Nov 8, 2010

Sys. of ODEs (some review and some new stuff) $x(t), y(t)$

$$\begin{cases} \dfrac{dx}{dt} = a_{11} x + a_{12} y \\ \dfrac{dy}{dt} = a_{21} x + a_{22} y \end{cases}$$

① showed that sys can be reduced to : 2nd order lin.

$$\frac{d^2y}{dt^2} - \beta \frac{dy}{dt} + \gamma y = 0 \qquad \beta = a_{11} + a_{22}$$
$$\gamma = a_{11} a_{22} - a_{12} a_{21}$$

$$y(t) = C e^{rt} \qquad r = \frac{\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}$$

$$\Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C \begin{pmatrix} \frac{r - a_{22}}{a_{21}} \\ 1 \end{pmatrix} e^{rt}$$

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② Lin. Alg. $\qquad \frac{d}{dt} \vec{X}(t) = M \cdot \vec{X}(t)$

$$\vec{X} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \qquad M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\beta = Tr(M) = a_{11} + a_{22}$$
$$\gamma = \det(M) = a_{11} a_{22} - a_{12} a_{21}$$

suppose; solns $\vec{X}(t) = \boxed{\vec{v}\, \boxed{e^{rt}}} \Rightarrow r\vec{v} = M \cdot \vec{v}$

r an eigenval $\leftarrow (M - rI) \cdot \vec{v} = 0$

$\vec{v}$ " eigenvect. of M.

Q: Do we always reduce the system to a single ODE?

Ans: No, not necessarily! In fact, we can use some linear algebra to solve this problem in its original form.

$$\begin{cases} \dfrac{dx}{dt} = a_{11}x + a_{12}y \\ \dfrac{dy}{dt} = a_{21}x + a_{22}y \end{cases} \rightarrow \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{d\vec{x}}{dt} = M\vec{x}$$

where $\vec{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ $\quad M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Assume solns in form $\vec{x} = \vec{V}e^{rt}$ where $\vec{V} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ are constants

Then $\dfrac{d\vec{x}}{dt} = \vec{V} r e^{rt}$ so

$$\vec{V} r e^{rt} = \underbrace{M}_{\text{matrix multiplication}} \cdot \vec{V} e^{rt}$$

2x2 matrix · 2x1 Vector

$e^{rt} \neq 0 \Rightarrow$

$$r\vec{V} = M \cdot \vec{V}$$

Remark: In the book's notation
matrix $M \rightarrow A$
Const. vector $\vec{V} \rightarrow \vec{\xi}$
See pp 390 -- B+D 9th ed (Prima)

rewrite as

$$M \cdot \vec{V} - r\vec{V} = 0$$

or as:

$$(M - rI) \cdot \vec{V} = 0$$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix

$$\begin{pmatrix} a_{11}-r & a_{12} \\ a_{21} & a_{22}-r \end{pmatrix} \cdot \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is just a system of linear algebraic equations in the "unknowns" V_1, V_2. One solution is just $V_1 = V_2 = 0$ (← the uninteresting trivial soln), which is usually unique, is the only soln!

The only way to have nontrivial solns to this algebraic system is if

$$\det (M - rI) = 0 \qquad \text{i.e. } \det \begin{pmatrix} a_{11}-r & a_{12} \\ a_{21} & a_{22}-r \end{pmatrix} = 0$$

This is true only for certain values of r, that we call <u>eigenvalues</u>

The corresponding values of $\vec{v}$ satisfy $M \cdot \vec{v} = r\vec{v}$ are called <u>eigenvectors</u>

Q: How do we find those eigenvalues?

Answ: ① eigenvalues: $\det \begin{pmatrix} a_{11}-r & a_{12} \\ a_{21} & a_{22}-r \end{pmatrix} = (a_{11}-r)(a_{22}-r) - a_{12}a_{21} = 0$

$$a_{11}a_{22} - a_{11}r - a_{22}r - r^2 - a_{12}a_{21} = 0$$

$$r^2 - \underbrace{(a_{11}+a_{22})}_{\beta}r + \underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{\gamma} = 0$$

$$r^2 - \beta r + \gamma = 0 \qquad \text{where } \beta = a_{11}+a_{22}, \quad \gamma = a_{11}a_{22} - a_{12}a_{21}$$

Note: We have arrived at the same characteristic eqn as we got earlier (when we reduced the system of ODEs to the 2nd order ODE).

Moreover, we now recognize that

$$\beta = a_{11} + a_{22} = \text{Trace of matrix } M = Tr(M)$$

(sum of diagonal elements)

$$\gamma = a_{11}a_{22} - a_{12}a_{21} = \text{determinant of matrix } M = \det(M)$$

eigenvalues are thus $r_{12} = \dfrac{+\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}$ ← $\boxed{\text{for sys of 2 ODEs we will have TWO EIGENVALUES}}$

(We again have many cases to consider for the behaviour of $e^{r_1 t}, e^{r_2 t}$).

Q : How do we find the eigenvectors?

Suppose $\vec{V} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ is an eigenvector. Then

$$r\vec{V} = M \cdot \vec{V} \quad \Rightarrow \quad \begin{pmatrix} rV_1 \\ rV_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

$$\Rightarrow \left\{ \begin{aligned} rV_1 &= a_{11}V_1 + a_{12}V_2 \\ rV_2 &= a_{21}V_1 + a_{22}V_2 \end{aligned} \right\}$$

← However, since we are solving this system when $\det(M - rI) = 0$, the two eqns are not linearly independent (i.e. they "duplicate" the information)

So take any <u>one</u> of these, e.g. 2nd eqn:

$$rV_2 = a_{21}V_1 + a_{22}V_2 \quad \text{i.e.} \rightarrow (r - a_{22})V_2 = a_{21}V_1$$

Let us (arbitrarily) set $V_2 = 1$ and find V_1. Then

$$(r - a_{22}) \cdot 1 = a_{21}V_1 \qquad V_1 = \frac{r - a_{22}}{a_{21}}$$

Thus we have found that each soln is of the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} e^{rt} = \begin{pmatrix} \frac{r - a_{22}}{a_{21}} \\ 1 \end{pmatrix} e^{rt}$$

← but there are <u>two</u> of these! One for $r = r_1$, one for $r = r_2$!

The two solns will look like:

$$\begin{pmatrix} \frac{r_1 - a_{22}}{a_{21}} \\ 1 \end{pmatrix} e^{r_1 t}, \qquad \begin{pmatrix} \frac{r_2 - a_{22}}{a_{21}} \\ 1 \end{pmatrix} e^{r_2 t}$$

corresponding to eigenvalue r_1 and to eigenvalue r_2

The general soln will be a LINEAR SUPERPOSITION OF these, i.e.

$$\boxed{\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} \frac{r_1 - a_{22}}{a_{21}} \\ 1 \end{pmatrix} e^{r_1 t} + c_2 \begin{pmatrix} \frac{r_2 - a_{22}}{a_{21}} \\ 1 \end{pmatrix} e^{r_2 t}}$$

Examples:

(1) $$\boxed{\begin{aligned} \frac{dx}{dt} &= 7x - 9y \\ \frac{dy}{dt} &= 2x - 2y \end{aligned}}$$

$$M = \begin{bmatrix} 7 & -9 \\ 2 & -2 \end{bmatrix}$$

$\beta = 7 - 2 = 5$

$\gamma = -14 + 18 = 4$

$\beta^2 - 4\gamma = 25 - 16 = 9 > 0$

eigenval (Roots) $r_{1,2} = \dfrac{5 \pm \sqrt{9}}{2} = \dfrac{5 \pm 3}{2} = 4, 1$

two real positive roots, solns: exponentially growing

eigenvectors: for $r_1 = 4$ $\quad V_1 = \begin{pmatrix} \frac{4+2}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

for $r_2 = 1$ $\quad V_2 = \begin{pmatrix} \frac{1+2}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

gen'l solns:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{4t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

suppose Initial cond's are $x(0) = 1 \quad y(0) = 0$

then

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \Rightarrow \begin{aligned} 1 &= 3c_1 + 3c_2 \\ 0 &= c_1 + 2c_2 \end{aligned}$$

$c_1 = -2c_2 \qquad c_2 = -\frac{1}{3} \quad c_1 = \frac{2}{3}$

Soln: $$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{2}{3} e^{4t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \frac{1}{3} e^{t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

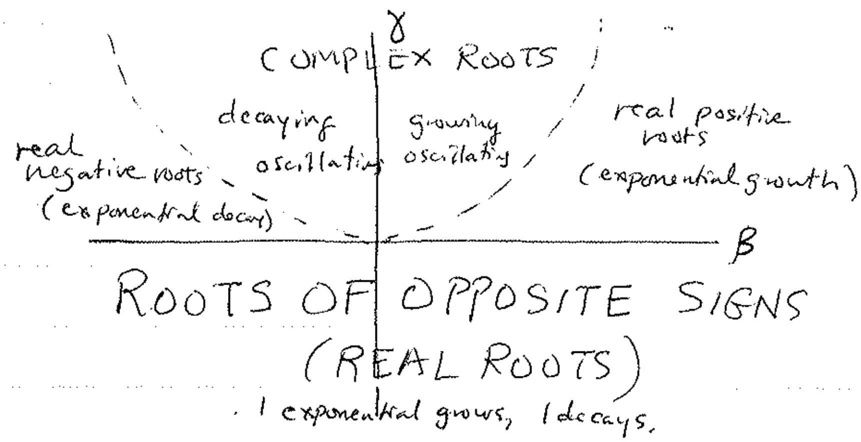
$$\begin{cases} \dfrac{dx}{dt} = a_{11} x + a_{12} y \\ \dfrac{dy}{dt} = a_{21} x + a_{22} y \end{cases}$$

$$M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$\beta = \operatorname{Tr} M = a_{11} + a_{22}$

$\gamma = \det M = a_{11} a_{22} - a_{12} a_{21}$

roots : $r_{1,2} = \dfrac{\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}$ (eigenvalues)



eigenvectors $\vec{v}_i = \begin{pmatrix} \dfrac{r_i - a_{22}}{a_{21}} \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ \dfrac{r - a_{11}}{a_{12}} \end{pmatrix}$