HOMEWORK 5: MATH 265 Due in class on Oct 27

Problem 1: Improper integrals.

(a) Consider the integral $I = \int_1^\infty \frac{1}{t^p}\,dt$. Show that this improper integral converges for $p > 1$ and find its value. Show that it diverges for $p = 1$. What happens when $p < 1$?

(b) Consider the integral $I = \int_0^\infty e^{-(5-k)t}\,dt$. Does this integral converge? To what value and under what condition(s) if any.

(c) Consider the integral $I = \int_0^\infty \frac{x}{10^5}\,dx$. Determine whether this integral converges or diverges and explain why.

Solution to Problem 1:

(a) Provided $p \neq 1$ we can write $I = \displaystyle\int_1^\infty \frac{1}{t^p}\,dt = \int_1^\infty t^{-p}\,dt = \frac{1}{-p+1}t^{(-p+1)}\Big|_1^\infty = \frac{1}{1-p}[\lim_{t\to\infty} t^{(1-p)} - 1]$.

If $p > 1$ then the limit in the above expression is 0 and we get $I = \dfrac{1}{1-p}[0-1] = \dfrac{1}{p-1}$. If $p < 1$ then the power of $t^(1-p)$ is positive, and hence the limit $\lim_{t\to\infty} t^{(1-p)} = \infty$ so the integral diverges.

If $p = 1$ then the integral is different, indeed $I = \displaystyle\int_1^\infty \frac{1}{t}\,dt = \ln(t)\Big|_1^\infty$. As discussed in class, this integral diverges since $\ln(t) \to \infty$ as $t \to \infty$.

(b) $I = \displaystyle\int_0^\infty e^{-(5-k)t}\,dt = \frac{1}{-(5-k)}e^{-(5-k)t}\Big|_0^\infty = \frac{1}{-(5-k)}\left[\lim_{t\to\infty} e^{-(5-k)t} - 1\right]$. The limit in the expression exists only if the exponent is negative, i.e. if $5 - k > 0$ namely when $k < 5$. In that case $I = \frac{-1}{-(5-k)} = \frac{1}{(5-k)}$

(c) Consider the integral $I = \int_0^\infty \frac{x}{10^5}\,dx = \frac{1}{10^5}\int_0^\infty x\,dx$. This integral cannot possibly converge since the integrand is an increasing function. Only improper integrals of functions that decrease to zero have any hope of converging.

Problem 2: A *Comparison Theorem* states the following two facts:

(1) If $0 \leq g(x) \leq f(x)$ and $I_1 = \int_0^\infty f(x)\,dx$ converges, then $I_2 = \int_0^\infty g(x)\,dx$ also converges.

(2) If $0 \leq g(x) \leq f(x)$ and $I_2 = \int_0^\infty g(x)\,dx$ diverges, then $I_1 = \int_0^\infty f(x)\,dx$ also diverges.

Use these theorems, together with the results of Problem 1 to determine which of the following integrals converges and briefly explain your answer.

(a) $I = \int_1^\infty \frac{1}{1+x^3}\,dx$

(b) $I = \int_1^\infty \frac{1}{\sqrt{x-0.5}}\,dx$

Solution to Problem 2:

(a) $I = \int_1^\infty \frac{1}{x^3}\,dx$ converges and $\frac{1}{1+x^3} < \frac{1}{x^3}$ so thus by the first part of the comparison theorem, $I = \int_1^\infty \frac{1}{1+x^3}\,dx$ converges.

(b) By problem 1 part (1) we know that $I = \int_1^\infty \frac{1}{t^p}\,dt$ diverges for $p < 1$, and in particular for $p = 1/2$, i.e. the integral $I = \int_1^\infty \frac{1}{\sqrt{x}}\,dx = \int_1^\infty \frac{1}{x^{1/2}}\,dx$ diverges. But $\frac{1}{\sqrt{x-0.5}} > \frac{1}{\sqrt{x}}$. Thus by the second part of the comparison theorem, both integrals diverge.

Problem 3: Show the detailed steps in computing the Laplace transform of the following functions. (You can check your answer using a table of Laplace transforms, but you are required to actually do the integration to verify your answers.)

(a) $f(t) = Kt$ where K is a constant.

(b) $f(t) = e^{-t/\tau}$ where τ is a constant.

(c) $f(t) = \sin(at)$ and $f(t) = \cos(at)$ (Note: you will find that this double integration by parts will allow you to find both of these "together".)

The solutions to this and other problems are on an accompanying handwritten scan.

Problem 4: Suppose that $F(s) = \mathcal{L}\{f(t)\}$ is the Laplace transform of $f(t)$. In class we showed that

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

(a) Show that $\mathcal{L}\{f''(t)\} = s^2F(s) - sf'(0) - f(0)$.

(b) Let $f^{(n)}(t)$ be the nth derivative of the function $f(t)$ and let $M_n = \mathcal{L}\{f^{(n)}(t)\}$ be its Laplace transform. Show that $M_n = sM_{n-1} - f^{(n-1)}(0)$.

(c) **Bonus**: Use the above *recursion relation* between M_n and M_{n-1} to arrive at the general "formula" for $\mathcal{L}\{f^{(n)}(t)\}$.

Problem 5:

(a) Find the Laplace transform for the following (discontinuous) function:

$$f(t) = \begin{cases} 0, & 0 \le t < 3 \\ 2, & t \ge 3 \end{cases}$$

(b) Prove the following theorem about shift (translation) of the Laplace transform:

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a).$$

(c) Use your result in part (b) to find the inverse Laplace transform for the $F(s) = \dfrac{1}{6(s-1)^3}$.

Problem 6: Find the solution to the following ODEs using the Laplace transform

(a) $y'' - 3y' + 2y = 12e^{4t}$ with $y(0) = 1, y'(0) = 0$.

(b) $y'' + y' - 2y = 4e^t + 1$ with $y(0) = 1, y'(0) = 0$. (Note: this is the example we discussed in class where there is a "hard way" to do the algebra and a somewhat easier way.)

(c) $y'' + 4y' - 5y = te^t$ with $y(0) = 1, y'(0) = 0$. (Hint: the shift theorem will be useful. There will be some algebra in this one.)