# Mechanical Vibrations



→ Recall, we have been studying the nonhomog. 2nd order ODE:

$$ay'' + by' + cy = g(t) \quad \text{with initial cond's: } y(0)=y_0,\ y'(0)=y_0'$$

and applications to

- spring-mass system
- LRC circuit (see HW3)
- pendulum (see MT1)

} See also Sec. 3.7 Boyce + Diprima 9th ed.

To understand its behaviour, first we consider corresponding homog. ODE

$$\boxed{ay'' + by' + cy = 0}$$

its char. eqn: $ar^2 + br + c = 0$ whose roots are: $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Vibrations (oscillations) occur when $b^2 - 4ac < 0$ : $r = \sigma \pm i\mu$

with $\sigma = \dfrac{-b}{2a}$ $\mu = \dfrac{\sqrt{4ac - b^2}}{2a}$

soln to homog eqn is

$$y(t) = e^{\sigma t}\left(c_1 \cos(\mu t) + c_2 \sin(\mu t)\right)$$

$\mu = \dfrac{\sqrt{4ac - b^2}}{2a}$ called "quasi frequency"

radians per unit time

$\dfrac{\mu}{2\pi}$ = cycles per unit time

$T = \dfrac{2\pi}{\mu}$ = quasi frequency

In an undamped system, $b \approx 0$ $r = \pm \dfrac{\sqrt{-4ac}}{2a} = \pm \sqrt{\dfrac{c}{a}}\, i = \pm \omega_0 i$

where $\omega_0$ is the (undamped) "natural frequency" of the system.

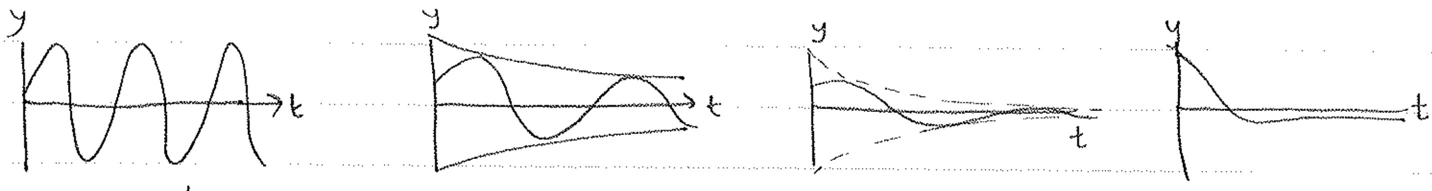
Note: the quasifreq. $\mu$ is always lower than the undamped frequency

$$\mu < \omega_0$$

Remark: you should be able to characterize the three regimes:

$$\begin{cases} \text{under damped} & b^2-4ac<0 \\ \text{critically damped} & b^2-4ac=0 \\ \text{over damped} & b^2-4ac>0 \end{cases}$$

How does increased damping affect the behavior ?




no damping — a lot of damping

——— $b$ (damping) as increases ———→

– – – quasi-frequency / $\mu$ decreases →

– – – amplitude decays more quickly ($\sigma$ larger negative value) →

we see this from the dependence of $\sigma$ and $\mu$ on $b$:

$$y(t) = e^{\sigma t}\left(c_1\cos(\mu t) + c_2\sin(\mu t)\right)$$

$$\sigma = \frac{-b}{2a} \qquad \mu = \frac{\sqrt{4ac-b^2}}{2a}$$

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Now return to the nonhomog. problem, and consider spring-mass system

$$\underset{a}{m}y'' + \underset{b}{\gamma}y' + \underset{c}{k}y = \underbrace{F\cos(\omega t)}_{g(t)}$$

Forcing function with frequency $\omega$

In each case, $y(t) =$ soln to homog problem + partic soln.

$= c_1y_1 + c_2y_2 + Y_p$

We now consider <u>THREE</u> cases

Case ① undamped $(\gamma=0)$ → (a) $\omega \neq \sqrt{\frac{k}{m}}$ ; → (b) $\omega = \sqrt{\frac{k}{m}}$

Case ② damped $(\gamma\neq 0)$

Remark: in undamped system ($\gamma=0$), $\boxed{\omega_0=\sqrt{\frac{k}{m}}}$ is the natural frequency

Case 1a (undamped, $\gamma=0$) $\omega \neq \omega_0$ ← Driving force frequency different from natural freq.

$$y(t) = \underbrace{c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)}_{\text{natural freq.}} + Y_p(t)$$

$$Y_p(t) = A\cos(\underbrace{\omega t}_{\text{forcing freq}}) + B\sin(\omega t)$$

... a bunch of algebra later (using undetermined coeffs) find

$$A = \frac{F}{k-\omega^2 m} = \frac{F}{m(\omega_0^2-\omega^2)}$$

particular soln: $$Y_p(t) = \frac{F}{m(\omega_0^2-\omega^2)}\cos(\omega t)$$

see details on Oct 4 lecture

Suppose initial conds are $y(0)=0$ $y'(0)=0$

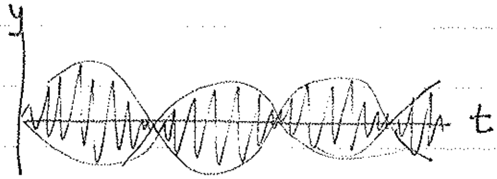
Then we find that $c_2=0$, $c_1 = \dfrac{-F}{m(\omega_0^2-\omega^2)}$

so solution is

$$y(t) = \frac{F}{m(\omega_0^2-\omega^2)}\left(\underbrace{\cos(\omega t)}_{\text{from partic. soln}} - \underbrace{\cos(\omega_0 t)}_{\text{from soln to hom. sys.}}\right)$$

natural freq. driving force freq. natural freq.

For $\omega \neq \omega_0$: In HW4, you are asked to show that this gives rise to behavior like this:




"beats"

Case ① b (undamped, $\gamma = 0$) $\omega = \omega_0$ ← driving force frequency same as natural freq.

particular soln has to be in the form

$$Y_p(t) = t\,[A\cos(\omega t) + B\sin(\omega t)]$$

↑ this factor of $t$ needed to avoid duplicating soln to hom. sys.

after a lot of algebra, find $A = 0$, $B = \dfrac{F}{2m\omega}$

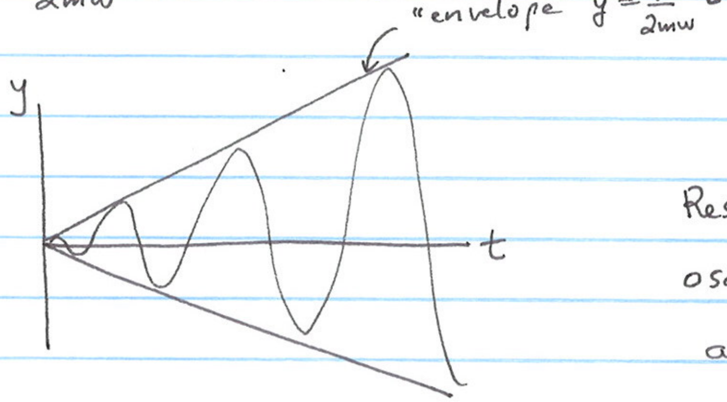
$$y(t) = c_1\cos(\omega t) + c_2\sin(\omega t) + Y_p(t)$$

$$= c_1\cos(\omega t) + c_2\sin(\omega t) + \frac{F}{2m\omega}\, t\sin(\omega t)$$

See details on Oct4 lecture

after using initial cond's $y(0)=0$, $y'(0)=0$, find $c_1 = c_2 = 0$

$$y(t) = \frac{F}{2m\omega}\, t\sin(\omega t)$$




Resonance
=
oscillations of growing amplitude.

Case (2) damped $\gamma \neq 0$ $\gamma^2 - 4mk < 0$

$$my'' + \gamma y' + ky = F\cos(\omega t)$$

We already know soln to hom eqn

$$y(t) = e^{\sigma t}\left(c_1 \cos(\mu t) + c_2 \sin(\mu t)\right)$$

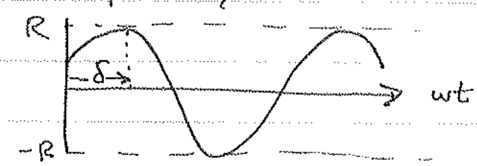
$\sigma = -\dfrac{\gamma}{2m}$ $\mu =$ quasi freq $= \dfrac{\sqrt{4mk - b^2}}{2a}$

partic. soln : $Y_p(t) = A\cos(\omega t) + B\sin(\omega t)$ ← form is ok, as no duplication of the hom. eqn solns (even if $\omega = \mu$) due to $e^{\sigma t}$ term.

could solve for $A, B$ as before. However, to get amplitude of soln, ~~A~~ "more convenient form" for $Y_p(t)$, (using a trig identity) is

$$Y_p(t) = R\cos(\omega t - \delta)$$

$R =$ amplitude, $\delta =$ shift




In HW4, you are asked to show that this form of the particular soln leads to

$$R = \frac{F}{\sqrt{\omega^2\gamma^2 - m^2(\omega_0^2 - \omega^2)^2}} \quad \text{where } \omega_0^2 = \frac{k}{m}$$

(this is how the amplitude of partic. soln depends on the forcing frequency)

See also plot of $\dfrac{R}{(F/k)}$ vs $\omega$ in book

$$\frac{R}{F/k} = \frac{1}{\sqrt{\frac{\omega^2}{\omega_0^2}\Gamma + \left(1 - \frac{\omega^2}{\omega_0^2}\right)^2}}$$

where $\Gamma = \dfrac{\gamma^2}{mk}$

Full solution to damped forced spring-mass sys.

$$y(t) = \underbrace{e^{\sigma t}\left(c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)\right)}_{\text{decaying oscillations of unforced system}} + \underbrace{R\cos(\omega t - \delta)}_{\text{forced system response}}$$

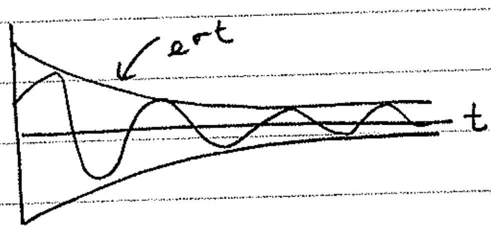
$\omega_0 = \frac{1}{2m}\sqrt{\gamma^2 - 2mk}$

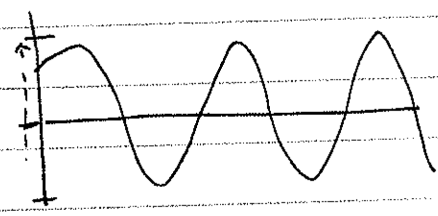
$\sigma = -\frac{b}{2a} < 0$ (in general)

$= -\frac{\gamma}{2m} < 0$ (for spring-mass)

amplitude $R$
frequency $\omega$
shift $\delta$

"transient solution"




After some time, this transient will go away and we'll be left with this

Q: How does the amplitude (R) of the forced oscillation depend on the forcing frequency?



Details are part of HW4