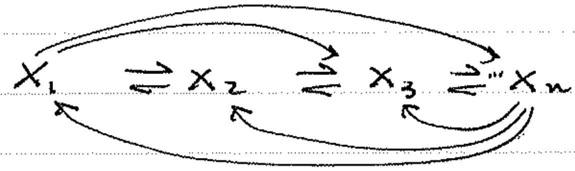
Nov 29, 2010 Math 265

Generalizing our Results to:
- Systems of n eqns
- Nonhomogeneous sys. of ODEs.

Larger Systems



← n variables with various interactions / interconversions

This may result in a system of n ODEs:

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots a_{1n}x_n \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \dots \qquad a_{nn}x_n \end{aligned}\right\} \text{ n coupled first order linear ODEs.}$$

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad \frac{d\vec{x}}{dt} = M\vec{x}, \qquad M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix}$$

$n \times n$ matrix

$$\vec{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{pmatrix} \equiv \vec{x}_0$$

The solns are still in the form

$$\vec{x}(t) = \vec{v}e^{rt}$$

eigenvalue

eigenvector (now an n-vector)

where r satisfies $\boxed{M\cdot\vec{v} = r\vec{v}}$

$$\det(M - rI) = 0 \qquad (1)$$

(i.e. r is still an eigenvalue.

$\vec{v}$ satisfies $(M - rI)\cdot\vec{v} = 0 \qquad (2)$

Now it will be harder to find r: its characteristic eqn, obtained from (1) is an <u>nth degree polynomial</u>:

$$r^n + \alpha_1 r^{n-1} + \alpha_2 r^{n-2} + \dots \alpha_n = 0$$

(whose roots may be hard to compute analytically). $r = r_1, r_2, \dots, r_n$

- some roots could be real
- some " " complex ← these always come in (complex conjugate.. pairs

$$r = \sigma_j \pm \mu_j i$$

The general solution is typically: (for roots with none repeated)

$$\boxed{X(t) = c_1 \vec{v}_1 e^{r_1 t} + c_2 \vec{v}_2 e^{r_2 t} + \dots \quad c_n \vec{v}_n e^{r_n t}}$$

Special cases:

- complex roots $r_{1,2} = \sigma \pm \mu i$:
  - Form the real valued solns for each such pair
  - (see lecture on complex roots for procedure).
- repeated roots: follow procedure as described for 2x2 case.

To find the values of $c_1, c_2 \dots c_n$: use IC's, plug in $t=0$ into general soln.

Remark: The n solns $\vec{x}_1(t) = \vec{v}_1 e^{r_1 t}, \dots \vec{x}_n(t) = \vec{v}_n e^{r_n t}$ should, as usual, be a (linearly independent) set of solns, i.e. a fundamental set of solns. We can test this by checking that the Wronskian

$$W = \det\left[\vec{x}_1 \ \vec{x}_2 \dots \vec{x}_n\right] \neq 0$$

i.e. that $\underbrace{\det\left[\vec{v}_1 \ \vec{v}_2 \dots \vec{v}_n\right]}_{\text{so it suffices to check that this part} \neq 0.} \underbrace{e^{r_1 t} e^{r_2 t} \dots e^{r_n t}}_{\text{this stuff is never } 0} \neq 0$

Example of a 3x3 system

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} x+2y-z \\ y+z \\ -y+z \end{pmatrix}.$$

Can write in usual matrix form:

$$\vec{X}' = M\vec{X} \qquad M = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

As usual, find eigenvalues + eigenvectors.

eigenvalues:

$$\det(M - rI) = \det\begin{pmatrix} 1-r & 2 & -1 \\ 0 & 1-r & 1 \\ 0 & -1 & 1-r \end{pmatrix}$$

$$= 1-r \begin{vmatrix} 1-r & 1 \\ -1 & 1-r \end{vmatrix} = (1-r)\left((1-r)^2+1\right) = 0$$

Char. eqn: $(1-r)\left((1-r)^2+1\right) = 0 \quad \Rightarrow$ either $1-r=0 \Rightarrow r=1$

or $\left((1-r)^2+1\right)=0 \Rightarrow 1-r = \pm\sqrt{-1} = \pm i$

$r = 1 \pm i$

eigenvalues $r = 1,\ \underbrace{1+i,\ 1-i}_{\text{(complex conjugate pair)}}$

eigenvectors: $(M - rI)\cdot\vec{v} = 0$

$$\begin{pmatrix} 1-r & 2 & -1 \\ 0 & 1-r & 1 \\ 0 & -1 & 1-r \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (1-r)v_1 + 2v_2 - v_3 &= 0 \\ (1-r)v_2 + v_3 &= 0 \\ -1\,v_2 + (1-r)v_3 &= 0 \end{aligned}\right\} \text{eqns are always redundant}$$

for eigenvalue $r_1 = 1$ get

$$\begin{aligned} 2v_2 - v_3 &= 0 \\ v_3 &= 0 \\ -v_2 &= 0 \end{aligned}$$

pick v_1 arbitrarily, e.g. $v_1 = 1$ and note that $v_2 = 0$, $v_3 = 0$

$\Rightarrow$ eigenvector $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\vec{x}_1(t) = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{1\cdot t}}_{\vec{v}_1 e^{r_1 t}}$$

for eigenvalues $r_{2,3} = 1 \pm i$ get eigenvectors $\vec{V}_{2,3} = \begin{pmatrix} 2 \mp i \\ 0 \pm i \\ -1 \pm 0i \end{pmatrix}$

$$\vec{V}_{2,3} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \pm i \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \equiv \vec{a} \pm \vec{b}i$$

where: $\vec{a}$ is $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$, $\vec{b}$ is $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$\vec{x}_{2,3} = \vec{V}_{2,3}\, e^{r_{2,3} t}$ are complex valued

but as usual, we can form two real valued solns from the real/imag. parts of $\vec{V}_{2,3}$, $e^{r_{2,3}t}$:

General soln is

$$\vec{X}(t) = c_1 \vec{V}_1 e^{r_1 t} + c_2 \overbrace{e^{\sigma t}(\vec{a}\cos\mu t - \vec{b}\sin\mu t)}^{\text{"}\vec{u}(t)\text{"}} + c_3 \underbrace{e^{\sigma t}(\vec{b}\cos\mu t + \vec{a}\sin\mu t)}_{\text{"}\vec{v}(t)\text{"}}$$

where $\sigma = Re(r_{2,3}) = 1$ $\qquad$ $\vec{u}, \vec{v}$ real valued.

$\mu = Im(r_{2,3}) = 1$

so

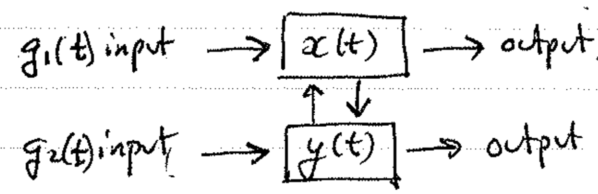
$$\vec{X}(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t + c_2 e^t \left[\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \cos t - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \sin t \right] + c_3 e^t \left[\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \sin t \right]$$

Components of soln:

$$\vec{X}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} c_1 + 2c_2(\cos t + \sin t) + c_3(\cos t + 2\sin t) \\ -c_2 \sin t + c_3 \cos t \\ -c_2 \cos t - c_3 \sin t \end{pmatrix} e^t$$

Brief view of Nonhomogeneous Systems of eqns.

(- as in 2nd order eqn, this arises from some input or forcing function



Example:

$$\frac{d}{dt}\vec{x}(t) = M\vec{x} + \vec{g}(t)$$

See Boyce + Diprima pp 432-435 for Undet'd coeffs soln methods
438-39 " Laplace Tr. " "

Here we'll consider only Undet'd Coeffs and one example.

p440 # 7

$$\frac{d\vec{X}}{dt} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\vec{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t \leftarrow \vec{g}(t)$$

first let us examine the eigenvalues of the homog. system (to ensure no duplication of soln to hom sys.)

$M = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ eigenvalues: $\det\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix} = (1-r)^2 - 4 = 0$

$(1-r)^2 = 4 \Rightarrow 1-r = \pm 2$

$r = 3, -1$

So hom soln will be of the form

$c_1 \vec{V}_1 e^{3t} + c_2 \vec{V}_2 e^{-t}$, and not duplicated by any particular soln in the form

$\vec{x}_p(t) = \vec{A}e^t \leftarrow$ i.e. form similar to $\vec{g}(t)$

Thus, this is a good guess for a partic. soln. where $\vec{A} = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$

Now compute $\frac{d\vec{x}_p}{dt} = \vec{A}e^t$, plug into ODE:

$$\vec{A}\cancel{e^t} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\vec{A}\cancel{e^t} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}\cancel{e^t}$$

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} a_0 + b_0 \\ 4a_0 + b_0 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} -b_0 \\ -4a_0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \Rightarrow \begin{matrix} a_0 = 1/4 \\ b_0 = -2 \end{matrix}$$

Conclude: $\vec{A} = \begin{pmatrix} 1/4 \\ -2 \end{pmatrix}$, so $\vec{x}_p = \begin{pmatrix} 1/4 \\ -2 \end{pmatrix} e^{t}$

It turns out (verify!) that soln to hom sys is

$$\vec{x}_h(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

the particular soln we just found

So soln to full problem is: $\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + \begin{pmatrix} 1/4 \\ -2 \end{pmatrix} e^{t}$$

Remarks: ① if the input $(g(t))$ has a form similar to parts of $\vec{x}_h(t)$ then have to revise the guess. See Ex. 2 p 435

e.g.: If $g(t) = \vec{c}\, e^{-t}$, which duplicates, then, Rather than take $\vec{A} e^{-t}$ we'd have to consider

$$\vec{A}\, t e^{-t} + \vec{B} e^{-t} \quad !!$$

② Undet'd coeffs, as usual applies only to simple inputs such as polynom., exp, sine/cosine, or their products.

③ for constant inputs, things are easy! We'll restrict our goals to those simplest cases.