Math 201 Lab Questions

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Lab 1: Textbook Sections 2.2, 2.6

Question 1.1. *Find the general solution for the differential equation*

$$\frac{dy}{dx} = 1 + \frac{1}{y^2} \tag{1}$$

Solution. Rearranging the equation, we get

$$-1 + \frac{1}{1 + \frac{1}{y^2}} \frac{dy}{dx} = 0$$

Which shows that equation (1) is separable. Integrating,

$$\int M dx = \int -1 dx = -x$$

and

$$\begin{aligned} \int N dx &= \int \frac{1}{1 + \frac{1}{y^2}} \frac{dy}{dx} dy \\ &= \int \frac{y^2}{y^2 + 1} dy \\ &= \int \frac{1 - 1 + y^2}{y^2 + 1} dy \\ &= \int 1 - \frac{1}{y^2 + 1} dy \\ &= y - \arctan(y) \end{aligned}$$

So the general solution for equation (1) is

$$y(x) = \arctan(y) + x + C$$

□

Question 1.2. *Find the general solution for the differential equation*

$$2x^2y^4 + 3xy^3 + (x^3y^3 - xy^3)\frac{dy}{dx} = 0 \tag{2}$$

Solution. Observe that we can make this equation simpler by factoring out xy^3. This shows that $y(x) = 0$ is clearly a solution, and that if we assume that $y(x) \neq 0$, other solutions to this problem are identically solutions to

$$2xy + 3 + (x^2 - 1)\frac{dy}{dx} = 0 \tag{3}$$

In this case,

$$\frac{\partial}{\partial y}M = 2x, \qquad \frac{\partial}{\partial x}N = 2x$$

so this equation is exact. Let

$$F(x, y) = \int M(x, y)dx = \int 2xy + 3dx = x^2y + 3x + g(y)$$

and now we find $g(y)$:

$$\begin{aligned} \frac{\partial}{\partial y}F &= N \\ \frac{\partial}{\partial y}\left(x^2y + 3x + g(y)\right) &= x^2 - 1 \\ \frac{\partial}{\partial y}\left(x^2y + g(y)\right) &= x^2 - 1 \\ x^2y + g(y) &= x^2y - y \\ g(y) &= -y \end{aligned}$$

So the non-zero solutions to this equation are given by

$$C = x^2y + 3x - y$$

which we can actually rearrange to solve for y (but this wasn't required for marks). With that, the general solution to equation (2) is

$$y(x) = 0, \qquad y(x) = \frac{C - 3x}{x^2 - 1}$$

□

Remark 1.3. *In Question 1.2, there was not really a way to know that factoring out xy^3 would lead to an exact equation. I just tried to make the equation simpler by factoring, and in this case was left with an exact equation. The way to do this in general is to find the integrating factor. It works out the same way, just takes a bit more effort. You'll find that the methods with $Z(y)$ and $Z(x)$ do not work, so the only thing we know to do so far is guess that the integrating factor is $\mu(x, y) = x^ay^b$. Once multiplied by this μ, the equation is of the form*

$$2x^{a+2}y^{b+4} + 3x^{a+1}y^{b+3} + \left(x^{a+3}y^{b+3} - x^{a+1}y^{b+3}\right)\frac{dy}{dx} = 0$$

Taking the partial derivatives gives

$$\frac{\partial}{\partial y}\mu M = 2(b+4)x^{a+2}y^{b+3} + 3(b+3)x^{a+1}y^{b+2}, \qquad \frac{\partial}{\partial x}\mu N = (a+3)x^{a+2}y^{b+3} - (a+1)x^ay^{b+3}$$

and equating them results in $b + 3 = 0$ and $a + 1 = 0$. Thus,

$$a = -1 \qquad b = -3$$

Once you multiply equation (2) by $\mu(x, y) = x^{-1}y^{-3}$, you just get equation (3) again, and from there the question is solved in the same way.

Lab 2: Textbook Sections 3.1, 3.2, 3.3

Question 2.1. *Solve the initial value problem*

$$y'' + 2y' + 4y = 0 \tag{4}$$

with initial conditions $y(0) = 1$ *and* $y'(0) = 2$.

Solution. Note the characteristic polynomial is

$$r^2 + 2r + 4$$

which has roots $r = -1 \pm i\sqrt{3}$. Thus the general solution is

$$y(x) = c_1 e^{-t} \sin(\sqrt{3}t) + c_2 e^{-t} \cos(\sqrt{3}t)$$

Solving for the constants, we get

$$1 = y(0) = c_1 e^0 \sin(0) + c_2 e^0 \cos(0) = c_2$$

and

$$2 = y'(0) = -c_1 e^0 \sin(0) + \sqrt{3} c_1 e^0 \cos(0) - \sqrt{3} c_2 e^0 \sin(0) - c_2 e^0 \cos(0) = \sqrt{3} c_1 - c_2 = \sqrt{3} c_1 - 1$$

Thus the solution to equation (4) is

$$y(x) = \sqrt{3} e^{-t} \sin(\sqrt{3}t) + e^{-t} \cos(\sqrt{3}t)$$

□

Question 2.2. *Solve the initial value problem*

$$y'' + 4y' + 5y = 0 \tag{5}$$

with initial conditions $y(\pi) = e^{-\pi}$ *and* $y'(0) = \sqrt{\pi} + 2e^{\pi}$.

Solution. Note the characteristic polynomial is

$$r^2 + 4r + 5 = 0$$

which has roots $r = -2 \pm i$. Thus the general solution is

$$y(x) = c_1 e^{-2t} \sin(x) + c_2 e^{-2t} \cos(x)$$

Solving for the constants, we get

$$e^{-\pi} = y(\pi) = c_1 e^{-2\pi} \sin(\pi) + c_2 e^{-2\pi} \cos(\pi) = -c_2 e^{-2\pi}$$

so $c_2 = -e^{\pi}$, and

$$\sqrt{\pi} + 2e^{\pi} = y'(0) = -2c_1 e^0 \sin(0) + c_1 e^0 \cos(0) - c_2 e^0 \sin(0) - 2c_2 e^0 \cos(0) = c_1 - 2c_2 = c_1 + 2e^{\pi}$$

Thus the solution to equation (5) is

$$y(x) = \sqrt{\pi} e^{-2t} \sin(x) - e^{\pi} e^{-2t} \cos(x)$$

□

Lab 3: Textbook Sections 3.4, 3.5

Question 3.1. *Find the general solution to the differential equation*

$$4y'' - 4y' + y = 16t^2 e^{t/2} \tag{6}$$

Solution. First solve the homogeneous equation. The characteristic equation is

$$4r^2 - 4r + 1 = (2r - 1)(2r - 1)$$

so $r = \frac{1}{2}$, and the two solutions of the homogeneous equation are $y_1 = e^{t/2}$ and $y_2 = te^{t/2}$. Next we look to the table for the correct way to choose a particular solution, and find that we ought to choose

$$y_p = \left(C_2t^2 + C_1t + C_0\right) e^{t/2}$$

however, this will not work since the C_0 term is a linear combination of y_1 and y_2. So, we try multiplying by t, but that doesn't work either in this case! We need to multiply by t twice to escape combinations of y_1 and y_2, so the particular solution in the end is

$$y_p = t^2 \left(C_2t^2 + C_1t + C_0\right) e^{t/2} = \left(C_2t^4 + C_1t^3 + C_0t^2\right) e^{t/2}$$

finding the derivatives:

$$\begin{aligned}
y_p' &= e^{t/2}\left(4\ell_2t^3 + 3\ell_1t^2 + 2\ell_0t\right) + \frac{1}{2}e^{t/2}\left(\ell_2t^4 + \ell_1t^3 + \ell_0t^2\right) \\
&= \frac{1}{2}e^{t/2}\left(\ell_2t^4 + (8\ell_2 + \ell_1)t^3 + (6\ell_1 + \ell_0)t^2 + \ell_0t\right) \\
y_p'' &= \frac{1}{2}e^{t/2}\left(4\ell_2t^3 + 3(\ell_1 + 8\ell_2)t^2 + 2(\ell_0 + 6\ell_1)t + \ell_0\right) + \frac{1}{4}e^{t/2}\left(\ell_2t^4 + (\ell_1 + 8\ell_2)t^3 + (\ell_0 + 6\ell_1)t^2 + \ell_0t\right) \\
&= \frac{1}{4}e^{t/2}\left(C_2t^4 + (16C_2 + C_1)\,t^3 + (48C_2 + 12C_1 + C_0)\,t^2 + (24C_1 + 5C_0)\,t + 2C_0\right)
\end{aligned}$$

Putting these into the equation, we get the following relations by combining terms with the same powers of t:

$$\begin{aligned}
C_2 - 2C_2 + C_2 &= 0 : t^4 \text{ terms} \\
16C_2 + C_1 - 16C_2 - 2C_1 + C_1 &= 0 : t^3 \text{ terms} \\
48C_2 + 12C_1 + C_0 - 12C_1 - 2C_0 + C_0 &= 16 : t^2 \text{ terms} \\
24C_1 + 5C_0 - 2C_0 &= 0 : t \text{ terms} \\
2C_0 &= 0 : \text{constant terms}
\end{aligned}$$

Interestingly the first two equations cancel out entirely and don't give any information. The last equation tells us that $C_0 = 0$, and then the second last tells us that $C_1 = 0$. The middle equation then shows that $C_2 = \frac{1}{3}$, and so we have found the particular equation:

$$y_p(t) = \frac{1}{3}e^{t/2}t^4$$

So the general solution is

$$y(t) = c_2e^{t/2}t + c_1e^{t/2} + \frac{1}{3}e^{t/2}t^4$$

□

Remark 3.2. *In question 3.1 we saw that the first two guesses y_p and ty_p had terms equal to $C_0 y_1$ and $C_0 y_2$ which is an interesting property. The equation shown here can be solved with general constants as well! If we are looking at an equation of the form*

$$y'' - 2ay' + a^2 y = bt^2 e^{at} \tag{7}$$

then the characteristic equation is $(r-a)(r-a)$, so finding the general solution we get $y_1 = e^{at}$, $y_2 = te^{at}$, and $y_p = (C_2 t^2 + C_1 t + C_0)t^2 e^{at}$. Solving for these coefficients is not any more difficult than solving for the coefficients in question 3.1, and we eventually get to the solution

$$y(t) = \frac{b}{12} t^4 e^{at} + c_2 t e^{at} + c_1 e^{at}$$

and note that in question 3.1, $a = \frac{1}{2}$ and $b = 4$. The next question asks you to solve a different sort of differential equation we saw in a general form.

Question 3.3. *Given a constant $\alpha \in \mathbb{R}$, find the general solution of*

$$t^2 y'' + \alpha t y' + \frac{(\alpha - 1)^2}{4} y = 0, \qquad t > 0 \tag{8}$$

given that one of the solutions is $t^{(1-\alpha)/2}$.

Solution. We did an example similar to this in the lab, although the constants there were fixed values. Let's try the same method. For this, we suppose that

$$y_2(t) = v(t) y_1(t) = v(t) t^{(1-\alpha)/2}$$

and try to find a proper function for $v(t)$. First, we must find the derivatives:

$$\begin{aligned}
y_2'(t) &= v'(t) t^{(1-\alpha)/2} + \frac{1-\alpha}{2} v(t) t^{(-1-\alpha)/2} \\
y_2''(t) &= t^{(1-\alpha)/2} v''(t) + \frac{1-\alpha}{2} t^{(-1-\alpha)/2} v'(t) + \frac{1-\alpha}{2} t^{(-1-\alpha)/2} v'(t) + \frac{1}{4}(-1-\alpha)(1-\alpha) t^{(-3-\alpha)/2} v(t) \\
&= t^{(1-\alpha)/2} v''(t) + (1-\alpha) t^{(-1-\alpha)/2} v'(t) + \frac{1}{4}(-1-\alpha)(1-\alpha) t^{(-3-\alpha)/2} v(t)
\end{aligned}$$

Now, plugging all this into equation (8), we get

$$\begin{aligned}
t^2 \left(t^{(1-\alpha)/2} v''(t) + (1-\alpha) t^{(-1-\alpha)/2} v'(t) + \frac{1}{4}(-1-\alpha)(1-\alpha) t^{(-3-\alpha)/2} v(t) \right) & \\
+ \alpha t \left(v'(t) t^{(1-\alpha)/2} + \frac{1-\alpha}{2} v(t) t^{(-1-\alpha)/2} \right) + \frac{(\alpha-1)^2}{4} \left(v(t) t^{(1-\alpha)/2} \right) &= 0 \\
t^{(5-\alpha)/2} v''(t) + (1-\alpha) t^{(3-\alpha)/2} v'(t) + \frac{1}{4}(-1-\alpha)(1-\alpha) t^{(1-\alpha)/2} v(t) & \\
+ \alpha v'(t) t^{(3-\alpha)/2} + \alpha \frac{1-\alpha}{2} v(t) t^{(1-\alpha)/2} + \frac{(\alpha-1)^2}{4} v(t) t^{(1-\alpha)/2} &= 0 \\
t^{(5-\alpha)/2} v''(t) + (1-\alpha-\alpha)\, t^{(3-\alpha)/2} v'(t) + \left(\frac{1}{4}(-1-\alpha)(1-\alpha) + \alpha \frac{1-\alpha}{2} + \frac{(\alpha-1)^2}{4} \right) t^{(1-\alpha)/2} v(t) &= 0
\end{aligned}$$

which thankfully simplifies a whole lot. In fact, expanding all the terms you can see that the entire factor that is multiplied by $t^{(1-\alpha)/2} v(t)$ cancels out, resulting in

$$t^{(5-\alpha)/2} v''(t) + t^{(3-\alpha)/2} v'(t) = 0$$

Next we make the substitution $w = v'$ to transform this equation into a first order problem. We get

$$t^{(5-\alpha)/2}w'(t) + t^{(3-\alpha)/2}w(t) = 0$$

which is separable! This can be solved easily by moving the $w(t)$ term to the other side, isolating $w'(t)$, and integrating. The result will be $w(t) = \frac{C}{t}$. We can then recover $v(t)$ by integrating,

$$v(t) = \int w(t)dt = \int \frac{C}{t}dt = C\ln(t) + K$$

Since the constants C and K will each give us a viable result for y_2, we will choose the y_2 that comes from setting $C = 1$, $K = 0$. Thus we have

$$y_2(t) = v(t)y_1(t) = t^{(1-\alpha)/2}\ln(t)$$

and so finally, the general solution to equation (8) is

$$y(t) = c_1 t^{(1-\alpha)/2} + c_2 t^{(1-\alpha)/2}\ln(t)$$

□

Lab 4: Textbook Sections 3.6, 3.7

Question 4.1.

a. *Find the general solution to the equation*

$$y''(t) + 4y(t) = \csc(2t) \tag{9}$$

using variation of parameters.

b. *You might find it difficult to solve very similar equations using variation of parameters. Why can't we solve the next equation fully like in part a?*

$$y''(t) + 4y(t) = \csc(7t) \tag{10}$$

Solution.

a. First solve the homogeneous equation. The characteristic equation is

$$r^2 + 4 = 0$$

so roots are $\pm 2i$, and the solutions and their derivatives are

$$\begin{array}{ll} y_1 = \cos(2t) & y_2 = \sin(2t) \\ y_1' = -2\sin(2t) & y_2' = 2\cos(2t) \end{array}$$

So the Wronskian is

$$2\cos^2(2t) + 2\sin^2(2t) = 2$$

We then solve for y_p using the formula,

$$\begin{aligned} y_p &= -y_1 \int \frac{fy_2}{w}dt + y_2 \int \frac{fy_1}{w}dt \\ &= -\cos(2t)\int \frac{\csc(2t)\sin(2t)}{2}dt + \sin(2t)\int \frac{\csc(2t)\cos(2t)}{2}dt \\ &= -\frac{1}{2}\cos(2t)\int 1dt + \frac{1}{2}\sin(2t)\int \cot(2t)dt \\ &= -\frac{1}{2}t\cos(2t) + \frac{1}{2}\sin(2t)\ln|\sin(2t)| \end{aligned}$$

Thus the general solution for equation (9) is

$$y(t) = c_2 \sin(2t) + c_1 \cos(2t) - \frac{1}{2} t \cos(2t) + \frac{1}{4} \sin(2t) \ln(\sin(2t))$$

b. The homogeneous solutions are the same, as is the Wronskian. So we can skip to the part where we find the integrals. The particular solution is

$$\begin{aligned} y_p &= -y_1 \int \frac{f y_2}{w} dt + y_2 \int \frac{f y_1}{w} dt \\ &= -\cos(2t) \int \frac{\csc(7t)\sin(2t)}{2} dt + \sin(2t) \int \frac{\csc(7t)\cos(2t)}{2} dt \\ &= -\cos(2t) \int \frac{\sin(2t)}{\sin(7t)} dt + \sin(2t) \int \frac{\cos(2t)}{\sin(7t)} dt \end{aligned}$$

but we are forced to stop here, because there is no way to solve the integrals. Note: I didn't ask you this part of the question since you are not going to be asked something like this on an assignment or test. I just wanted to illustrate how this method looks when it doesn't completely work.

□

Question 4.2. *Solve the general problem of a free harmonic oscillator where* $m = 1$*,* $k = 1$*,* $0 < \gamma < 2$*,* $y(0) = 1$*, and* $y'(0) = b$*. In other words, solve the differential equation*

$$y''(t) + 2ay'(t) + y(t) = 0 \tag{11}$$

with $0 < a < 1$*,* $y(0) = 1$*, and* $y'(0) = b$*. This will show you how to plot the* $0 < a < 1$ *case like in the diagram from the lab.*

Solution. First, find the characteristic equation, which is

$$r^2 + 2ar + 1 = 0$$

The quadratic equation gives us the two roots,

$$\begin{aligned} \frac{-2a \pm \sqrt{(2a)^2 - 4}}{2} &= -a \pm \sqrt{a^2 - 1} \\ &= a \pm \sqrt{(-1)(1 - a^2)} \\ &= a \pm i\sqrt{1 - a^2} \end{aligned}$$

We did these few extra steps since $1 - a^2$ is positive, so this makes sure that we have a real number plus/minus i times a real number. This allows us to get the general solution to (11),

$$y(t) = c_1 e^{-at} \cos(t\sqrt{1 - a^2}) + c_2 e^{-at} \sin(t\sqrt{1 - a^2})$$

Now, we satisfy the initial conditions.

$$\begin{aligned} 1 = y(0) &= c_1 e^0 \cos(0) + c_2 e^0 \sin(0) = c_1 \\ b = y'(0) &= -ac_1 e^0 \cos(0) + \sqrt{1 - a^2} c_2 e^0 \cos(0) \\ &= -a + \sqrt{1 - a^2} c_2 \end{aligned}$$

and so solving for c_2 we get $c_2 = \frac{a+b}{\sqrt{1-a^2}}$, and the final solution to equation (11) is

$$y(t) = e^{-at} \cos(t\sqrt{1 - a^2}) + \frac{a + b}{\sqrt{1 - a^2}} e^{-at} \sin(t\sqrt{1 - a^2})$$

□

Lab 5: Textbook Sections 3.8, 5.1

Question 5.1.

a. *Use a Taylor polynomial about $x = 0$ to find two solutions to*

$$y'' - 2xy' + \lambda y = 0 \tag{12}$$

(Hint: solving equations with series is something covered this week, so I'll give the steps for you. Start by plugging $y(x) = \sum_{n=0}^{\infty} a_n x^n$ into the equation. Then combine everything into a single sum with an x^n term. Then use induction to solve and prove the resulting recurrence relation for the coefficients to find a_n. You may find this notation useful:

$$\prod_{n=0}^{m} f(n) = f(0) \cdot f(1) \cdot f(2) \cdot \ldots \cdot f(m)$$

for example, $\prod_{n=1}^{m} n = m!$)

b. *You will find that if λ is an even natural number, one of your two solutions is a polynomial. What is the degree of this polynomial in terms of λ?*

Solution.

a. We set our solution to be

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

plugging this in to equation (12),

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2x \sum_{n=0}^{\infty} n a_n x^{n-1} + \lambda \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} -2n a_n x^n + \sum_{n=0}^{\infty} \lambda a_n x^n \\ &= \sum_{n=0}^{\infty} \left((n+2)(n+1)a_{n+2} - 2n a_n + \lambda a_n\right) x^n \end{aligned}$$

So we now must solve the recurrence relation

$$a_{n+2} = \frac{2n - \lambda}{(n+2)(n+1)} a_n$$

So we have two different cases. Let's first suppose that n is even, with $n = 2k$. You might need to analyze the first few elements in this sequence to come up with a good guess, but I'm just going to state it. I claim that

$$a_n = a_{2k} = \frac{\prod_{m=0}^{k-1}(4m - \lambda)}{(2k)!} a_0$$

is the right answer. Let's prove this with induction. The base case is clearly true (an 'empty product' is always 1, i.e. $\prod_{m=0}^{-1} f(m) = 1$), so let's move to the induction step.

$$\begin{aligned} a_{2(k+1)} &= \frac{4k - \lambda}{(2k+2)(2k+1)} a_{2k} \\ &= \frac{4k - \lambda}{(2k+2)(2k+1)} \frac{\prod_{m=0}^{k-1}(4m - \lambda)}{(2k)!} a_0 \\ &= \frac{\prod_{m=0}^{k}(4m - \lambda)}{(2k+2)!} a_0 \end{aligned}$$

And so, by induction, the equivalence relation is solved. By the same means we get to the same kind of solution if $n = 2k + 1$. In this case,

$$a_n = a_{2k+1} = \frac{\prod_{m=0}^{k-1}(4m + 2 - \lambda)}{(2k + 1)!} a_1$$

So the general solution to equation (12) in series form is

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{\prod_{m=0}^{k-1}(4m - \lambda)}{(2k)!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{\prod_{m=0}^{k-1}(4m + 2 - \lambda)}{(2k + 1)!} x^{2k+1}$$

b. If λ is an even number, there are two possibilities. We see that if $\lambda = 4s$ for some $s \in \mathbb{N}$, then the first summation will have a 0 in every product as soon as $k - 1 \geq s$. This is because when $m = s$ is a term in the product, $4m - \lambda = 0$, causing every term after that point to be zero. To find the degree, note that $k = s$ at most, and the degree is $d = 2k$ as that is the power x is raised to. So,

$$d = 2s = \frac{\lambda}{2}$$

On the other hand, if $\lambda = 4s + 2$, then the second summation is the one that will terminate. This still happens right when $k - 1 \geq s$, so $k = s$ is the highest degree term. In this function, the degree of this term will be $2k + 1$, so the degree is

$$d = 2k + 1 = 2s + 1 = \frac{4s + 2}{2} = \frac{\lambda}{2}$$

□

Lab 6: Textbook Sections 5.2, 5.3

Question 6.1. *Solve the differential equation*

$$(\lambda - x^2)y'' + 2y = 0 \tag{13}$$

with a Taylor series about $x = 0$. What is the radius of convergence?

Solution. First note that since we have to use a Taylor Series at $x = 0$, it's implicit that $x = 0$ is an ordinary point, so λ can't be 0. We suppose $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and try to find a_n. Plugging this in,

$$0 = (\lambda - x^2) \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 2 \sum_{n=0}^{\infty} a_n x^n$$

$$0 = \lambda \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n$$

$$0 = \lambda \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n$$

$$0 = \sum_{n=0}^{\infty} \left(\lambda(n+2)(n+1)a_{n+2} + (2 - n(n-1))\, a_n\right) x^n$$

and we need all the coefficients to be 0 for this to hold, thus

$$a_{n+2} = \frac{n^2 - n - 2}{\lambda(n+2)(n+1)} a_n$$

So we have the recurrence relation

$$a_{n+2} = \frac{1}{\lambda} \cdot \frac{n-2}{n+2} a_n$$

First, suppose $n = 2k$. Then

$$a_{2(k+1)} = \frac{1}{\lambda} \cdot \frac{2k-2}{2k+2} a_{2k}$$

So

$$a_2 = \frac{1}{\lambda} \cdot \frac{-2}{2} a_0 = -\frac{1}{\lambda} a_0$$
$$a_4 = \frac{1}{\lambda} \cdot \frac{2-2}{2+2} a_0 = 0$$

and all of the terms after this point are therefore going to be 0. This means there are only two non-zero terms in this solution. Next consider what happens if $n = 2k+1$.

$$a_3 = \frac{(-1)}{(3)\lambda} a_1 = -\frac{1}{3 \cdot \lambda} a_1$$
$$a_5 = \frac{(1)}{(5)\lambda} a_3 = -\frac{1}{3 \cdot 5 \cdot \lambda^2} a_1$$
$$a_7 = \frac{(3)}{(7)\lambda} a_5 = -\frac{1}{5 \cdot 7 \cdot \lambda^3} a_1$$

So in this case the relation is solved by

$$a_n = a_{2k+1} = \frac{1}{(2k+1)(2k-1)\lambda^k} a_1 = \frac{1}{(4k^2-1)\lambda^k} a_1$$

So the Taylor series solution to equation (13) at $x = 0$ is

$$y(x) = a_1 \sum_{k=0}^{\infty} \frac{1}{(4k^2-1)\lambda^k} x^{2k+1} + a_0 \left(1 - \frac{1}{\lambda} x^2\right)$$

Now the second solution is a polynomial and thus it is clearly valid for all $x \in \mathbb{R}$. Let's check the radius of convergence of the first solution.

$$\begin{aligned} L &= \lim_{k\to\infty} \left| \frac{x^{2k+3}}{(2k+3)(2k+1)\lambda^{k+1}} \frac{(2k+1)(2k-1)\lambda^k}{x^{2k+1}} \right| \\ &= \lim_{k\to\infty} \left| \frac{2k-1}{2k+3} \cdot \frac{x^2}{\lambda} \right| \\ &= \left| \frac{x^2}{\lambda} \right| \lim_{k\to\infty} \left| \frac{2k-1}{2k+3} \right| \\ &= \left| \frac{x^2}{\lambda} \right| \end{aligned}$$

This converges when $x^2 < |\lambda|$, so the radius of convergence is in fact $\sqrt{|\lambda|}$. □

Remark 6.2. *An interesting thing about the radius of convergence of the first solution of equation* (13) *is that it depends on* λ*. Observe that this type of differential equation has a power series solution which can be made to converge on an arbitrarily small (or large) window by changing* λ*! It is intuitive that the radius of convergence would shrink as* λ *does, since once* $\lambda = 0$*, the equation is*

$$-x^2 y'' + 2y = 0$$

which no longer has an ordinary point at $x = 0$*.*

Question 6.3. *Attempt to solve the differential equation*

$$\left(x^2 + x\right) y'' + y = 0 \tag{14}$$

with a Taylor series about $x = 1$. Your solution can end once you clearly state the recurrence relation, since it will unfortunately be unsolvable. Give a lower bound for the radius of convergence of the Taylor series solution.

Solution. We suppose $y(x) = \sum_{n=0}^{\infty} a_n \left(x-1\right)^n$ and try to find a_n. Plugging this in, we get

$$0 = \left(x^2 + x\right) \sum_{n=0}^{\infty} n(n-1)a_n \left(x-1\right)^{n-2} + \sum_{n=0}^{\infty} a_n \left(x-1\right)^n \tag{15}$$

Now let's try to deal with the first term. We have to replace all instances of x in the expression by $x - 1$, so we start by fixing the highest power and work our way down. We know $(x-1)^2 = x^2 - 2x + 1$, so we need to start by adding and subtracting a $2x$ and a 1 from the expression.

$$\begin{aligned} x^2 + x &= x^2 + x - 2x + 2x + 1 - 1 \\ &= (x-1)^2 + 3x - 1 \\ &= (x-1)^2 + 3x - 3 + 2 \\ &= (x-1)^2 + 3(x-1) + 2 \end{aligned}$$

And at this point, the expression is a polynomial in $x - 1$, so we can now deal with the first sum:

$$\begin{aligned} &x^2 \sum_{n=0}^{\infty} n(n-1)a_n \left(x-1\right)^{n-2} \\ &= \left((x-1)^2 + 3(x-1) + 2\right) \sum_{n=0}^{\infty} n(n-1)a_n \left(x-1\right)^{n-2} \\ &= \sum_{n=0}^{\infty} n(n-1)a_n \left(x-1\right)^n + 3\sum_{n=0}^{\infty} n(n-1)a_n \left(x-1\right)^{n-1} + 2\sum_{n=0}^{\infty} n(n-1)a_n \left(x-1\right)^{n-2} \\ &= \sum_{n=0}^{\infty} n(n-1)a_n \left(x-1\right)^n + 3\sum_{n=0}^{\infty} (n+1)na_{n+1} \left(x-1\right)^n + 2\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} \left(x-1\right)^n \\ &= \sum_{n=0}^{\infty} \left(2(n+2)(n+1)a_{n+2} + 3n(n+1)a_{n+1} + n(n-1)a_n\right) \left(x-1\right)^n \end{aligned}$$

And finally, we can add back the second summation from equation (15) to get

$$0 = \sum_{n=0}^{\infty} \left(2(n+2)(n+1)a_{n+2} + 3n(n+1)a_{n+1} + \left(n(n-1) + 1\right) a_n\right) \left(x-1\right)^n$$

So the recurrence relation is given by

$$a_{n+2} = -\frac{3n}{2(n+2)}a_{n+1} - \frac{n^2 - n + 1}{2n^2 + 6n + 4}a_n$$

which is indeed 2 scary 4 me. As for the radius of convergence, the leading coefficient in equation (14) is $x^2 + x = x(x+1)$ which means $x = 0$ and $x = -1$ are singular points. Since the center of the Taylor series was $x = 1$, we can tell that the radius of convergence is at least 1. □

Lab 7: Textbook Sections 6.1, 6.2

Question 7.1. *Solve the initial value problem*

$$y'' + y = e^t, \quad y(0) = 1, \quad y'(0) = \lambda \tag{16}$$

using the method of Laplace transforms.

Solution. Taking the Laplace transform of both sides, we get

$$\begin{aligned}
\mathscr{L}\{y'' + y\} &= \mathscr{L}\{e^t\} \\
\mathscr{L}\{y''\} + \mathscr{L}\{y\} &= \mathscr{L}\{e^t\} \\
s^2\mathscr{L}\{y\} - sy(0) - y'(0) + \mathscr{L}\{y\} &= \frac{1}{s-1} \\
s^2\mathscr{L}\{y\} - s - \lambda + \mathscr{L}\{y\} &= \frac{1}{s-1}
\end{aligned}$$

Now, Isolating $\mathscr{L}\{y\}$,

$$\begin{aligned}
(s^2+1)\mathscr{L}\{y\} &= \frac{1 + (s-1)(s+\lambda)}{s-1} \\
\mathscr{L}\{y\} &= \frac{s^2 + (\lambda-1)s + 1 - \lambda}{(s-1)(s^2+1)}
\end{aligned}$$

And now we must use partial fraction decomposition on the r.h.s. to simplify. To start, let's write out what we're trying to find:

$$\frac{s^2 + (\lambda-1)s + 1 - \lambda}{(s-1)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{C}{s-1} \tag{17}$$

Now, put the fractions on the r.h.s. together. Let's look at what happens to the numerator of the r.h.s.

$$\begin{aligned}
(As+B)(s-1) + C(s^2+1) &= As^2 - As + Bs - B + Cs^2 + C \\
&= (A+C)s^2 + (B-A)s + C - B
\end{aligned}$$

Finally, comparing this to the l.h.s. of equation (17), we get the following system of equation by comparing terms with the same power:

$$\begin{aligned}
A + C &= 1 \\
B - A &= \lambda - 1 \\
C - B &= 1 - \lambda
\end{aligned}$$

Adding all three equations together, we get $2C = 1$. This tells us that $C = \frac{1}{2}$, $A = \frac{1}{2}$, and $B = \lambda - \frac{1}{2}$. So with this new expression, we have

$$\begin{aligned}
\mathscr{L}\{y\} &= \frac{s + 2\lambda - 1}{2(s^2+1)} + \frac{1}{2(s-1)} \\
&= \frac{2\lambda - 1}{2} \cdot \frac{1}{s^2+1} + \frac{1}{2} \cdot \frac{s}{s^2+1} + \frac{1}{2} \cdot \frac{1}{s-1}
\end{aligned}$$

So, taking the inverse Laplace transform of each side, we get the solution to the initial value problem in equation (16),

$$y(x) = \frac{2\lambda - 1}{2}\sin(t) + \frac{1}{2}\cos(t) + \frac{1}{2}e^t$$

□

Lab 8: Textbook Sections 6.3, 6.4

Question 8.1. *Solve the initial value problem*

$$y''(t) + y(t) = u_\pi(t) - u_{2\pi}(t), \qquad y(0) = -1, \qquad y'(0) = 0 \tag{18}$$

for $y(t)$ using Laplace transforms.

Solution. As usual, we start by taking the Laplace transform of both sides of the equation, and isolating for $\mathscr{L}\{y\}$.

$$\begin{aligned}
\mathscr{L}\{y''(t)\} + \mathscr{L}\{y(t)\} &= \mathscr{L}\{u_\pi(t) - u_{2\pi}(t)\} \\
s^2\mathscr{L}\{y(t)\} + s + \mathscr{L}\{y(t)\} &= \frac{e^{-\pi t}}{s} - \frac{e^{-2\pi t}}{s} \\
(s^2+1)\mathscr{L}\{y(t)\} &= \frac{e^{-\pi t} - e^{-2\pi t}}{s} - s \\
\mathscr{L}\{y(t)\} &= e^{-\pi t}\frac{1}{s(s^2+1)} - e^{-2\pi t}\frac{1}{s(s^2+1)} - \frac{s}{s^2+1}
\end{aligned}$$

And now, take the inverse Laplace transform of both sides. Before we can do this, we need to break up the fractions here to match them with something from the table. To decompose this, let

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{As^2 + A + Bs^2 + Cs}{s(s^2+1)}$$

Comparing the left most and right most sides of this equation, it's clear that $A = 1$, $C = 0$, and $B = -1$. So we have

$$\mathscr{L}\{y(t)\} = (e^{-\pi t} - e^{-2\pi t})\left(\frac{1}{s} - \frac{s}{s^2+1}\right) - \frac{s}{s^2+1}$$

and at this point, we are able to take the inverse Laplace transform using the table to get the solution to equation (18),

$$\begin{aligned}
y(t) &= u_\pi(t)\left(1 - \cos(t-\pi)\right) - u_{2\pi}(t)\left(1 - \cos(t - 2\pi)\right) - \cos(t) \\
&= u_\pi(t)\left(1 + \cos(t)\right) - u_{2\pi}(t)\left(1 - \cos(t)\right) - \cos(t) \\
&= \cos(t)\left(u_\pi(t) + u_{2\pi}(t) - 1\right) + u_\pi(t) - u_{2\pi}(t)
\end{aligned}$$

□

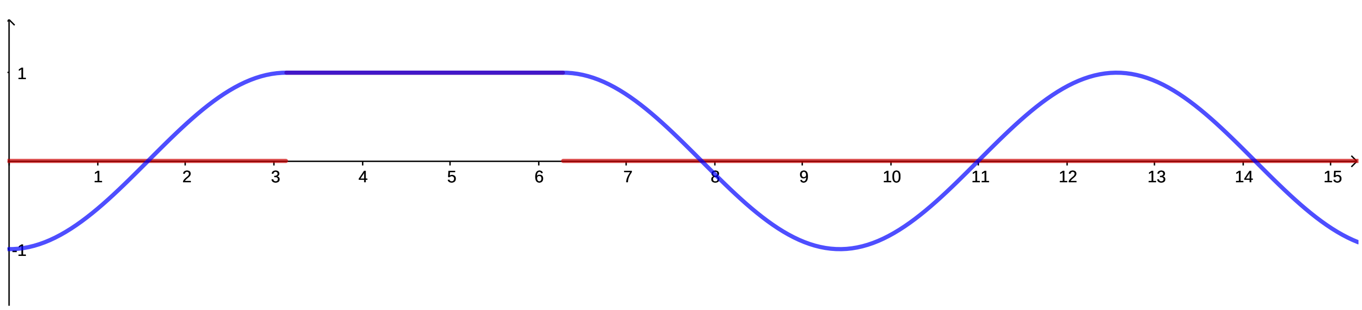

Figure 1: Solution to question 8.1 (blue) vs. its driver function (red)

Lab 9: Textbook Sections 6.5, 6.6

Question 9.1. *Find values a and b such that the solution to the equation*

$$y''(t) + 2y'(t) + y(t) = a\delta(t-2) \qquad y(0) = b \qquad y'(0) = -9 \tag{19}$$

is identically 0 *for all* $t \geq 2$.

Solution. Taking the Laplace transform of both sides, we get

$$\begin{aligned} \mathscr{L}\left\{y'' + 2y' + y\right\} &= \mathscr{L}\left\{a\delta(t-2)\right\} \\ s^2\mathscr{L}\left\{y\right\} - bs + 9 + 2s\mathscr{L}\left\{y\right\} - 2b + \mathscr{L}\left\{y\right\} &= ae^{-2s} \\ (s^2+2s+1)\mathscr{L}\left\{y\right\} &= ae^{-2s} + bs - 9 + 2b \\ \mathscr{L}\left\{y\right\} &= ae^{-2s}\frac{1}{(s+1)^2} + b\frac{s}{(s+1)^2} + (2b-9)\frac{1}{(s+1)^2} \end{aligned}$$

In order to get this into a form that is easy to take the inverse Laplace transform of, first write

$$\frac{s}{(s+1)^2} = \frac{s+1-1}{(s+1)^2} = \frac{1}{(s+1)^2} - \frac{1}{s+1}$$

Now, we get the solution for equation (19),

$$y(t) = au(t-2)(t-2)e^{2-t} + be^{-t}(1-t) + (2b-9)te^{-t}$$

We want this function to be identically 0 after the point 2, so everything should cancel out. For this to happen, let $t \geq 2$ and set

$$\begin{aligned} 0 &= a(t-2)e^{2-t} + be^{-t}(1-t) + (2b-9)te^{-t} \\ &= \left(ae^2 - b + 2b - 9\right)te^{-t} + \left(-2ae^2 + b\right)e^{-t} \end{aligned}$$

so we get two equations for a and b,

$$\begin{aligned} ae^2 + b - 9 &= 0 \\ -2ae^2 + b &= 0 \end{aligned}$$

which shows that $b = 6$, and then $a = \frac{3}{e^2}$. So the final solution with the desired property is

$$y(t) = 3u(t-2)(t-2)e^{-t} + 6e^{-t}(1-t) + 3te^{-t}$$

□

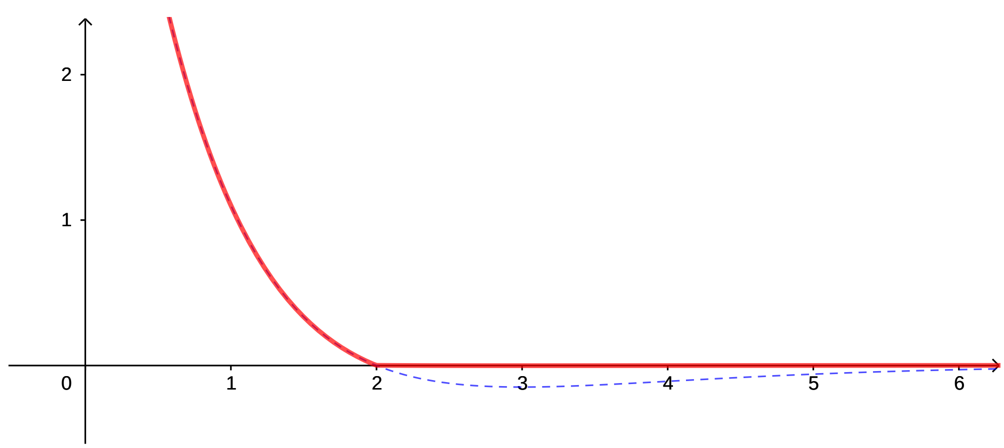

Figure 2: Solutions to problem 9.1 with $b = 6$ and $a = 0$ (blue) vs. $a = \frac{3}{e^2}$ (red)

Question 9.2. *Using Laplace transforms, find constants a and b such that the solutions $y(t)$, $y_{a,b}(t)$ to the initial value problems*

$$y'' + y = 0, \qquad y(0) = 1, \qquad y'(0) = 1$$
$$y''_{a,b} + y_{a,b} = a\left(\delta(t-r) - \delta(t-r-b)\right), \qquad y_{a,b}(0) = 1, \qquad y'_{a,b}(0) = 1 \tag{20}$$

are identical for all $t \in \mathbb{R}$, except in the interval $t \in [r, r+b]$ where r is the smallest positive number such that $y(r) = 0$. In that interval, $f_{a,b}(t)$ is identically 0.

Solution. Taking the Laplace transform of both sides, we get

$$\mathscr{L}\left\{y''_{a,b} + y_{a,b}\right\} = \mathscr{L}\left\{a\left(\delta(t-r) - \delta(t-r-b)\right)\right\}$$
$$s^2\mathscr{L}\left\{y_{a,b}\right\} - 1 - s + \mathscr{L}\left\{y_{a,b}\right\} = a\left(e^{-rs} - e^{-(r+b)s}\right)$$
$$\mathscr{L}\left\{y_{a,b}\right\} = a \cdot e^{-rs}\frac{1}{s^2+1} - ae^{-(r+b)s}\frac{1}{s^2+1} + \frac{1}{s^2+1} + \frac{s}{s^2+1}$$

Now, taking the inverse Laplace transform of the r.h.s. is not hard, and we get that the general solution for equation (20) is

$$y_{a,b}(t) = \sin(t) + \cos(t) + a\sin(t-r)u_r(t) + a\sin(t-r-b)u_{r+b}(t)$$

We have two interesting cases: $t \in [r, r+b)$, or $t > r+b$. Respectively, these two situations give the following three equations:

$$0 = \sin(t) + \cos(t) + a\sin(t-r)$$
$$\sin(t) + \cos(t) = \sin(t) + \cos(t) + a\sin(t-r) + a\sin(t-r-b)$$

The second equation shows that $0 = \sin(t-r) + \sin(t-r-b)$, which means that $b = \pi + 2n\pi$, $n \geq 0$. The first equation shows that

$$0 = \sqrt{2}\left(\frac{\sqrt{2}}{2}\sin(t) + \frac{\sqrt{2}}{2}\cos(t)\right) + a\sin(t-r)$$
$$0 = \sqrt{2}\sin(t + \frac{\pi}{4}) + a\sin(t-r)$$

Observe that r is just the first positive solution to $\sin(r) + \cos(r) = 0$, so $r = \frac{3\pi}{4}$. Thus, $a = \sqrt{2}$. □

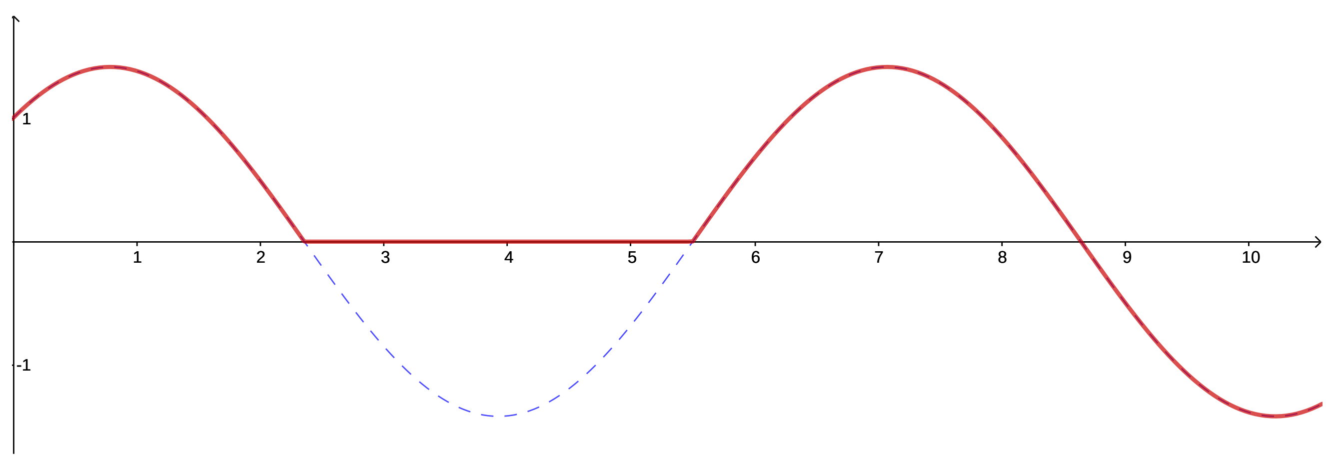

Figure 3: Solutions to equation (20) with $b = \pi$ and $a = 0$ (blue) vs. $a = \sqrt{2}$ (red)

Lab 10: Textbook Sections 10.1, 10.2

Question 10.1.

a. *Find the Fourier series of the function*

$$f_m(x) = m\left(u\left(x+\frac{1}{2m}\right) - u\left(x-\frac{1}{2m}\right)\right) \tag{21}$$

in the interval $(-\pi, \pi)$*, for an integer* $m \geq 1$*.*

b. *Find the limit of this Fourier Series as* $m \to \infty$ *(Note: the 'limit of this Fourier series' is just the Fourier series with coefficients* $\widetilde{a}_n = \lim_{m\to\infty} a_n$ *and* $\widetilde{b}_n = \lim_{m\to\infty} b_n$*).*

Solution.

a. First, note that f is an even function. Since we're integrating on a symmetric interval about 0 and sin is an odd function, all the a_n coefficients are going to be 0. Let's try to calculate the coefficients b_n. Integrating,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_m(x)\cos\left(\frac{n\pi x}{\pi}\right)dx \\ &= \frac{m}{\pi}\int_{-\frac{1}{2m}}^{\frac{1}{2m}} \cos(nx)\,dx \\ &= \frac{m}{\pi}\left(\frac{\sin(n\left(\frac{1}{2m}\right))}{n} - \frac{\sin(n\left(-\frac{1}{2m}\right))}{n}\right) \\ &= \frac{2m}{\pi n}\sin\left(\frac{n}{2m}\right) \end{aligned}$$

And finally, the leading coefficient a_0 is given by

$$\begin{aligned} a_0 =& \frac{1}{\pi}\int_{-\pi}^{\pi} f_m(x)dx \\ =& \frac{m}{\pi}\int_{-\frac{1}{2m}}^{\frac{1}{2m}} dx \\ =& \frac{m}{\pi}\left(\frac{1}{2m}+\frac{1}{2m}\right) \\ =& \frac{1}{\pi} \end{aligned}$$

Thus, the Fourier series of equation (21) is

$$f_m(x) \approx \frac{1}{2\pi} + \sum_{n=0}^{\infty}\left(\frac{2m}{\pi n}\sin\left(\frac{n}{2m}\right)\cos(nx)\right)$$

b. Since $a_n = 0$ for all n, $\widetilde{a}_n = 0$ as well. For $\widetilde{b}_n$, let $u = \frac{1}{2m}$. Then

$$\widetilde{b}_n = \lim_{m\to\infty} b_n = \lim_{u\to\infty}\frac{1}{\pi n}\cdot\frac{\sin(n\cdot u)}{u} = \frac{1}{\pi n}\cdot n = \frac{1}{\pi}$$

So the limit Fourier series is the same one from Example 3 in the lab with $c = 0$,

$$\lim_{m\to\infty} f_m(x) \approx \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=0}^{\infty}\cos(nx)$$

□

Lab 11: Textbook Sections 10.3, 10.4

Question 11.1. *Find the Fourier sine **and** cosine series of the function*

$$f(x) = e^x \tag{22}$$

in the interval $(-\pi, \pi)$.

Solution. The even extension is

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi e^x \cos(nx)dx \\ &= \frac{2}{\pi}\left(\frac{e^x(n\sin(nx)+\cos(nx))}{n^2+1}\right)\Big|_0^\pi \\ &= \frac{2}{\pi}\left(\frac{e^\pi(n\sin(n\pi)+\cos(n\pi))}{n^2+1} - \frac{e^0(n\sin(0)+\cos(0))}{n^2+1}\right) \\ &= \frac{2}{\pi}\frac{e^\pi(-1)^n-1}{n^2+1} \end{aligned}$$

In particular, note that $a_0 = \frac{2}{\pi}(e^\pi - 1)$. So we have the even representation of equation (22) given by

$$f(x) \approx \frac{e^\pi - 1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n e^\pi - 1}{n^2+1}\cos(nx)$$

The odd extension is

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi e^x \sin(nx)dx \\ &= \frac{2}{\pi}\left(\frac{e^x(\sin(nx)-n\cos(nx))}{n^2+1}\right)\Big|_0^\pi \\ &= \frac{2}{\pi}\left(\frac{e^\pi(\sin(n\pi)-n\cos(n\pi))}{n^2+1} - \frac{e^0(\sin(0)-n\cos(0))}{n^2+1}\right) \\ &= \frac{2}{\pi}\frac{-ne^\pi(-1)^n+n}{n^2+1} \end{aligned}$$

So we have the odd representation of equation (22) given by

$$f(x) \approx \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n - n(-1)^n e^\pi}{n^2+1}\sin(nx)$$

□

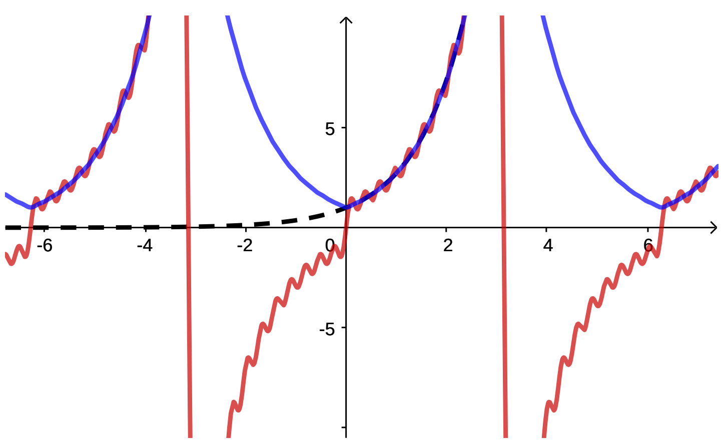

Figure 4: The exponential function (black) with Fourier cosine (blue) and sine (red) series.

Question 11.2. *Consider the function*

$$f(x) = x \cdot \cos(x) \tag{23}$$

a. Is this function even, odd, or neither? Make sure to justify.

b. If $f(x)$ is even, find its Fourier sine series. If it is odd, find its Fourier cosine series. If it is neither, find its Fourier series. Take the period in each case to be $L = \pi$.

Solution.

a. Since x is an odd function, and $\cos(x)$ is even, their product must be odd. Let's double check this by plugging in $-x$:

$$f(-x) = (-x) \cdot \cos(-x) = -(x \cdot \cos(x)) = -f(x)$$

which verifies that $f(x)$ is odd.

b. Since $f(x)$ is odd, we are going to find the Fourier cosine series. First, let's show another trig identity like the one in the lab. Using Euler's identity,

$$\begin{aligned}
\cos(x)\cos(nx) &= \frac{e^{-ix}+e^{ix}}{2} \cdot \frac{e^{-inx}+e^{inx}}{2} \\
&= \frac{1}{4}\left(e^{-inx-ix} + e^{ix-inx} + e^{inx-ix} + e^{inx+ix}\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\left(e^{-i(n+1)x} + e^{i(n+1)x)}\right) + \frac{1}{2}\left(e^{-i(n-1)x} + e^{i(n-1)x}\right)\right) \\
&= \frac{1}{2}\left(\cos(x(n+1)) + \cos(x(n-1))\right)
\end{aligned}$$

Using this identity, the coefficients are found by integrating. Once again, there are going to be two different cases depending on if $n = 1$ or not. Let's assume first that $n \neq 1$.

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi x\cos(x)\cos(nx)dx \\
&= \frac{1}{\pi}\int_0^\pi x\cos(x(n-1)) + x\cos(x(n+1))dx \\
&= \frac{1}{\pi}\left[-\frac{2x\sin(x)\cos(nx)}{n^2-1} + \frac{2nx\cos(x)\sin(nx)}{n^2-1} + \frac{\cos((n-1)x)}{(n-1)^2} + \frac{\cos((n+1)x)}{(n+1)^2}\right]_0^\pi \\
&= \frac{1}{\pi}\left(\frac{(-1)^n}{(n-1)^2} + \frac{(-1)^{n+2}}{(n+1)^2} - \frac{(-1)^{n-1}}{(n-1)^2} - \frac{(-1)^{n+1}}{(n+1)^2}\right) \\
&= -\frac{2}{\pi}\frac{\left((-1)^n+1\right)\left(n^2+1\right)}{\left(n^2-1\right)^2}
\end{aligned}$$

Finally, assume that $n = 1$ and find a_1.

$$\begin{aligned}
a_1 &= \frac{1}{\pi}\int_0^\pi x + x\cos(2x)dx \\
&= \frac{1}{\pi}\left[\frac{1}{2}x^2 + \frac{1}{4}\left(2x\sin(2x) + \cos(2x)\right)\right]_0^\pi \\
&= \frac{\pi}{2}
\end{aligned}$$

So we have all we need to write the Fourier cosine series of equation (23). Remember that a_1 should be multiplied by $\cos(x)$ even though it's not included in the summation.

$$-\frac{1}{\pi} + \frac{\pi}{2}\cos(x) - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{\left((-1)^n+1\right)\left(n^2+1\right)}{\left(n^2-1\right)^2}\cos(nx)$$

□

Lab 12: Textbook Sections 10.5, 10.6

Question 12.1. *Solve the heat equation*

$$\begin{aligned} u_t &= \beta u_{xx} \\ u(x,0) &= m\sin(x) - \sin(mx) \end{aligned} \tag{24}$$

for an integer $m > 1$*, under the boundary conditions* $u(0,t) = u(\pi,t) = 1$.

Solution. The boundary condition being 1 just means that we are going to use the function f minus the steady state, so the coefficients will be with respect to

$$g(x) = m\sin(x) - \sin(mx) - 1$$

and we'll add the steady state back on to the result. $g(x)$ is the sum of two functions which will need slightly different approaches to solve, so first let's solve for the coefficients with $g_1(x) = -1$. They are

$$\begin{aligned} T_n(0) &= -\frac{2}{\pi}\int_0^\pi \sin(nx)dx \\ &= -\frac{2}{n\pi}\left(\cos(nx)\right)_0^\pi \\ &= \frac{2}{n\pi}\left((-1)^n - 1\right) \end{aligned}$$

Next, solve for the coefficients with $g_2(x) = m\sin(x) - \sin(mx)$. This one is much simpler, since there are only 2 non-zero terms. The solution in this case would be

$$me^{-\beta t}\sin(x) - e^{-m^2\beta t}\sin(mx)$$

Now by linearity, the full solution is going to be the sum of these two. Thus, the complete solution to equation (24) is

$$u(x,t) = 1 + me^{-\beta t}\sin(x) - e^{-m^2\beta t}\sin(mx) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left((-1)^n - 1\right)e^{-n^2\beta t}\sin(nx)$$

□

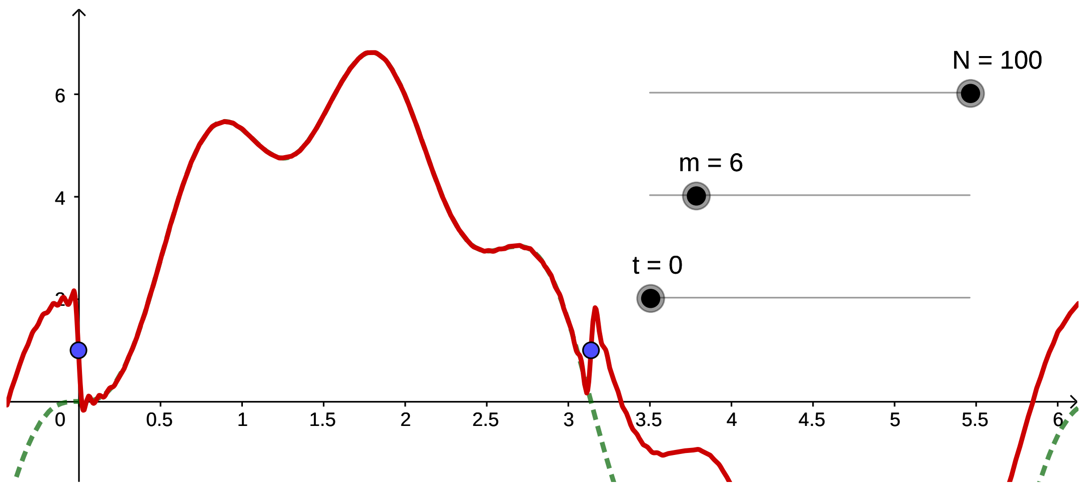

Figure 5: The solution to (24) (red) up to 100 terms vs. the initial condition (green).

Question 12.2. *Find conditions on the positive functions $f(t)$, $g(t)$, and $h(t)$ under which the P.D.E.*

$$u_{tt} + f(t)u_{xx} + g(t)u_x - h(t)u = 0 \tag{25}$$

can be separated into two O.D.E.s. Assuming those conditions hold, separated it.

Solution. Start by assuming it can be separated, and plugging in $X(x) \cdot T(t) = u(x,t)$. We get

$$0 = XT'' + f(t)X''T + g(t)X'T - h(t)XT$$

Now, simplifying,

$$\begin{aligned} 0 &= XT'' + f(t)X''T + g(t)X'T - h(t)XT \\ -XT'' &= T\big(f(t)X'' + g(t)X' - h(t)X\big) \\ \frac{T''}{T} &= h(t) - f(t)\frac{X'}{X} - g(t)\frac{X''}{X} \\ h(t) - \frac{T''}{T} &= f(t)\frac{X'}{X} + g(t)\frac{X''}{X} \end{aligned}$$

From this, we see that for the equation to be separable, it is necessary that $\frac{f(t)}{g(t)} =: \alpha$ be a constant function, i.e. g is a scalar multiple of f. Under this condition, we can finish separating this:

$$\begin{aligned} h(t) - \frac{T''}{T} &= f(t)\frac{X'}{X} + g(t)\frac{X''}{X} \\ \frac{h(t)}{g(t)} - \frac{T''}{g(t)T} &= \frac{f(t)}{g(t)}\frac{X'}{X} + \frac{X''}{X} \\ \frac{h(t)T - T''}{g(t)T} &= \frac{\alpha X' + X''}{X} \end{aligned}$$

And so, in order for this equation to hold we must have that each side of this equation is equal to the same constant λ. This means we can write this P.D.E. as two O.D.E.s, in particular

$$\begin{aligned} \frac{\alpha X' + X''}{X} &= \lambda \\ \frac{h(t)T - T''}{g(t)T} &= \lambda \end{aligned}$$

which, after some rearranging, is just

$$\begin{aligned} X'' + \alpha X' - \lambda X &= 0 \\ T'' + (\lambda g(t) - h(t))\,T &= 0 \end{aligned}$$

□

Best of luck on finals and have a good summer!