Using Initial Conditions to solve 2nd order, Lin. ODE

Example for Sept 20

Solve $\underbrace{2y'' + y' - y = 0}_{\text{2nd order linear ODE}}$ $\qquad \underbrace{y(0) = 3 \quad y'(0) = -1}_{\text{initial conditions (I.C.'s)}}$

Look for solutions of form $y(t) = e^{rt}$. Plug y and its derivatives into the ODE, cancel common factor of e^{rt} to get

$$2r^2 + r - 1 = 0 \qquad \text{characteristic eqn.}$$

$$r = \frac{-1 \pm \sqrt{1 + 4 \cdot 2}}{2 \cdot 2} = \frac{-1 \pm \sqrt{9}}{4} = \frac{-1 \pm 3}{4} = -1, \tfrac{1}{2}$$

Solutions :

$$f_1(t) = e^{-t}, \quad f_2(t) = e^{\frac{1}{2}t}$$

so $\boxed{y(t) = c_1 e^{-t} + c_2 e^{\frac{1}{2}t}}$ ← the general solution (includes two arbitrary constants, c_1, c_2)

Now use I.C's to find c_1, c_2

$y(0) = 3 \implies 3 = c_1 e^0 + c_2 e^0 = c_1 + c_2 \qquad (1)$

$y'(0) = 1 \implies$ note $y'(t) = -c_1 e^{-t} + \frac{1}{2} c_2 e^{\frac{1}{2}t}$

$\implies 1 = y'(0) = -c_1 \cdot e^0 + \frac{1}{2} c_2 e^0$

$= -c_1 + \frac{1}{2} c_2 \qquad (2)$

We have two algebraic eqns for the two constants:

$$\left.\begin{array}{ll} (1) & c_1 + c_2 = 3 \\ (2) & -c_1 + \frac{1}{2} c_2 = -1 \end{array}\right\} \text{solve for } c_1, c_2$$

(1)+(2) : $\frac{3}{2} c_2 = 2 \implies c_2 = \frac{4}{3}$

(1) : $c_1 = 3 - c_2 = 3 - \frac{4}{3} = \frac{5}{3}$

So the solution we want is $\boxed{y(t) = \frac{5}{3} e^{-t} + \frac{4}{3} e^{\frac{1}{2}t}}$