

Problem 1: In each case, solve for $y(t)$:

- (a) $y' + 3y = 2e^{t/2}$ with $y(0) = 1$.
- (b) $y' - 4y = t$ with $y(1) = 0$.
- (c) $ty' + 2y = \sin t$ with $y(\pi/2) = 0$.
- (d) $t^2y' + 2ty = t^3 + 1$ with $y(1) = 1$.

Solution to Problem 1:

- (a) The integrating factor is $\mu(t) = e^{3t}$. The solution is $y(t) = \frac{4}{7}e^{7t/2} + Ce^{-3t}$. Using the initial condition, the constant is found and the solution is $y(t) = \frac{4}{7}e^{7t/2} + \frac{3}{7}e^{-3t}$.
- (b) The integrating factor is $\mu(t) = e^{-4t}$. Integration by parts is needed to find that the solution is $y(t) = -\frac{1}{16} - \frac{1}{4} + Ce^{4t}$. Using the initial condition, we find the constant. The result is

$$y(t) = -\frac{1}{16} - \frac{1}{4} + \frac{5}{16e^4}e^{4t}$$

- (c) The integrating factor is $\mu(t) = t^2$ (**typo corrected**). Integration by parts is needed to find that the solution is

$$y(t) = \frac{\sin(t)}{t^2} - \frac{\cos(t)}{t} + \frac{C}{t^2}$$

The constant is found to be $C = -1$ (**typo corrected**).

- (d) The LHS is actually simply $d(t^2y(t))/dt$ so we do not need an integrating factor. We can simply integrate both sides. We get $y(t) = \frac{1}{t} + \frac{t^2}{4} + C\frac{1}{t^2}$. Using the initial condition, we find that the constant is $C = -1/4$.

Problem 2: Draw the direction fields for each of the below and sketch the solution curve corresponding to the indicated initial condition.

- (a) $y' = y - 2\cos(t) + 1$, $y(0) = 0$
- (b) $y' = \sqrt{y} - ty$, $y(0) = 2$

Solution to Problem 2: For part (a) we note that the directions of tangent lines to solution curves will all be flat (have slope zero) whenever $0 = y - 2\cos(t) + 1$. We can sketch this set of points. It may be easiest to first rewrite it as $y = 2\cos(t) - 1$, as we see that it is a cosine of amplitude 2 shifted down the y axis by 1 (dotted line in Fig 1(a)). All along this locus, we sketch horizontal “dashes”. This locus separates portion of the yt plane for which solutions are increasing (i.e. where $y'(t) > 0$) from the portion

where solutions are decreasing ($y'(t) < 0$). We can go further by noting that the slope of tangent lines will have a value M all along the locus $M = y - 2 \cos(t) + 1$. We can similarly sketch such sets of points ($y = 2 \cos(t) + M - 1$) and assign to them directions of slope M for various values of M to get a fairly good picture. The (computer generated) direction field, together with one solution curve satisfying the initial condition is shown in Fig 1(a).

For part (b) the equation can be written as $y' = \sqrt{y} - ty = \sqrt{y}(1 - t\sqrt{y})$. First we see that y always has to be positive for real values, so only the first two quadrants are relevant. If $t < 0$ then $(1 - t\sqrt{y})$ is positive. This means arrows have positive slope in the 2nd quadrant. On the curve $(1 - t\sqrt{y}) = 0$ the slopes are zero (dotted line in Fig 1(b)), and this curve separates the 1st quadrant into a portion with directions of positive slope and a portion with directions of negative slope. Another way to express the same curve is $t = y^{-1/2}$. It has as asymptotes the y axis and the t axis. A table with a few other values can be used to fill in the picture. The (computer generated) direction field, together with one solution curve satisfying the initial condition is shown in 1(b).

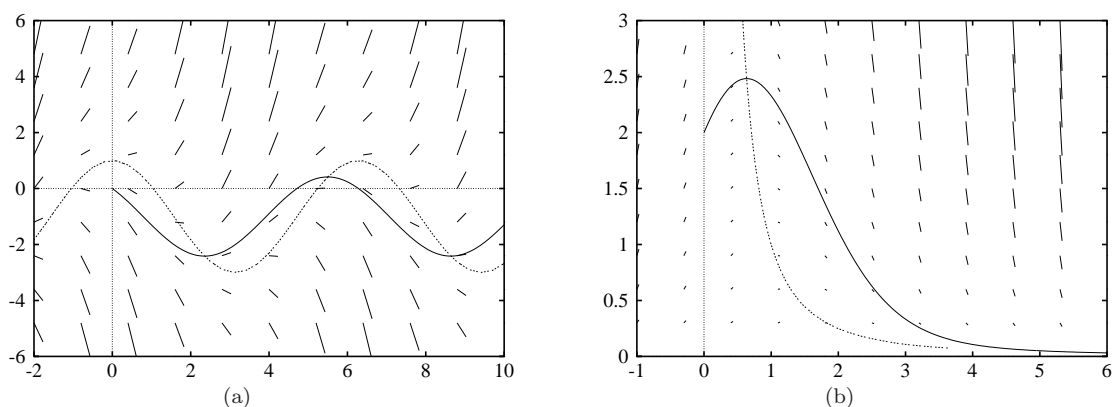


Figure 1: Solutions to parts (a) and (b) of Problem 2

Problem 3: The following equation, called the **logistic equation**, is often used to describe the growth of a population $N(t) \geq 0$ with limited resources:

$$\frac{dN}{dt} = rN \frac{(K - N)}{K}.$$

Here $r, K > 0$ are constants (called the intrinsic growth rate and the carrying capacity, respectively).

- Define a new variable, $y(t) = N(t)/K$ and rewrite the logistic equation in terms of this new variable.
- Sketch a direction field for $t \geq 0, y \geq 0$ and include on it the solution curves for initial condition $y(0) = 0, y(0) = 0.2, y(0) = 0.8, y(0) = 1.6$. What happens as $t \rightarrow \infty$? Interpret in terms of the growing population (i.e. in terms of the original variable $N(t)$.)
- Find a mathematical argument that supports the following statement: “Only solutions curves with $y(0) \leq 1/2$ have an inflection point.”

Solution to Problem 3:

- (a) Divide both sides of the equation by K , and write

$$\frac{1}{K} \frac{dN}{dt} = \frac{d(N/K)}{dt} = r \frac{N}{K} (1 - N/K).$$

Now sub in $y(t) = N(t)/K$ to get

$$\frac{dy}{dt} = ry(1 - y)$$

- (b) See Fig 2.

- (c) The ODE can be written as $y'(t) = ry(1 - y) = ry - ry^2$. To understand where there are inflection points on the solution curves $y(t)$, we have to determine when the second derivative $y''(t)$ changes sign. (At such points the **concavity** of the curve $y(t)$ changes from up to down or vice versa. NOTE: $y''(t) = 0$ is necessary, but not sufficient to guarantee an inflection point. For example, the function $y(t) = t^4$ satisfies $y''(0) = 0$ but is always concave up, i.e. has no inflection points anywhere, not even at $t = 0$.) Taking a derivative of each side of the ODE **with respect to time** gives $y''(t) = r(y'(t)) - 2ry(y'(t)) = ry'(t)(1 - 2y)$. (**Avoid the common error of forgetting the $y'(t)$ here - it comes from the Chain Rule of calculus.**) Using the ODE we can replace $y'(t)$ by $ry(1 - y)$ to obtain $y''(t) = r[ry(1 - y)](1 - 2y)$. This means that we should examine the possibilities of inflection points at $y = 0, y = 1, y = 1/2$ as all these are values at which the expression $y(1 - y)(1 - 2y)$ changes sign. But from the direction field of part (b) we know that any solution that starts at $y = 0$ never changes, and the same for $y = 1$, so those values are constant, and have no curvature nor concavity, and thus no inflection point. The only remaining possibility is that solution curves have an inflection point when they go through the value $y = 1/2$. From the direction field, we know that solutions that start in $0 \leq y(0) \leq 1$ are increasing towards $y = 1$. Thus solution curves that start at $y(0) < 1/2$ will increase, cross $y = 1/2$ and they all have an inflection point as they do so.

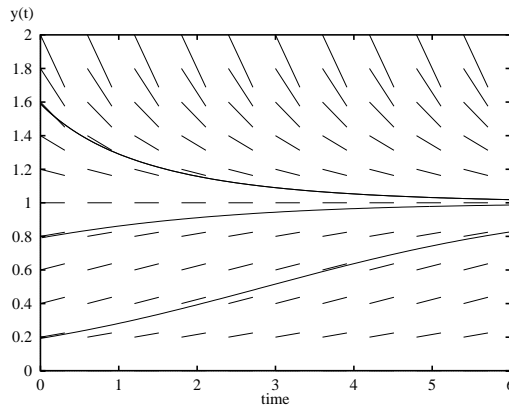


Figure 2: Solution to problem 3b

Problem 4: According to Torricelli's Law, the height of fluid in a container above a hole (through which the fluid is escaping) is governed by a differential equation:

$$\frac{dh}{dt} = -k\sqrt{h}.$$

where $k \geq 0$ is a constant. Suppose the height of the fluid is initially $h(0) = h_0$. How long does it take for the fluid to drain to the level of the hole?

Solution to Problem 4: Solve the DE using separation of variables

$$\begin{aligned} \frac{dh}{\sqrt{h}} &= -k dt \\ \int_{h_0}^{h(T)} h^{-1/2} dh &= \frac{h^{1/2}}{(1/2)} \Big|_{h_0}^{h(T)} = -k \int_0^T dt = -kT \\ 2 \left(\sqrt{h(T)} - \sqrt{h_0} \right) &= -kT \\ h(T) &= \left(\sqrt{h_0} - k \frac{T}{2} \right)^2. \end{aligned}$$

The tank will be empty when $h(t) = 0$, i.e. when $\left(\sqrt{h_0} - k \frac{t}{2} \right)^2 = 0$. Solving this equation for the time t , we get

$$k \frac{t}{2} = \sqrt{h_0}, \quad \Rightarrow \quad t = \frac{2\sqrt{h_0}}{k}.$$

Problem 5: Set up the following two problems as initial value problems (i.e. differential equation and initial condition) and solve each one.

- (a) In the LR circuit shown in Fig 3(a), $R, L, V \geq 0$ are constant. Find the current $i(t)$ given that $i(0) = 0$.
- (b) In the RC circuit shown in Fig 3(b), $R, C \geq 0$ are constant and the voltage is time dependent, $V(t) = e^{-t}$. Find the charge on the capacitor $q(t)$ given that $q(0) = 0$.

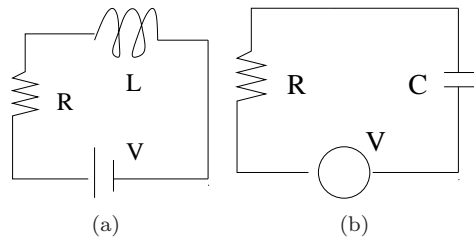


Figure 3: For problem 5 (a) and 5 (b)

Solution to Problem 5 We will use the circuit equation $L \frac{di}{dt} + Ri(t) + \frac{q(t)}{C} = V(t)$ for both parts, but some circuit elements are missing in each part of the question.

- (a) The ODE for circuit (a) is $L\frac{di}{dt} + Ri(t) = V = \text{constant}$. We write it in the standard form $\frac{di}{dt} + \frac{R}{L}i(t) = \frac{V}{L}$. We find that the integrating factor is $\mu(t) = e^{Rt/L}$ and we get the solution $i(t) = \frac{V}{R} + Ce^{-Rt/L}$. Using the initial condition $i(0) = 0$ we get $i(t) = \frac{V}{R}(1 - e^{-Rt/L})$.
- (b) There is a resistor and capacitor; recall that $i(t) = dq/dt$ so, the equation we want is $R\frac{dq}{dt} + \frac{1}{C}q(t) = V(t)$. We are also told that $V(t) = e^{-t}$. In standard form the ODE is $\frac{dq}{dt} + \frac{1}{RC}q(t) = \frac{e^{-t}}{R}$. The integrating factor is $\mu(t) = e^{t/RC}$ and the solution works out to $q(t) = \frac{C}{1-RC}e^{-t} + Ke^{-t/RC}$ where K is an arbitrary constant. Using the initial condition leads to the final answer $q(t) = \frac{C}{1-RC}[e^{-t} - e^{-t/RC}]$.

Problem 6: Consider a stirred tank reactor that initially contains a volume $V(0) = V_0$ of water. Now suppose that a stock solution of salt (at concentration S gm/Litre) is pumped in at rate $F_{in} = F$ Litres/hr and the well-stirred mixture is pumped out at a slightly faster rate, $F_{out} = (F + f)$ Litres/hr where $f > 0$. [Note that the volume of the fluid will not be constant.] Let $C(t)$ denote the concentration of salt inside the tank.

- (a) Set up this problem as a differential equation problem for the volume of fluid $V(t)$ and the salt concentration $C(t)$.
- (b) Determine the volume of fluid $V(t)$ for $t > 0$. Over what period of time $0 \leq t \leq T$ is this result valid?
- (c) Use your result in (b) to set up one ODE that depends only on the variable $C(t)$ and solve that equation.

Solution to Problem 6

- (a) The volume equation: $dV/dt = F_{in} - F_{out} = -f$. (Note that this tells us the volume is decreasing, as $f > 0$. For the salt equation, we still need to use conservation of mass, but note that the volume is not constant $d[V(t)C(t)]/dt = SF_{in} - C(t)F_{out}$.)
- (b) From (a) $V(t) = -ft + V_0$, but this is reasonable only for $t \leq V_0/f$ as later on, the tank has drained. (A negative volume is meaningless here!)
- (c) From (a,b) (b) we have (using the product rule) $d[V(t)C(t)]/dt = V(t)dC/dt + C(t)dV/dt = SF_{in} - C(t)F_{out} = SF - C(t)(F + f)$. We know both $V(t)$ and dV/dt from (a). Substitute in dV/dt to get $-fC(t) + V(t)dC/dt = SF - FC(t) - fC(t)$. Cancel common terms: $V(t)dC/dt = SF - FC(t)$. Divide through by $V(t) = V_0 - ft$ and rearrange terms. The equation we get (in standard form) is:

$$\frac{dC}{dt} + C\frac{F}{V_0 - ft} = \frac{SF}{V_0 - ft}.$$

We find that the integrating factor is $\mu(t) = \exp\left[\int \frac{F}{V_0 - ft}\right]$ The integral is done by substitution:

$$u = V_0 - ft, \quad du = -f dt, \quad \text{so } \mu(t) = \exp[-(F/f)\ln(V_0 - ft)] = \exp[\ln(V_0 - ft)^{-F/f}] = \frac{1}{(V_0 - ft)^{F/f}}.$$

[Remark: I will not usually show such details, and it is up to you to review integration and calculus techniques required to handle these old technical steps]. Thus, the general solution is $C(t) = S + K(V_0 - ft)^{F/f}$ where K is an arbitrary constant. Using the initial condition that $C(0) = 0$ we get

$$C(t) = S \left[1 - \left(\frac{(V_0 - ft)}{V_0} \right)^{F/f} \right]$$

This solution is valid only as long as $0 \leq t \leq V_0/f$.