Math 532 (Algebraic Geometry) Notes

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Lecture 2

Definition 2.1. *let* $Z \subset \mathbb{A}^n = \{(z_1, \dots, z_n) : z_i \in \mathbb{C}\}$ *is an* algebraic set *or a* closed set *if it is the zero set* $Z = Z(S) = \{p \in \mathbb{A}^n, f'(p) = 0 \ \forall \ f \in S \subset \mathbb{C}[x_1, \dots, x_n]\}$.

Some examples,

- $\mathbb{A}^n = Z(0)$: the 0 polynomial, or $\emptyset = Z(1)$: the 1
- Any point $(a_1, \dots, a_n)$ is a n algebraic set for the zero set $(a_1, \dots, a_n) = Z(x_1 - a_1, \dots, x_n - a_n)$
- any linear subspace
- $\{f(x, y) = 0\} \subset \mathbb{A}^n$: plane curves

Question 2.2. *How does* $Z(S)$ *depend on* S*?*

The first observation is that in general, you may replace the subset S with the ideal generated by S, since for two functions in S, the linear combination $g_1 f_1 + g_2 f_2$ for arbitrary polynomials g_1, g_2 is also in there. That is, $Z(I(S)) = Z(S)$ so we always consider $S \in \mathbb{C}[x_1, \dots, x_n]$ which are ideals. The Hilbert Basis Theorem shows that all ideals in $\mathbb{C}[x_1, \dots, x_n]$ are finitely generated, so any algebraic set is the solution to a finite set of polynomials.

The following is a result in commutative algebra.

Lemma 2.3. *Let* R *be a ring (in this class, we always assume rings are commutative and with identity). Then the following are equivalent.*

1. *Every ideal in* R *is finitely generated*
2. *Every infinite ascending chain of ideals is stationary (a stationary chain of ideals is one where eventually the inclusions all become equalities. For* $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ *there is an* n *sufficiently large so that* $I_m = I_{m+1}$ *for all* $m \geq n$*).*

Such rings are called Noetherian

Proof.

(1) $\implies$ (2) Suppose $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ Then $\bigcup_i I_i$ is an ideal, so $I = (f_1, \dots, f_m)$ where f_i is in some I_{k_i}. Then $I = I_m$

(2) $\implies$ (1) Suppose there was an I which was not finitely generated, and let $f_1 \in I$ be any non-zero element, and let $f_{i+1} \in I - (f_1, \dots, f_i)$. Then $(f_1) \subset (f_1, f_2) \subset \dots$ is nonstationary, which is a contradiction.

$\square$

Note that If R is Noetherian, then $R[x]$ is also Noetherian, which implies the earlier Hilbert basis theorem.

Theorem 2.4. *If R is Noetherian, then $R[x]$ is Neotherian.*

Proof. Suppose $I \subset R[x]$ was not finitely generated. Construct a $f_0, f_1, \dots \in I$ where f_0 is non-zero and minimal degree, choosing $f_{i+1} \in I - (f_1, \dots, f_i)$ to be of minimal degree. Let a_i be the leading coefficient of f_i, which creates an ascending sequence of ideals $I_i := (a_1, \dots, a_i) \subset R$, which makes an ascending sequence which must be stationary. This means there is an n so that for all $m > n$, $a_{m+1} \in (a_1, \dots, a_n)$. But then $a_{m+1} = \sum_{i=0}^{m} r_i a_i$. So we let

$$f := f_{m+1} - \sum_{i=0}^{m} x^{deg f_{m+1} - deg f_i} r_i f_1 \tag{2.1}$$

Now, the second part is in $(f_0, \dots, f_m)$, and the first is in $I - (f_0, \dots, f_m)$, which means $f \in I - f(f_0, \dots, f_m)$, but the degree $deg f < deg f_{m+1}$ since the coefficient of $x^{deg f_{m+1}}$ of f is

$$a_{m+1} - \sum_{i=0}^{m} r_i a_i = 0 \tag{2.2}$$

which contradicts that f_{m+1} was chosen to be the minimal degree. $\square$

Now, algebraic sets do not exactly determine a unique ideal. That is, any algebraic set may not be the zeros of some polynomials. For example, let $0 \in \mathbb{A}^1$. This is $\{0\} = Z(x) = Z(x^2)$, but $(x) \neq (x^2)$. More generally, $Z(f_1, \dots, f_k)$ and $Z(f_1^{i_1}, \dots, f_k^{i_k})$ are the same.

The assignment $S \mapsto Z(S)$ reverses inclusions.

1. If $S_1 \subset S_2 \subset \mathbb{C}[x_1, .., x_n]$ then $Z(S_2) \subset Z(S_1) \subset \mathbb{A}^n$.
2. $Z(\bigcup_i S_i) = \bigcap_i Z(S_i)$
3. $Z(S_1) \cup Z(S_2) = Z(S_1 \cdot S_2)$

For number three, suppose $p \in Z(S_1) \cup Z(S_2)$, then for all $f_1 \cdot f_2 \in S_1 \cdot S_2$, one of f_1 and f_2 vanishes so their product does too. So $p \in Z(S_1 \cdot S_2)$.

Next suppose $p \notin Z(S_1) \cup Z(S_2)$, then there is $f_1 \in S_1$, $f_2 \in S_2$ so that $f_i(p) \neq 0$, so their product is not zero as well, so $p \notin Z(S_1 \cdot S_2)$.

Definition 2.5. *Recall that we can define a topology on X by specifying what the closed sets are, just requiring that*

1. *$\emptyset$, X are closed*
2. *Arbitrary intersections and finite unions of closed sets are closed*

This means we can specify a topology by specifying that the algebraic sets are closed, which results in the Zariski Topology*.*

Using the subspace topology, the Zariski topology induces a topology on any algebraic set. Note that the Zarkisi topology is very coarse. That is, the open sets are very big; they are all dense. This topology is very different from analytic topology. Compactness is useless, every closed set is compact. Any set bijection is continuous as well, so continuity is not useful. Be careful with intuition!

Definition 2.6. *For $X \subset \mathbb{A}^n$, let $I(X) = \{f \in \mathbb{C}[x_1, \ldots, x_n] : f(p) = 0 \ \forall \ p \in X\}$ be the ideal of functions that vanish on X. Then*

$$I : \left\{\textit{Algebraic sets in } \mathbb{A}^N\right\} \mapsto \{\textit{Ideals in } \mathbb{C}[x_1, \ldots, x_n]\} \tag{2.3}$$

and Z is a map in the other direction.

Example 2.7. *Let $Z = x\text{-axis} \cup y\text{-axis} \cup z\text{-axis}$. Note that $x\text{-axis} \cup y\text{-axis} = Z(z, xy)$, and $z\text{-axis} = Z(x, y)$, and so*

$$Z(z, xy) \cup Z(x, y) = Z(xz, x^2y, zy, xy^2) \tag{2.4}$$

and as well $Z(ZX) = \{xy\text{-plane}\} \cup \{yz\text{-plane}\}$, so

$$Z = Z(ZX) \cup Z(XY) \cup Z(YZ) = Z(ZX, XY, YZ) \tag{2.5}$$

but $I_1 \neq I_2$, even though $Z(I_i) = Z$. It turns out $I_1 \subset I_2$, but $xy \in I_2 - I_1$. Note that $x^2y^2 \in I^2$.

Lecture 3

Definition 3.1. *Given some ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$, then the* radical *of the ideal is*

$$\sqrt{I} = \{f \in \mathbb{C}[x_1, \ldots, x_n] : f^r \in I \textit{ for some } r\} \tag{3.1}$$

Theorem 3.2 (Nullstellensatz)**.**

1. *If $X \subset \mathbb{A}^n$ is algebraic, then $Z(I(X)) = X$.*
2. *If $I \subset \mathbb{C}[x_1, \ldots, x_n]$, then $I(Z(X)) = \sqrt{I}$.*

Proof.

1. True by definition. But can you think of a non-algebraic set where this fails? One example is any interval. Any infinite set of points in fact!

2. "$\supset$" is easy: if f^r vanishes, then f vanishes.

   "$\subset$": if f vanishes on $Z(I)$, then $f^r \in I$ for some r. To see this, we invoke the following lemma from commutative algebra. Suppose f vanishes on $Z(I)$. Consider $J = I + (f \cdot t - 1) \subset \mathbb{C}[x_1, \ldots, x_n, t]$. Then $Z(J) \subset \mathbb{A}^{n+1}$ is $\emptyset$, since the x coordinates lie in $Z(I)$ and $f \cdot t - 1 = 0$ turns into $-1 = 0$ at such point and thus cannot have solutions. By the corollary, we have $1 = g_0(f \cdot t - 1) + \sum g_i f i$ where $I = (f_1, \ldots, f_k)$, and $g_i \in \mathbb{C}[x_1, \ldots, x_n, t]$. Now, let N be the highest degree of the powers of t occurring in the functions g_i. Then

$$G_i = f^N g_i \in \mathbb{C}[x_1, \ldots, x_n, ft] \tag{3.2}$$

which is a polynomial in the x_i and ft. Then

$$f^N = G_0 \cdot (f \cdot t - 1) + \sum_{G_i} f_i \in \mathbb{C}[x_1, \dots, x_n, ft] \tag{3.3}$$

Now, setting $ft = 1$, we get a polynomial equation

$$f^N = \sum G_i(x_1, \dots, x_n, 1) f_i \in I \tag{3.4}$$

□

Lemma 3.3. *The maximal ideals in $\mathbb{C}[x_1, \dots, x_n]$ are all of the form $m = (x_1 - a_1, \dots, x_n - a_n)$ for $(a_1, \dots, a_n) \subset \mathbb{A}^n$.*

Corollary 3.4. *If $Z(I) = \emptyset$, then $I = (1)$ (I is everything. If $I \neq (1)$, then it would be contained in some maximal ideal $I \subset M$ whcih means there is an $(a_1, \dots, a_n) \in Z(I)$).*

Note that the lemma fails for example $\mathbb{R}[x]$ since $(x^2 + 1) \subset \mathbb{R}[x]$ is maximal.

Definition 3.5. *A set X is reducible if $X = X_1 \cup X_2$, X_i closed and $X_i \neq X$. If X is not redicible it is called irreducible. An irreducible algebraic set $X \subset \mathbb{A}^n$ is called an* affine variety.

For example, let $X \subset \mathbb{A}^2$ where $X = Z(x^2 - y^2) = Z((x - y)(x + y)) = Z(x - y) \cup Z(x + y)$.

Definition 3.6. *A topological space is Noetherian if every descending chain of closed sets is stationary.*

A corollary of this is that every algebraic set can be written as a finite union of affine varieties, just using a similar logic to Lemma 2.3 and noting that $Z(\cdot)$ reverses inclusions.

$P \subset R$ is a prime ideal if $ab \in P \implies a, b \in P$. This happens iff R/P is an integral domain, where $a \cdot b = 0$ in R/P implies $a = 0$ or $b = 0$. There are correspondences between some concepts, given in the following list.

- Geometry in $\mathbb{A}^n$
  1. Points in $\mathbb{A}^n$
  2. Affine varaieties
  3. Closed sets
- Algebra (Ideals in $\mathbb{C}[x_1, \dots, x_n]$)
  1. maximal ideals in $\mathbb{C}[x_1, .., x_n]$
  2. prime ideals
  3. radical ideals

Definition 3.7. *Let $X \subseteq \mathbb{A}^n$ be a variety. The dimension of X is the largest d s.t. there is a chain*

$$\emptyset \neq X_0 \subset X \subset \dots \subset X_d = X \tag{3.5}$$

where each X_i are irreducible closed sets. By definition, the dimension of any closed set is the highest dimension of its components (Jargon: if $X = X_1 \cup \dots \cup X_n$, a union of irreducible sets X_i, the X_i are called the components of X).

For example, the closed sets of $\mathbb{A}^1$ are $\emptyset$, $\mathbb{A}^1$ or finite sets of points. Thus $\dim \mathbb{A}^1 = 1$. It is also true that $\dim \mathbb{A}^n = n$, however it is quite hard to prove that directly from the definition.

3.1 Projective space and projective varieties

As a set, complex projective space $\mathbb{CP}^n$ (or just $\mathbb{P}^n$) is equivalence classes of $(n+1)$-tuples

$$\mathbb{CP}^n = \left(\mathbb{C}^{n+1} - \{0\}\right) / \mathbb{C}^\times = \{(a_1, \dots, a_n) \neq (0, \dots, 0)\} / \sim \tag{3.6}$$

where $(a_1, \dots, a_n) \sim (\lambda a_1, \dots, \lambda a_n)$ for any $\lambda \in \mathbb{C}^\times$. You can think of this as the set of lines in $\mathbb{A}^{n+1}$, whcih are in bijective correspondence with the copy of $\mathbb{A}^n$ given by the subset of $\mathbb{A}^{n+1}$ where $a_0 = 1$ through stereographic projection, except those where $a_0 = 0$.

Now, in a way $\mathbb{CP}^n$ is $\mathbb{A}^n$ with some points added at ∞. Let's consider $U_0 \subset \mathbb{P}^n$, where $U_0 = \{(a_0, \dots, a_n) : a_0 \neq 0\}$. We have a bijection $U_0 \to \mathbb{A}^n$ where $(a_0, \dots, a_n) \mapsto \left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right)$, $(1 : b_1, \dots, b_n) \leftarrow (b_1, \dots, b_n)$. Similarly, define $U_i \subset \mathbb{P}^n$ where the $a_i \neq 0$. This makes $\mathbb{CP}^n$ into an n-dimensional complex manifold with $U_i \cong \mathbb{C}^n$ as coordinate charts (you could see that the change of coordinates are holomorphic).

In the analytic topology, $\mathbb{CP}^n$ is compact and has a natural metric:

$$\mathbb{CP}^n = \left\{ (a_, \dots, a_n) : \sum_i |a_i|^2 = 1 \right\} / \sim \tag{3.7}$$

if $(a_1, \dots, a_n) \sim (\lambda a_1, \dots, \lambda a_n)$ for any $\lambda \in \mathbb{C}^\times$ where $|\lambda| = 1$. THus

$$\mathbb{CP}^n \cong \mathbb{S}^{2n+1} / \mathbb{S}^1 \tag{3.8}$$

For example, let's look at $\mathbb{CP}^1 \cong S^2$. Then $U_0 \cong \mathbb{C} \ni z = \frac{a_1}{a_0}$ and $U_1 \cong \mathbb{C} \ni w = \frac{a_0}{a_1}$. So, setwise, $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Note also that $\mathbb{S}^2 = \mathbb{S}^3 / \mathbb{S}^1$.

Definition 3.8. *$f(x_1, \dots, x_n)$ is homogeneous if $f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n)$.*

Lecture 4

Definition 4.1. *A polynomial $f \in \mathbb{C}[x_0, \dots, x_n]$ is* homogeneous *of degree d if for all $\lambda \in \mathbb{C}$,*

$$f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n) \tag{4.1}$$

Now, let $(f_1, \dots, f_k)$ be a collection of homogeneous polynomials (not necessarily of the same degree). Then

$$Z(f_1, \dots, f_k) = \{p \in \mathbb{CP}^n : f_i(p) = 0\} \tag{4.2}$$

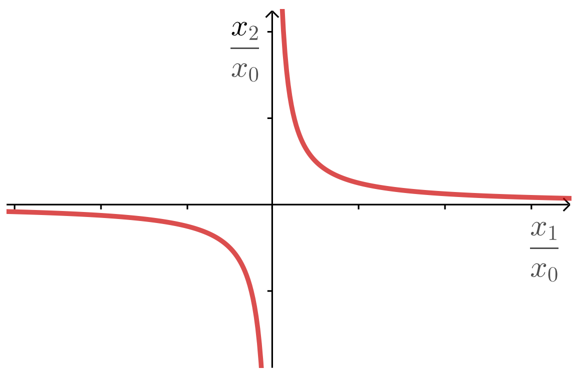
Note that a homopoly f is not a function on $\mathbb{CP}^n$ (the value $f(x_0, \dots, x_n)$ is not well defined), however the 0 set is well defined.

Definition 4.2. *$Z \subset \mathbb{CP}^n$ is an algebraic set (a.k.k. a closed set) if there are homogeneous $f_1, \dots, f_k$ such that $Z = Z(f_1, \dots, f_k)$.*

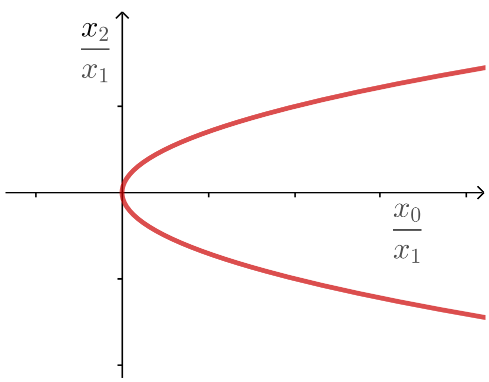
If all the f_i are linear, then their zero set Z is also called linear. It's a linear subset of $\mathbb{CP}^n$, and for example a linear subspace of $\mathbb{CP}^2$ is called a line.

Example 4.3. *Consider $Z(x_0^2 - x_1 x_2) \subset \mathbb{CP}^2$. This is a* conic plane curve. *What does this look like in the 3 affine coordinate patches it has?*

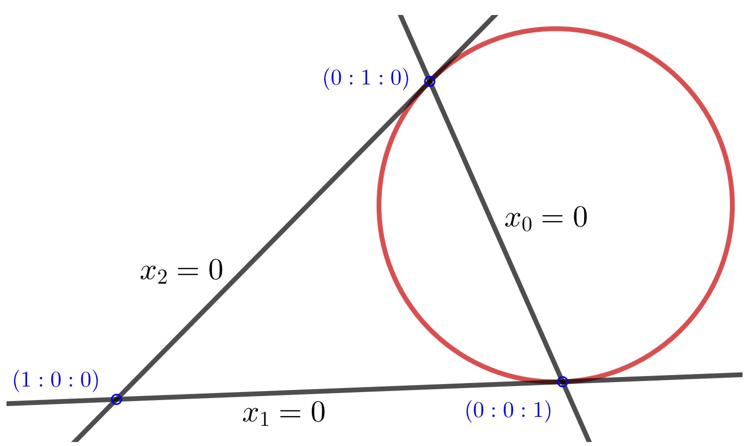
1. $x_0^2 = x_1 x_2$ in U_0, we have affine coordinates $\frac{x_1}{x_0}$, $\frac{x_2}{x_0}$, so U_0 where $x_0 \neq 0$ is the set where $1 = \left(\frac{x_1}{x_0}\right)\left(\frac{x_2}{x_0}\right)$



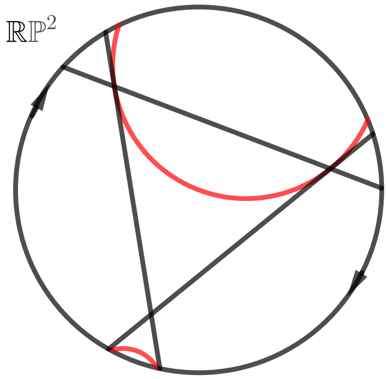
2. In U_1, $x_1 \neq 0$, and the equation is $\left(\frac{x_0}{x_1}\right)^2 = \frac{x_2}{x_1}$

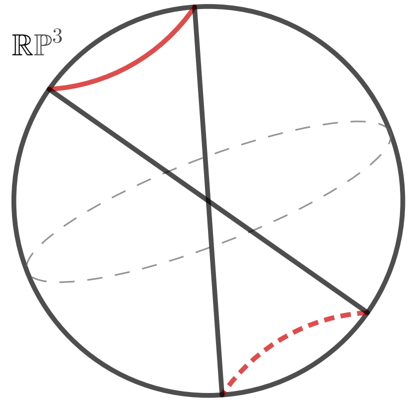


3. Schematically, we can encapsulate all the equations:



We can also get a look at this in $\mathbb{RP}^2$, which form lines in $\mathbb{RP}^3$,





Example 4.4 (twisted cubic)**.** *Let $X \in \mathbb{CP}^3$ be the image of the set map $\mathbb{CP}^1 \mapsto \mathbb{CP}^3$, $(x_0 : x_1) \mapsto (x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3)$. This map is well defined, just check equivalence classes; multiplying by lambda thing. You also need to check not all coordinates in the image are 0 at any point.*

I claim X is a closed algebraic set, so it must be the 0 set of 3 polynomials (f_0, f_1, f_2). Let $\mathbb{CP}^3$ have homogeneous coords $(z_0 : z_1 : z_2 : z_3)$. Then let

- $f_0 = z_0 z_2 - z_1^2$
- $f_1 = z_1 z_3 - z_2^2$
- $f_2 = z_0 z_3 - z_1 z_2$

So, clearly $X \subseteq Z(f_0, f_1, f_2)$. Suppose $(z_0 : z_1 : z_2 : z_3) \in f(f_0, f_1, f_2)$. Then either z_0 or $z_3 \neq 0$; shouldn't be hard to see this from the equations. If $z_0 \neq 0$, then

$$\begin{aligned}(z_0 : z_1 : z_2 : z_3) &\sim (z_0^3 : z_0^2 z_1 : z_0^2 z_2 : z_0^2 z_3) \\ &= (z_0^3 : z_0^2 z_1 : z_0 z_1^2 : z_0 z_1 z_2) \\ &= (z_0^3 : z_0^2 z_1 : z_0 z_1^2 : z_1^3)\end{aligned} \tag{4.3}$$

Now, in U_0 $(x_0 \neq 0)$, let $t = \frac{x_1}{x_0}$. we have the map $\mathbb{A}^1 \mapsto \mathbb{A}^3 = U_0$, $t \mapsto (t, t^2, t^3)$.

Surprisingly, we show in the homework that the twisted cubic $X \subset \mathbb{CP}^3$ cannot be described by fewer than 3 equations. Jargon: X is not a *complete intersection.* Consider the minors of the matrix

$$\begin{bmatrix} z_0 & z_1 & z_2 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

Aye, they are equivalent to the three equations for X. Them all being 0 is equivalent to this matrix having rank exactly 1.

Like our correspondence

$$\{\text{closed sets in } \mathbb{A}^n\} \longleftrightarrow \{\text{radical ideals } I \subset \mathbb{C}[x_1, \dots, x_n]\} \tag{4.4}$$

we would like similar statements for sets $X \subset \mathbb{CP}^n$. $I \subset \mathbb{C}[x_0, .., x_n]$ is a homogeneous ideals if it is generated by homo polys. $Z(f_1, .., f_k) = Z(I)$ for $I = (f_1, \dots, f_k)$. Conversely, given $Z \subset \mathbb{CP}^n$, $I(Z)$ is the ideal generated by homogeneous f such that $f(p) = 0$ for all $p \in Z$.

Theorem 4.5. *$Z(\cdot)$ and $I(\cdot)$ give a bijection*

$$\{Z \subset \mathbb{CP}^n \text{ closed algebraic sets}\} \longleftrightarrow \left\{ \text{homogeneous } I \subset \mathbb{C}[x_0, \dots, x_n] \text{ where } \sqrt{I} = I, \text{ except } (x_0, x_1, \dots, x_n) \right\} \tag{4.5}$$

Definition 4.6. *A closed se $X \subset \mathbb{CP}^n$ is irreducible if $X \neq X_1 \cup X_2$ a non-trivial union of closed sets. A* projective variety *is an irreducible closed set in $\mathbb{CP}^n$.*

Passing between projective to affine varieties: Let $H \subset \mathbb{CP}^n$ be the hyperplane $H = \{x_0 = 0\}$. Then $\mathbb{CP}^n - H \cong \mathbb{A}^n$, where $(x_0, \dots, x_n) \mapsto \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$, $(1 : z_1 : \dots : x_n) \leftarrow (z_1, \dots, z_n)$ Suppose $X \subset \mathbb{CP}^n$ is an algebraic set defined by $I(X) = (f_1(x_0, \dots, x_n), \dots, f_k(x_0, \dots, x_n))$.

Let $X^0 = X \cap (\mathbb{CP}^n - H)$ so $X^0 \subset \mathbb{A}^n$. Then $I(X^0) = (f_1^0, \dots, f_k^0)$ where $f_i^0(z_1, \dots, z_n) = f_i(1, z_1, \dots, z_n)$.

Conversely suppose $X \subset \mathbb{A}^n$, $X = Z(f_1, \dots, f_k)$, define $\overline{X} \subset \mathbb{CP}^n$ be the Zariski closeure of $X \subset \mathbb{A}^n \subset \mathbb{CP}^n$.

$$\begin{array}{ccc} \{Z \in \mathbb{P}^n\} & \longrightarrow & \left\{ \begin{array}{c} I \subset \mathbb{C}[x_1, \dots, x_n] \\ I = \sqrt{I},\ I \neq (x_1, \dots, x_n) \end{array} \right\} \\ \downarrow & \swarrow_{\text{Ordinary Nullstellensatz}} & \\ \left\{ \begin{array}{c} \widetilde{Z} \subset \mathbb{A}^{n+1} \text{ s.t. if } (a_0, \dots, a_n) \subset \widetilde{Z}, \text{ then} \\ (\lambda a_0, \dots, \lambda a_n) \in \widetilde{Z} \text{ for all } \lambda \text{ and } \widetilde{Z} \neq \{(0, \dots, 0)\} \end{array} \right\} & & \end{array} \tag{4.6}$$

Definition 4.7. *Let $f \in \mathbb{C}[z_1, \dots, z_n]$ be of degree d, and let $f^{homo} \in \mathbb{C}[x_0, \dots, x_n]$ be*

$$f^{homo}(x_0, \dots, x_n) = x_0^d f_i\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \tag{4.7}$$

chich is homogeneous of degree d The claim in the (next) homework is that

$$\overline{X} = Z(I(X)^{homo}) \tag{4.8}$$

where $I \in \mathbb{C}[z_1, \dots, z_n]$, $I(X)^{homo} \subset \mathbb{C}[x_0, \dots, x_n]$ is the ideal generated by f^{homo} for $f \in I$. We will also show by example that for $I = (f_1, \dots, f_k)$, it is not necessary that $I^{homo} \neq \left(f_1^{homo}, \dots, f_k^{homo}\right)$.

Lecture 5

In general

$$H = \{L(x_0, \dots, x_n) = 0\} \tag{5.1}$$

where L is a linear polynomial and let $p \in \mathbb{CP}^n$, $p \notin H$. So $H \subset \mathbb{CP}^n$ is a hyperplane. There is a linear change of coordinates $x_0, \dots, x_n \to x'_0, \dots, x'_n$ so that $\widetilde{H} = \{x'_0 = 0\}$, and $\widetilde{p}$ is the x'_0-axis $= \{x'_1 = \dots = x'_n = 0\}$. Then $\mathbb{CP}^n \backslash H \cong \mathbb{A}^n$ and $p = 0 \in \mathbb{A}^n$. Thus you can really identify $H \cong$ lines in $\mathbb{CP}^n$ passing through p.

Consider $C \subset \mathbb{CP}^n$, where $C = Z(Q)$ where $Q(x_0, x_1, x_2)$ is an irreducible quadric polynomial. We would like to prove this is an isomorphism of projective varieties, but we need to define morphisms of projective varieties first.

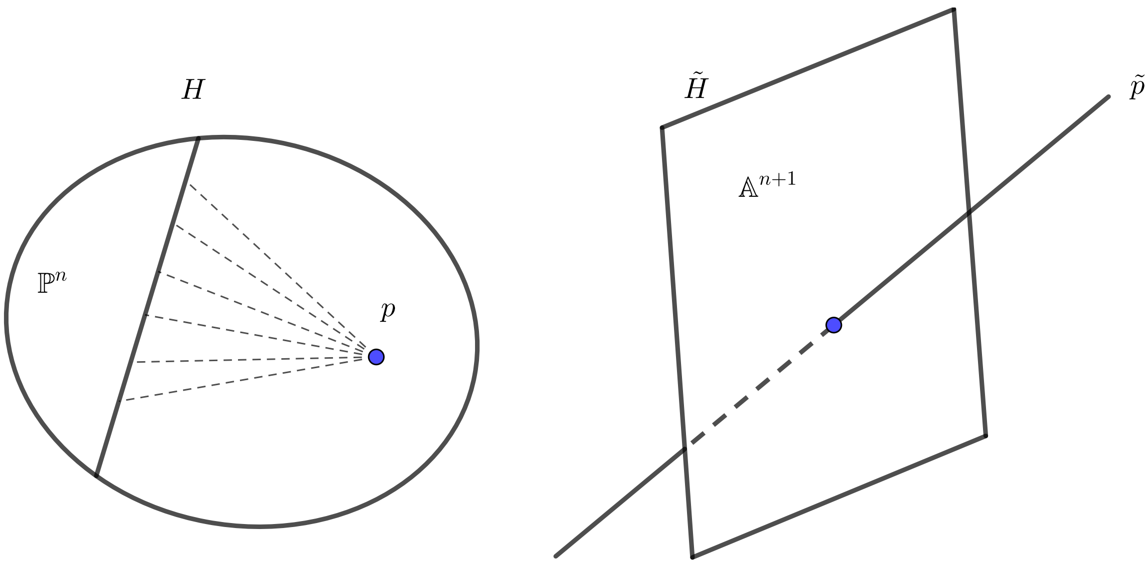
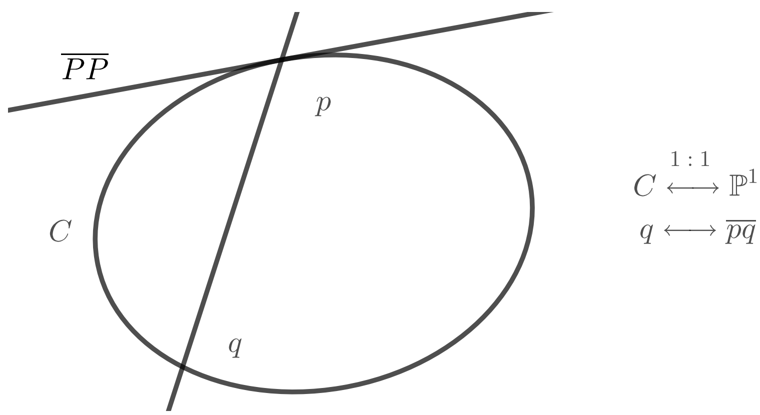


Figure 2: Correspondence between hyperplanes



Example 5.1. *From Last time, consider* $\mathbb{CP}^1 \mapsto \mathbb{CP}^3$, $(x_0 : x_1) \mapsto (x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3)$ *should be a morphism, right?*

Definition 5.2. *Let* $V \subset \mathbb{A}^n$, $W \subset \mathbb{A}^m$ *be affine varieties. A set map* $\alpha : V \mapsto W$ *is a* morphism *if there are polynomials* $f_1, \dots, f_m \in \mathbb{C}[x_1, \dots, x_n]$ *so* $\alpha(x_1, \dots, x_n) = (f_1(x1, \dots, x_n), \dots, f_m(x1, \dots, x_n))$. *I.e. morphisms are the restrictions of polynomial maps* $\mathbb{A}^n \mapsto \mathbb{A}^m$.

Definition 5.3. *Let* $V \subset \mathbb{A}^n$ *be an affine variety. The ring of functions on* V *(a.k.a. the affine coordinate ring of* V*)*

$$A(V) : \mathbb{C}[x_1, \dots, x_n]/I(V) \tag{5.2}$$

Also sometimes called $\mathbb{C}[V]$, $R[V]$, $\mathcal{O}_V$, *or* $\mathcal{O}_V(V)$.

Note that $f \in A(V)$ can be regarded as actual functions on V. Since $(x_1, \dots, x_n) \in V$, if $f = \widetilde{g}$ mod $I(V)$, then $f(x_1, \dots, x_n) = \widetilde{f}(x_1, \dots, x_n)$.

Given $\alpha : V \mapsto W$ morphism of varieties, we get a homomorphism of algebras

$$\begin{aligned} \alpha^* : A(W) &\mapsto A(V) \\ \alpha &\mapsto \alpha^*(g) = g \circ \alpha \end{aligned} \tag{5.3}$$

and the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & W \\ & \underset{\alpha^*(g)}{\searrow} & \downarrow g \\ & & \mathbb{C} \end{array} \tag{5.4}$$

Conversely let $\lambda : A(W) \mapsto A(V)$ be an algebra homomorphism. Then there is $\alpha : V \mapsto W$ such that $\alpha^* = \lambda$.

$$\lambda : \mathbb{C}[y_1, \dots, y_m]/I(W) \mapsto \mathbb{C}[x_1, .., x_n]/I(V) \tag{5.5}$$

so choose $f_1, \dots, f_m \in \mathbb{C}[x_1, \dots, x_n]$ such that $f_i = \lambda(y_i)$. This defines a map $\alpha : \mathbb{A}^n \mapsto \mathbb{A}^m$, $\alpha(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$.

If $(x_1, \dots, x_n) \in V$, then $\alpha(x_1, \dots, x_n) \in W$ since if $g \in I(W)$,

$$\begin{aligned} g(f_1((x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)) &= g(\lambda(y_1), \dots, \lambda(y_m)) \mod I(V) \\ &= \lambda(g(y_1, \dots, y_m)) \mod I(V) \\ &= 0 \end{aligned} \tag{5.6}$$

This construction is canonical and gives a bijective correspondence

$$\operatorname{Mor}(V, W) = \operatorname{Hom}(A(W), A(V)) \tag{5.7}$$

That is, we have a *categorical equivalence*.

$$\begin{gathered} \{\text{Category of affine varieties over } \mathbb{C}\} \longleftrightarrow \left\{ \begin{matrix} \text{Category of finitely generated algebras} \\ \text{over } \mathbb{C} \text{ which are integral domains} \end{matrix} \right\} \\ V \longrightarrow A(V) \\ (f : V \mapsto W) \longrightarrow (f^* : A[W] \mapsto A[V]) \end{gathered} \tag{5.8}$$

Another correspondence from this is

$$\begin{aligned} \text{Varieties} &\longleftrightarrow \text{prime ideals } I \subset \mathbb{C}[x_1, \dots, x_n] \\ \text{Variety } V &\longleftrightarrow A[V] = \mathbb{C}[x_1, \dots, x_n]/I(V) \end{aligned} \tag{5.9}$$

If A is a finitely generated algebra over $\mathbb{C}$ (which is an integral domain), then pick some set of generators $x_1, \dots, x_n$ so that

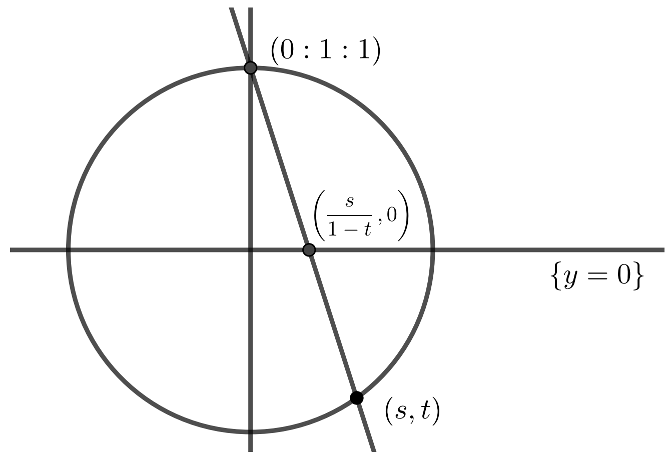
$$A = \mathbb{C}[x_1, \dots, x_n]/I \tag{5.10}$$

We could try doing the same strategy with projective varieties; define morphisms in terms of homogeneous coord rings. Consider $C \subset \mathbb{CP}^2$, $x^2 + y^2 - z^2 = 0$. Let's try to define a map $C \mapsto \mathbb{CP}^1$ by $(x : y : z) \mapsto (x : z - y)$. The problem is that this is not defined at the point $(0 : 1 : 1)$, so it doesn't give us a morphism. Now, let's look at the addine coordinate patch $z \neq 0$. The curve is

$$\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 = 0 \tag{5.11}$$

Let $\frac{x}{z} - s$, and $\frac{y}{z} = t$. Then the map is $\left(\frac{x}{z} : \frac{y}{z} : 1\right) \mapsto \frac{x}{z-y}$, or in s, t language, $(s, t) \mapsto \frac{s}{1-t}$, which kind of makes it stereographic projection. The missing point should be the point at infinity, so it should be well defined? Well, consider

$$\begin{aligned} (x : y : z) \mapsto &(x : z - y) \\ \sim &(x(z + y) : z^2 - y^2) \\ \sim &(x(z + y) : x^2) \\ \sim &(z + y : x) \end{aligned} \tag{5.12}$$

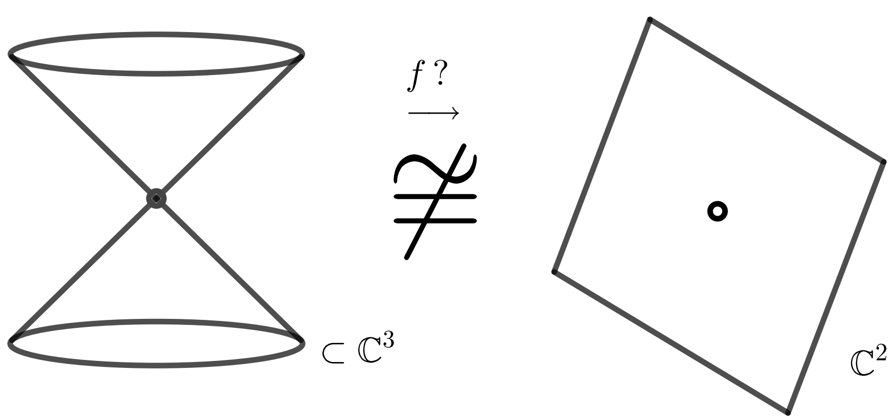


so we have "two" maps $(x : y : z) \mapsto (x : z - y)$, $(x : y : z) \mapsto (z + y : x)$ which agree as maps $C \mapsto \mathbb{CP}^1$ when defined, but the first is not defined at $(0 : 1 : 1)$, second not defined at $(0 : 1 : -1)$. So what we're finding that affine varieties are like coordinate charts! For morphisms of affine varieties we have a nice correspondence, but projective varieties are kind of like global morphisms.

Lecture 6

We have a contravariant equivalence of categories equation (5.8). We want to make sense of defining morphisms locally in a coherent way. In particular, the previous example $C \mapsto \mathbb{P}^1$ should be an isomorphism. Note that the homogeneous coordinate rings are not isomorphic.

Definition 6.1. $\mathbb{C}[x, y, z]/(x^2 + y^2 - z^2) \not\cong \mathbb{C}[x, y]$ *are not isomorphic. These are the coordinate rings of the corresponding cone-like affine varieties. The one on the left hand side is the homogeneous coordinate ring.*



Suppose that they were isomorphic. Then there is a polynomial map f which induces a homeomorphism of $\{x^2 + y^2 = z^2\} \mapsto \mathbb{C}^2$. But then we have a homeomorphism between $\{x^2 + y^2 = z^2\} - \{0, 0, 0\} \mapsto \mathbb{C}^2 - \{0, 0\}$.

Explicitly, we actually have

$$\left\{x^2 + y^2 = z^2\right\} - \{0, 0, 0\} \cong \left(\mathbb{C}^2 - \{0, 0\}\right) / \pm 1 \tag{6.1}$$

by the map

$$\begin{aligned}\left(\mathbb{C}^2 - \{0,0\}\right)/\pm 1 &\mapsto \left\{x^2+y^2=z^2\right\} - \{0,0,0\} \\ \pm(u,v) &\mapsto \left(u^2-v^2, 2uv, u^2+v^2\right)\end{aligned} \tag{6.2}$$

with inverse

$$\left(\sqrt{\frac{1}{2}(x+z)}, \sqrt{\frac{1}{2}(z-x)}\right) \leftarrow (x,y,z) \tag{6.3}$$

with the relative sign fixed by $\sqrt{\frac{1}{2}(x+z)} \cdot \sqrt{\frac{1}{2}(z-x)} = \frac{1}{2}y^2$. Now, the contradiction is that $\mathbb{C}^2 - \{(0,0)\}/\pm 1$ cannot be homeomorphic to $\mathbb{C}^2 - \{(0,0)\}$, since the action of ± 1 on $\mathbb{C}^2 - \{(0,0)\}$ is free so they have different π_1.

To talk about maps on Zariski open sets, we need to first talk about functions defined on open sets back in the affine case.

Let $X \subset \mathbb{A}^n$ be an affine variety. We know the functions on X are

$$A[X] = \mathbb{C}[x_1, \dots, x_n]/I(X) \tag{6.4}$$

But how should we define functions on some open set $U \subset \mathbb{A}^N$? Since X is a variety, (6.4) is an integral domain, so we can talk about its "field of fractions".

Definition 6.2. *The field of rational functions or field of fractions on X is called $K(X)$, and it is the quotient field of $A[X]$. That is,*

$$K(X) = \left\{\frac{f}{g} : f, g \in A[x], g \neq 0\right\} \tag{6.5}$$

modulo the usual equivalence of fractions, with the usual multiplication and addition of fractions.

Even though we call the elements of $K(X)$ "rational functions", they are not functions really since they are not defined everywhere. A point $p \in X$ is called a regular point of $\frac{f}{g}$ if $g(p) \neq 0$.

Definition 6.3. *For $p \in X \subset \mathbb{A}^n$, the local ring of X at p is*

$$\mathcal{O}_{X,p} = \left\{\frac{f}{g} \in K(X) : g(p) \neq 0\right\} \subseteq K(X) \tag{6.6}$$

Definition 6.4. *Let $U \subseteq X$ be Zariski open, and define the ring of regular functions on U be*

$$\mathcal{O}_X(U) = \bigcap_{p \in U} \mathcal{O}_{X,p} \tag{6.7}$$

Note that these $\frac{f}{g}$ functions are equivalence classes actually, so it's not quite right to just say $g(p) \neq 0$. You can't just refer to g specifically. What is meant by this notation is that there is *some representative* $\frac{f}{g}$ in the equivalence class for which $g(p) \neq 0$.

Remark 6.1. *The regular functions on U are not necessarily given by $\frac{f}{g}$ with $g(p) \neq 0$ for all $p \in U$. The equivalence classes bit can mess with this.*

Example 6.5. *Let $X \subseteq \mathbb{A}^4$ be given by $x_1x_4 = x_2x_3$, and let $U \subseteq X$ be*

$$U = X - \{x_2 = 0\} \cap \{x_4 = 0\} = X - Z((x_2, x_4)) \tag{6.8}$$

So Y is the set $(x_1, \dots, x_4) \in X$ so that $x_2 \neq 0$ or $x_4 \neq 0$. Then $\frac{x_1}{x_2} = \frac{x_3}{x_4} \in K(X)$ are regular on all of U.

Proposition 6.6. *Let $X \subseteq \mathbb{A}^n$ be an affine variety, $f \in A[X]$. Let*

$$X_f = \{p \in X : f(p) \neq 0\} = X - Z(f) \tag{6.9}$$

Then

$$\mathcal{O}_X(X_f) = A[X]_f = \left\{ \frac{h}{f^r} : h \in A[X], r \geq 0 \right\} \tag{6.10}$$

Proof. Clearly $A[X)]_f \subseteq \mathcal{O}_X(X_f)$. Conversely, suppose $\phi \in \mathcal{O}(X_f) \subset K(X)$. Let

$$J_\phi = \{g \in A[X] : g \cdot \phi \in A\} \subseteq A[X] \tag{6.11}$$

The ideal of functions clearing the denominator of ϕ. Then For all $p \in X_f$, there are $h, g \in A[X]$, we have $g(p) \neq 0$ and $\phi = \frac{h}{g}$. Thus, $g \in J$ and $g(p) \neq 0$. Then $Z(J) \subseteq Z(f)$ are closed sets in X, so $(f) \subseteq \sqrt{J} \implies f^r \in J$ for some r, which means $\phi = \frac{h}{f^r}$ and so $\mathcal{O}(X_f) \subseteq A[X]_f$. □

In the special case where $f = 1$, this implies $\mathcal{O}_X(X) = A[X]$.
Notice the open set $\mathbb{A}^1 - \{0\} \subset \mathbb{A}^1$ is isomorphic to a (closed) affine variety.

$$\mathcal{O}_{\mathbb{A}^1}\left(\mathbb{A}^1 - \{0\}\right) = \mathbb{C}[x]_X = \mathbb{C}[x, x^{-1}] \cong \mathbb{C}[x, y]/(xy - 1) \tag{6.12}$$

More generally, $X = Z(f, \dots, f_k) \subset \mathbb{A}^n$. If $g \in A[X]$, then $X_g = X - Z(g)$ is isomorphic to

$$Z\left(f_1(x_1, \dots, x_n), \dots, f)k(x_1, \dots, x_n), 1 - tg(x_1, \dots, x_n)\right) \subset \mathbb{A}^{n+1} \tag{6.13}$$

Distinguish open sets $X_g \subseteq X$ of affine varieties are isomorphic to (closed) affine varieties , however not all open sets are isomorphic to affine varieties. Let $X = \mathbb{A}^2 - \{(0,0)\} \subset \mathbb{A}^2$. Since

$$\mathbb{A}^2 - \{(0,0)\} = \left\{\mathbb{A}^2 - \{x = 0\}\right\} \cup \left\{\mathbb{A}^2 - \{y = 0\}\right\} \tag{6.14}$$

But then

$$\begin{aligned} \mathcal{O}_{\mathbb{A}^2}\left(\mathbb{A}^2 - \{(0,0)\}\right) &= \mathcal{O}_{\mathbb{A}^2}\left(\mathbb{A}^2 - \{x = 0\}\right) \cap \mathcal{O}_{\mathbb{A}^2}\left(\mathbb{A}^2 - \{y = 0\}\right) \\ &= \left\{\frac{f(x,y)}{x^r}\right\} \cap \left\{\frac{g(x,y)}{y^s}\right\} \end{aligned} \tag{6.15}$$

and

$$\mathcal{O}_{\mathbb{A}^2}\left(\mathbb{A}^2 - \{(0,0)\}\right) \cong \mathbb{C}[x, y] \tag{6.16}$$

So the $\mathbb{A}^2 - \{(0,0)\}$ cannot be isomorphic to an affine variety otherwise it would have to be $\mathbb{A}^2$, and $\mathbb{C}^2 - \{(0,0)\} \not\cong \mathbb{C}^2$ for topological reasons. So $\mathbb{A}^2 - \{(0,0)\}$ is not an affine variety but it is covered by open sets which are.

Lecture 7

Recall $X \subset \mathbb{A}^n$ is an affine variety $K(X) = \left\{ \frac{f}{g} : f, g \in \mathbb{A}[x] = \mathbb{C}[x_1, ..., x_n]/I(X), g \neq 0 \right\}$. The local ring $\mathcal{O}_{X,p} = \left\{ \frac{f}{g} \in K(X) : g(p) \neq 0 \right\}$ for $p \in X$. If $\mathcal{O}_X(U) = \bigcap_{p \in U} \mathcal{O}_{X,p}$. We have distinguished open sets $X_f \subset X$ which is $X_f = X \backslash Z(f)$ for $f \in \mathbb{A}[x] = \mathbb{C}[x_1, \dots, x_n]/I(X)$. The localized ring is $\mathcal{O}_X(X_f) =: \mathbb{A}[x]_f$. X_f is isomorphic to an affine variety. $X_f \subset \mathbb{A}^{n+1}$ with coordinates $(x_1, \dots, x_n, t)$. In particular $I(X_f) = (f_1, \dots, f_n, ft - 1)$. However for example

$$\mathcal{O}_{\mathbb{A}^2}\left(\mathbb{A}^2 \backslash \{(0,0)\}\right) \cong \mathbb{C}[x, y] \tag{7.1}$$

so the open set $\mathbb{A}^2 \backslash \{(0,0)\}$ is not isomorphic to an affine variety. However

$$\mathbb{A}^2 \backslash \{(0,0)\} = \mathbb{A}^2 \backslash \{x\text{-axis}\} \cup \mathbb{A}^2 \backslash \{y\text{-axis}\} \tag{7.2}$$

so it has an open covering by affine varieties. We want to build a category of abstract varieties – spaces with an open covering by affine varieties. We then want to know what the morphisms are. Our category should include projective varieties and open sets in affine varieties. The basic tool for this is "sheaves".

7.2 Sheaves

Let X be a topological space and consider the category *open*(X) where the objects are open sets $U \subset X$ and the morphisms are open inclusions $U \subset V$. Let *Rings* be the category of commutative rings with a unit.

Definition 7.1. *A* pre-sheaf $\mathcal{F}$ *is a contra-variant functor* $\mathcal{F} : \textit{open}(X) \mapsto \textit{Rings}$. *A functor just takes objects to objects and functors to functors, and contravarient means it reverses the arrows. If we have* $U \subseteq V$, *we get a homomorphism* $\mathcal{F}(V) \xrightarrow{\rho_{UV}} \mathcal{F}(U)$ *by a restriction* ρ_{UV} *such that if* $U \subset V \subset W$, *the diagram*

$$\begin{array}{ccc} \mathcal{F}(W) & \xrightarrow{\rho_{VW}} & \mathcal{F}(V) \\ & \underset{\rho_{UW}}{\searrow} & \downarrow \rho_{UV} \\ & & \mathcal{F}(U) \end{array} \tag{7.3}$$

commutes.

For example, if X is a smooth manifold, then C^∞ be the sheaf of smooth functions on X. so if $U \subset X$ is open, then $C^\infty(U) =$ ring of smooth functions on $U = \{f : U \mapsto \mathbb{R} : f \text{ is } C^\infty\}$. Now if we have $C^\infty(V) \mapsto C^\infty(U)$ for $U \subset V$, we write $f \mapsto f\Big|_U$ is the restriction map.

Another example, if $X \subset \mathbb{A}^n$ an affine variety, $\mathcal{O}_X$ is the sheaf of regular functions on X (called the structure sheaf). Here, $\mathcal{O}_X(U)$ is the ring of regular functions.

Some jargon: elements $f \in \mathcal{F}(U)$ are called "sections of $\mathcal{F}$ over U". We define the *stalk* of $\mathcal{F}$ at p to be the ring of equivalence classes $\mathcal{F}_p := \mathit{lim}_{U \ni p} \mathcal{F}(U)$, which "sort of" concretely = equivalence classes on (ϕ, U), $\phi \in \mathcal{F}(U)$, $p \in U$ where $(\phi_1, U_1) \sim (\phi_2, U_2)$ iff there is a $V \subseteq U_1 \cap U_2$ and $\psi \in \mathcal{F}(V)$ so that $\phi = \phi_1\Big|_V = \phi_2\Big|_V$.

For example, in $C^\infty(\mathbb{R}^n)$, $p \in \mathbb{R}$, $C_p^\infty =$ germ of functions at p.

For another example, $\mathcal{O}_{X,p}$ the stalk at p of the structure sheaf $\mathcal{O}_X$ is the local ring $\mathcal{O}_{X,p}$.

A third example, let X be a space, $p \in X$ is a point. The *skyscraper sheaf* $\mathcal{F}$ is the sheaf where

$$\mathcal{F}(U) = \begin{cases} 0 \text{ ring} & p \notin U \\ \mathbb{C} & p \in U \end{cases} \tag{7.4}$$

Definition 7.2. *A pre-sheaf is called a sheaf if it satisfies the following "gluing" property: If $U \in X$ is open and $\{U_i\}$ is an open cover of U and we have sections $f_i \in \mathcal{F}(U_i)$ are sections such that $f_i\Big|_{U_i \cap U_j} = f_j\Big|_{U_i \cap U_j}$ for all $i \neq j$, then there is a unique $f \in \mathcal{F}(U)$ such that $f\Big|_{U_i} = f_i$*

Some examples of presheaves that are not sheaves:

1. The presheaf of constant functions. To see this, suppose you had two disjoint sets with different constant functions on them. Then there is no global constant function which can satisfy both.

2. The presheaf of bounded real functions on $\mathbb{R}$ (or any non-compact space). Then just let e^x and cover $\mathbb{R}^n$ with bounded open sets; restricting e^x to each of them. Each are bounded and agree on overlaps, but cannot make a global function for all of them.

These examples tell you that *bounded* and *constant* are not local conditions.

Definition 7.3. *If $f : X \mapsto Y$ is a continuous map of topological spaces and $\mathcal{F}$ is a sheaf on X, then the* pushforward sheaf *$f_*\mathcal{F}$ is the sheaf on Y given by $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$.*

Sheaves can be used to define morphisms. For example, if X and Y are smooth manifolds and C^∞_X and C^∞_Y are the sheaves of smooth functions, then a continuous map $\phi : X \mapsto Y$ is smooth if it pulls back smooth functions to smooth functions, i.e. for all $U \subset Y$, $f \in C^\infty_Y(U)$, then $\phi^*(f) = f \circ \phi \in C^\infty_X(f^{-1}(U))$. In other words, ϕ is smooth if it induces a sheaf map $\phi^* : C^\infty_Y \mapsto \phi_* C^\infty_X$ (note: these are two sheaves on Y) meaning that homomorphisms

$$\phi^* : C^\infty_Y(U) \mapsto (\phi_* C^*_X)(U) \tag{7.5}$$

for all $U \subset Y$ that commute with restriction. We can formalize this idea.

Definition 7.4. *A topological space X equipped with a sheaf of rings $\mathcal{O}_X$ is called a ringed space. A morphism of ringed spaces is a pair (f, f^*) where $f : X \mapsto Y$ is continuous topologically, and $f^* : \mathcal{O}_Y \mapsto f_*\mathcal{O}_X$ is a sheaf map (i.e. a collection $\mathcal{O}_Y(U) \xrightarrow{f^*} \mathcal{O}_X(f^{-1}(U))$ which commutes with restriction).*

Proposition 7.5. *Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be affine varieties. Then $f : X \mapsto Y$ is a morphism if and only if (f, f^*) is a morphism of the ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$*

Proof. $\Leftarrow$ is easy. If (f, f^*) is a morphism $f^*L\mathcal{O}_Y \mapsto f_X\mathcal{O}_X$, apply this to

$$\begin{aligned} \mathcal{O}_Y(Y) &\xrightarrow{homo} \mathcal{O}_X(f^{-1}(Y)) \\ A[Y] &\xrightarrow{homo} A[x] \end{aligned} \tag{7.6}$$

"$\Rightarrow$", beginning with a homomorphism $\mathcal{O}_Y(Y) \xrightarrow{f^*} \mathcal{O}_X(X)$, we first show that F^* induces ring maps on local rungs for all p, $f^*L\mathcal{O}_{Y,f(p)} \mapsto \mathcal{O}_{X,p}$. Let $\phi = \frac{g}{h} \in \mathcal{O}_{y,f(p)}$, so $g, h \in \mathcal{O}_Y(Y)$ and $h(f(p)) \neq 0$. Then

$$f^*\left(\frac{g}{h}\right) = \frac{f^*(g)}{f^*(h)} \in K(X) \tag{7.7}$$

which is in $\mathcal{O}_{X,p}$ and thus $h(f(p)) \neq 0 \iff (f^*(h))(p) \neq 0$. Similar argument applies to $\mathcal{O}_Y(U) \xrightarrow{f^*} \mathcal{O}_X(f^{-1}(U))$. $\square$

Lecture 8

Definition 8.1. *A pre-variety is a connected ringed space $(X, \mathcal{O}_X)$ such that there exists a finite open cover $\{U_i\}$ of X such that $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic as a ringed space to an affine variety.*

A morphism of pre-varieties is a morphism of ringed spaces of this type.

Recall that if $X \subset \mathbb{A}^n$ is an affine variety, then $A[x] = \mathbb{C}[x_1, \dots, x_n]/I(X)$ is the coordinate ring, and

$$\mathcal{O}_X(U) = \left\{ \frac{f}{g} \in k(X) \mid f, g \in A[x],\, g \neq 0 \text{ for some } p \in U \right\} \tag{8.1}$$

We will have an additional condition to get rid of the "pre" that rule out some unpleasant cases. First, a few examples.

8.3 Projective varieties are (pre-)varieties

$$X = Z(f_1, \dots, f_k) \subset \mathbb{P}^n \tag{8.2}$$

for f_i homogeneous in $\mathbb{C}[x_0, \dots, x_n]$. Let $I(x) = (f_1, \dots, f_k)$ and

$$K(X) = \left\{ \frac{f}{g}, f, g \in \mathbb{C}[x_0, \dots, x_n]/I(X) \text{ homogeneous of the same degree, } g \not\equiv 0 \right\} \tag{8.3}$$

Note that at regular values $p \in X$, $\frac{f}{g}$ is an actual (rational) function.

$$\mathcal{O}_X(U) = \left\{ \frac{f}{g} \in K(X) \mid \frac{f}{g} \text{ is regular at } p \text{ for all } p \in U \right\} \tag{8.4}$$

Btw, something that is not obvious is that $\mathcal{O}_X(X)$ (global functions on X) are constant so $\mathcal{O}_X(X) \cong \mathbb{C}$. We may prove it later but it is hard.

To see that $(X, \mathcal{O}_X)$ for $X \in \mathbb{P}^n$ is a pre-variety, we show that $X_i = X \cap U_i$ ($U_i \cong \mathbb{A}^n$) so that $X_i \subset \mathbb{A}^n$. So all we need to do is check that

$$\left(X_i . \mathcal{O}_X \Big|_{X_i} \right) \cong (X_i, \mathcal{O}_{X_i}) \tag{8.5}$$

Let $Y \subset \mathbb{A}^n$ be

$$Y = Z(g_1(y_1, \dots, y_n), \dots, g_k(y_1, \dots, y_n))$$

where $y_i = \frac{x_i}{x_0}$ and $g_i(y_1, \dots, y_n) = f_i(1, y_1, \dots, y_n)$. Then define

$F : X_0 \mapsto Y$, $(x_0 : \dots : x_n) \mapsto \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$ and $F^{-1} : Y \mapsto X_0$, $(y_1, \dots, y_n) \mapsto (1, y_1 : \dots : y_n)$.

Then F is a morphism of ringed spaces: suppose $\frac{P(y_1, \dots, y_n)}{q(y_1, \dots, y_n)}$ is regular on $U \subset Y$. Then

$$F^* \left(\frac{P(y_1, \dots, y_n)}{q(y_1, \dots, y_n)} \right) = \frac{P(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})}{q(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})} \tag{8.6}$$

Now we can clear denominators with a x_0^d for some d. The result is a degree d homogeneous polynomial over a degree d homogeneous polynomial, and the denominator does not vanish on the preimage $F^{-1}(U)$. So $F^* \left(\frac{p}{q}\right) \in \mathcal{O}_{X_0}(F^{-1}(U))$. Proving the reverse inclusion is similar, and also "boring".

8.4 A very important idea

Let $X \subset \mathbb{P}^n$ be a projective variety and suppose $f_0, \dots, f_m \in \mathbb{C}[x_0, \dots, x_n]/I(X)$ homogeneous of the same degree, and suppose that for all $p \in X$, $f_i(p) \neq 0$ for some i (jargon: this is a "basepoint-free linear system). Then $f : X \mapsto \mathbb{P}^m$, $p \mapsto (f_0(p) : \dots : f_m(p))$ is a morphism.

First, this is well defined set theoretically:

$$\begin{aligned} p = (x_0 : \dots : x_n) &\mapsto f_0(x_0, \dots, x_n) : \dots : f_m(x_0, \dots, x_n) \\ (\lambda x_0 : \dots : \lambda x_n) &\mapsto \left(\lambda^d f_0(x_0, \dots, x_n) : \dots : \lambda^d f_m(x_0, \dots, x_n)\right) \end{aligned} \tag{8.7}$$

To check that $f : X \mapsto \mathbb{P}^m$ is a morphism, it suffices to check an open cover satisfies that

$$f^* \left(\mathcal{O}_{\mathbb{P}^m}(U_i)\right) \subseteq \mathcal{O}_X(f^{-1}(U_i)) \tag{8.8}$$

so let $W_0 = f^{-1}(U_0)$, $f : W_0 \mapsto U_0 \cong \mathbb{Q}^n$ be given by

$$(x_0 : \dots : x_n) \mapsto \left(\frac{f_1(x_0, \dots, x_n)}{f_0(x_0, \dots, x_n)}, \dots, \frac{f_m(x_0, \dots, x_n)}{f_0(x_0, \dots, x_n)}\right) \tag{8.9}$$

with $f(x_0, \dots, x_n) \neq 0$ for all $(x_0, \dots, x_n) \in W_0$. Given $g \in \mathcal{O}_{\mathbb{P}^m}(U_0) \cong \mathbb{C}[z_1, \dots, z_m]$, we have

$$f^*(g) = g\left(\frac{f_1(x_0, \dots, x_n)}{f_0(x_0, \dots, x_n)}, \dots, \frac{f_m(x_0, \dots, x_n)}{f_0(x_0, \dots, x_n)}\right) \tag{8.10}$$

and each of the $\frac{f_i(x_0, \dots, x_n)}{f_0(x_0, \dots, x_n)} \in \mathcal{O}_X(W_0)$ since f_0 is not 0 on W_0 and $\deg f_i = \deg f_0$.

A special case of this idea is the *Varonese embedding*: this is a map

$$\mathbb{P}^n \mapsto \mathbb{P}^{\binom{n}{d+n}-1}(x_0 : \dots : x_n) \mapsto \left(\dots : x_0^{i_0} x_1^{i_1} \dots x_n^{i_n} : \dots\right) \tag{8.11}$$

so the entries are monomials of degree d. One special-er case of this is the twisted cubic. Another could be $\mathbb{P}^2 \mapsto \mathbb{P}^5$, $(x : y : z) \mapsto (x^2 : y^2 : z^2 : xy : xz : yz)$.

A non-trivial example: let $C \subset \mathbb{P}^2$ be the cubic curve with the equation $zy^2 = x(x^2 - z^2)$. We can geometrically define a set map $\phi : C \mapsto C$ which we will show is a morphism. We will see it is also an involution ($\phi \circ \phi = id$). (This is secretly related to the group law on C, shhh)

Let $a = \frac{x}{z}$, $b = \frac{y}{z}$ be the affine coordinates on $U_2 \subset \mathbb{P}^2$, then $C_2 = C \cap U_2$ has equation $b^2 = a^3 - a = a(a-1)(a+1)$

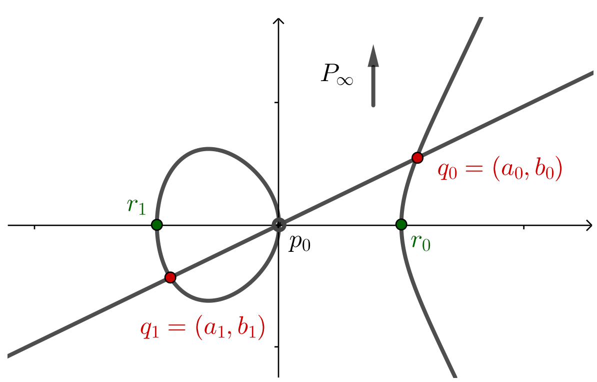


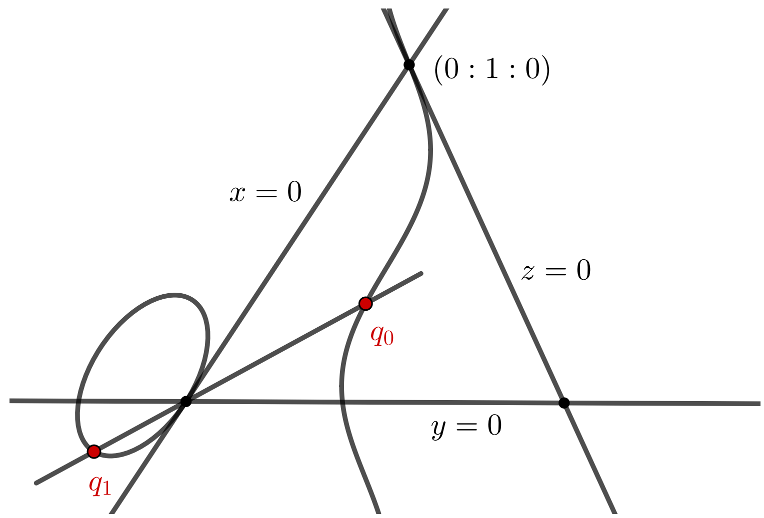
Figure 3: The curve $zy^2 = x(x^2 - z^2)$ defining a stereographic projection-like map ϕ

Now define $\phi(q_0) = q_1$ based on the above figure. We also have $\phi(p_0) = P_\infty$. So $C \backslash C_2 = (0 : 1 : 0) = P_\infty$, which lies on the $\{x = 0\}$ line which is the b-axis in affine $\mathbb{A}^2$.

Points on the line $\overline{p_0 q_0}$ are given by (ta_0, tb_0), so to solve for $q_1 = (a_1, b_1)$, it will be one of the solutions to

$$(tb_0)^2 = t^3 a_0^3 - ta_0 \qquad b_0^2 = a_0^3 - a_0 \tag{8.12}$$

Lecture 9



After some manipulating of the first equation there, we end up with the fact that $t = 1$ or $t = -\frac{1}{a_0^2}$, so basically if $q_0 = (a_0, b_0)$, we have $q_1 = \left(-\frac{1}{a_0}, -\frac{b_0}{a_0^2}\right)$. So we can write

$$\phi : \left(\frac{x}{z} : \frac{y}{z} : 1\right) \mapsto \left(-\frac{z}{x} : -\frac{y}{z}\frac{z^2}{x^2} : 1\right) \tag{9.1}$$

thus

$$\phi(x : y : z) = \left(-zx : -yz : x^2\right) \tag{9.2}$$

This is well defined except at $(0 : 0 : 1)$ and $(0 : 1 : 0)$. However we can rewrite it as $(yz : x^2 - z^2 : -yz)$ which is defined at $(0 : 0 : 1)$, this point gets sent to $(0 : 1 : 0)$. We could have also written $\phi(x : y : z) = (x^2 : xy : -y^2 - z^2)$ shows that the other point is defined as well.

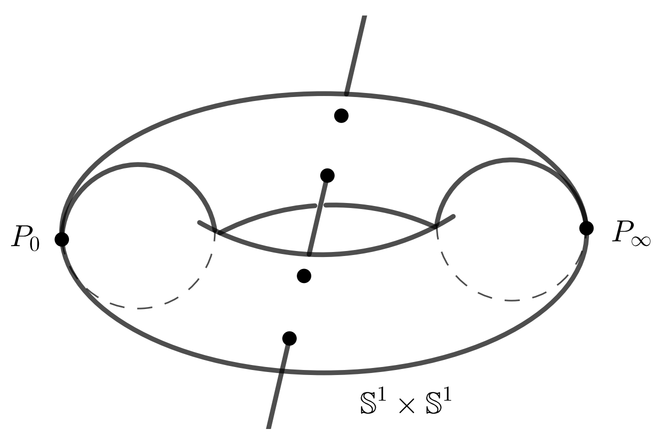


Figure 4: Under ϕ, there are 4 fixed points.

So we need to show for an open affine cover of C, ϕ pulls back regular functions to regular functions. Let $U_0, U_1, U_2 \subset \mathbb{P}^2$ affine opens $C_i = C \cap U_i$, $\{c_i\}$ an affine cover. Note $\phi^{-1}(c_i)$ is open in C. In particular let

- $c_0 = C - \{p_0, p_\infty\}$
- $c_1 = C - \{r_0, p_0, r_1\}$
- $c_2 = C - \{p_\infty\}$

so just $\{c_1, c_2\}$ give an open cover. Note $\phi^{-1}(C_1) = C - \{p_\infty, r_0, r_1\}$ and $\phi^{-1}(c_2) = C - \{p_0\})$. So $\phi^{-1}(c_1) \subset C_2$ and $\phi^{-1}(c_2) \subset C_2 \cup C_1$.

Consider

$$\begin{aligned} \phi : \phi^{-1}(c_1) \mapsto & C_1 \\ (a,b) \mapsto & (xz : yz : -x^2) \\ = & \left(\frac{x}{y} : 1 : \frac{-x^2}{yz}\right) \\ = & \left(\frac{b}{a^2-1} : 1 : \frac{-ab}{a^2-1}\right) \end{aligned}$$

So if $c = \frac{x}{y}$ and $d = \frac{z}{y}$ are coordinates on c_1, and $g(c,d) \in \mathcal{O}_C(C_1)$, then

$$(\phi^*(g))(a,b) = g\left(\frac{b}{a^2-1}, \frac{-ab}{a^2-1}\right) \in \mathcal{O}_C\left(\phi^{-1}(c_1)\right) \tag{9.3}$$

$\frac{b}{a^2-1}$ and $\frac{-ab}{a^2-1}$ are regular on $C_2 - \{r_0, r_1\}$, so any polynomial expression in them is as well. This kind of argument works similarly for the other chart.

By the same argument, let's look at

$$\begin{aligned} \widetilde{\phi} : C \mapsto & \; C \\ (a,b) \mapsto & \left(-\frac{1}{a}, \frac{b}{a^2}\right) \end{aligned}$$

which is comparable to $\phi(a,b) = \left(-\frac{1}{a}, -\frac{b}{a^2}\right)$..? Then $\widetilde{\phi}(p_0) = p_\infty$ and $\widetilde{\phi}(p_\infty) = p_0$ as well.

Foreshadowing, (C, p_0) is an abelian group with p_∞ the origin in the group. An involution gives rise to a quotient $\frac{C}{\phi}$ as a set of equivalence classes $p \sim \phi(p)$. In general, if X is a pre-variety and G is a finite group which acts by isomorphims on X then we can define a new pre-variety $Y = X/G$ is the "orbit space", the topological space of orbits $Y \cong X/G$ with $\pi : X \mapsto Y$, $\mathcal{O}_Y(U) := \left(\mathcal{O}_X(\pi^{-1}(U))\right)^G$ (the ring of G invariant functions)

Problem 9.1. *Show that $C/\phi \cong \mathbb{P}^1$, and $C/\widetilde{\phi} \cong C$*

To do this, we need to find a morphism $\pi : C \mapsto C$ such that for every point in C, the preimage is the point p and $\widetilde{\phi}(p)$ as well.

Lecture 10

Proposition 10.1. *Let $X \subset \mathbb{P}^n$ be a projective variety. $\mathcal{O}_X(X) = \mathbb{C}$ i.e., the constant functions are the only global ones.*

Proof. Recall $\mathcal{O}_X(X) \subset K(X) = \left\{\frac{f}{g} : f, g \in S(X) = \mathbb{C}[x_0, \dots, x_n]/I(X),\ \deg\left(\frac{f}{g}\right) = 0\right\}$. Now, $\phi \in \mathcal{O}(X)$ if ϕ is regular at all points. There are in general many possible expressions of one equivalence class $\phi = \frac{f_1}{g_1} = \frac{f_2}{g_2} = \dots$, but the possible denominators form an ideal. Define $I_\phi \subset S(X)$, $I_\phi = \{g : g\phi \in S(X)\}$. Since ϕ is regular at all points of X means that $Z(I_\phi) = \emptyset$.

Observe that if projective nullstellensatz was the same as affine nullstellensatz, we would get that $\sqrt{I_\phi} = (1)$ which implies $1 \in I_\phi$, and thus $\phi \in S(X)$. However, all we actually know is that $(x_0, \dots, x_n) \subset \sqrt{I_\phi}$. This gives

$$\phi = \frac{f_0}{x_0^N} = \frac{f_0}{x_1^N} = \dots = \frac{f_0}{x_n^N} \tag{10.1}$$

for some N. Let $d \geq (n+1)N+1$, and then let $g \in S(X)_d$ (that's polynomials of degree d in $S(X)$). Each monomials of g are divisible by some x_i^N by the pigeon hole principle. Thus $g\phi \in S(X)_d$ which implies $g\phi^2 \in S(X)_d$ and in fact $g\phi^q \in S(X)_d$ for all q. (Logic: we can just iterate this: ϕ is degree 0, so we can take $g_1 = g\phi$ which is another degree d thing, and multiply that by ϕ and the same property holds.)

In particular, $x_0^d\phi^q \in S(X)_d$ for all q. consider the $S(X)$ module $x_0^{-d}S(X)$. Then $\phi^q \in x_0^{-d}S(X)$ for all q. Finitely generated modules over a noetherian ring satsify the descending chain condition on submodules. Here we have

$$S(X) \subseteq S(X) + \phi S(X) \subset S(X) + \phi S(X) + \phi^2 S(X) \subseteq \dots \subseteq x_0^{-d}S(X) \tag{10.2}$$

so the sequence must stabalize, and at that point we can write

$$\phi^m = g_{m-1}\phi^{m-1} + \dots + g_1\phi + g_0 \tag{10.3}$$

for $g_i \in S(X)$. This equation holds in $K(X)$, so in particular it holds in each degree. In degree 0, it says that $\phi^m = a_{m-1}\phi^{m-1} + \dots + a_0$ where $a_i \in S(X)_0 = \mathbb{C}$ are the degree 0 (constant) bits of g_i. $\mathbb{C}$ is algebraically closed, so

$$\prod_{i=1}^{m} (\phi - r_i) = 0 \tag{10.4}$$

for some $r_i \in \mathbb{C}$. Thus $\phi = r_i$ for some i. $\square$

Now, here are some "bad" pre-varieties. Consider

$$\begin{array}{ccccc} \mathbb{P}^1 & = & A^1 & \cup & \mathbb{A}^1 \\ (x_0, x_1) & & z = \frac{x_0}{x_1} & & w = \frac{1}{z} = \frac{x_1}{x_0} \end{array} \tag{10.5}$$

We have "made" $\mathbb{P}^1$ by gluing two affine varieties $\mathbb{A}^1 \cup \mathbb{A}^1$ with the gluing map $\mathbb{A}^1 - \{0\} \mapsto \mathbb{A}^1 - \{0\}$, $z \mapsto \frac{1}{z}$. However, we could have gotten a prevariety by just by $X = \mathbb{A}^1 \cup \mathbb{A}^1$ gluing with $z \to z$ away from 0. This is like an affine line but with two origins. This is an example of a non-hausdorff in the analytic topology btw. In algebraic geometry, we call this situation *non-separated*. To get a useful characterization of this "hausdorff-like" property, we need a notion of products.

Let $X = Z(f_1, \dots, f_k) \subset \mathbb{A}^n$, $f_i \in \mathbb{C}[x_1, \dots, x_n]$, $Y = Z(g_1, \dots, g_\ell) \subset \mathbb{A}^m$, $g_j \in \mathbb{C}[y_1, \dots, y_m]$, then we define

$$I(X \times Y) = (f_1, \dots, f_k, g_1, \dots g_\ell) \subset \mathbb{C}[x_1, \dots, x_n, y_1, \dots y_m] \tag{10.6}$$

and

$$\begin{aligned} A(X \times Y) &= \mathbb{C}[x_1, \dots, x_n, y_1, \dots y_m]/(f_1, \dots, f_k, g_1, \dots g_\ell) \\ &= \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_k) \otimes \mathbb{C}[y_1, \dots, y_m]/(g_1, \dots, g_\ell) \\ &=: R \otimes S \end{aligned} \tag{10.7}$$

and from commutative algebra, we know the tensor product of integral domains is an integral domain. This notion of product of affine varieties is a product in the categorical sense. The morphisms $X \times Y \mapsto X$, $X \times Y \mapsto Y$ (which correspond to the inclusions of rings $R \mapsto R \otimes S$, $S \mapsto R \otimes S$) satisfy the following universal property, saying that whenever there is a morphism from W to both X and Y, there must be a unique on from W to $X \times Y$.

$$\begin{array}{ccccc} & \swarrow & W & \searrow & \\ & & \downarrow \exists! & & \\ X & \longleftarrow & X \times Y & \longrightarrow & Y \end{array} \tag{10.8}$$

This follows from the fact that tensor product is a "categorical co-product", which is the fact that if we have maps $R \to A$, and $S \to A$, then there exists a unique one from $R \otimes S \to A$.

$$\begin{array}{ccccc} & \nearrow & W & \nwarrow & \\ & & \uparrow \exists! & & \\ R & \longrightarrow & R \otimes S & \longleftarrow & S \end{array} \tag{10.9}$$

Let $(X, \mathcal{O}_x)$ and $(Y, \mathcal{O}_Y)$ be pre-varieties. Then define $(X \times Y, \mathcal{O}_{X \times Y})$ as follows. $X \times Y$ is Cartesian product as a set basis for topology $\{U_i \times V_j\}$ for $\{U_i\}$, $\{V_j\}$ open covers of X and Y.

Proposition 10.2. *The product of projective varieties is a projective variety.*

Proof. Since any irreducible closed set of a projective variety is a projective variety, we need only prove that $\mathbb{P}^n \times \mathbb{P}^m$ is a projective variety. Define the map

$$\begin{aligned} I : \mathbb{P}^n \times \mathbb{P}^m &\mapsto \mathbb{P}^{(n+1)(m+1)-1} \\ (x_0, \dots, x_n), (y_0, \dots, y_m) &\mapsto (\dots : x_i y_j : \dots) \end{aligned} \tag{10.10}$$

let z_{ij} be homogeneous coordinates on $\mathbb{P}^{(n+1)(m+1)-1}$. Let $U_{ij} = \{z_{ij} \neq 0\}$. then

$$\begin{array}{ccccc} I_{-1}(U_{ij}) & = & U_i & \times & U_j \\ \uparrow & & \uparrow & & \uparrow \\ z_{ij} \neq 0 & & x_i \neq 0 & & y_i \neq 0 \end{array} \tag{10.11}$$

Then consider

$$\begin{array}{ccccc} I : U_0 & \times & U_0 & \mapsto & U_{00} \\ \uparrow & & \uparrow & & \uparrow \\ s_1, \dots, s_n & & t_1, \dots, t_m & & r_{ij} := s_i t_j \\ & & & & \\ s_i = \frac{x_i}{x_0} & & t_j = \frac{y_j}{y_0} & & r_{ij} = \frac{z_{ij}}{z_{00}} \end{array} \tag{10.12}$$

Where $i = 0, \dots, n$, $j = 0, \dots, m$, and $(i, j) \neq (0, 0)$, and

$$I(s_1, \dots, s_n, t_1, \dots, t_m) = \begin{cases} r_{ij} = s_i t_j & \text{if } i \neq 0 \ \& \ j \neq 0 \\ r_{i0} = s_i \\ r_{0j} = t_j \end{cases} \tag{10.13}$$

So the image is the set of points satisfying $r_{i0} \cdot r_{0j} = r_{ij}$. Finally,

$$\mathbb{C}[r_{ij}]/(r_{ij} - r_{i0} r_{0j}) \cong \mathbb{C}[r_{10}, \dots, r_{n0}, r_{01}, \dots, r_{0m}] \tag{10.14}$$

□

Remark 10.2. *Closed subvarieties of* $\mathbb{P}^n \times \mathbb{P}^m$ *are given by zeros of bihomogeneous polynomials* $f(x_0, \dots, x_n, y_0, \dots, y_m)$, $f(\lambda x_0, \dots, \lambda x_n, \mu y_0, \dots, \mu y_m) = \lambda^{d_1} \mu^{d_2} f(x_0, \dots, x_n, y_0, \dots, y_m)$

Lecture 11

Recall: a prevariety is a ringed space $(X, \mathcal{O}_X)$ which is locally an affine variety. We want our varieties to have a "Hausdorff" property.

Definition 11.1. *A prevariaty* X *is called a variety if for every pair of morphisms* f, $g : Y \mapsto X$, *the set* $\{p \in Y : f(p) = g(p)\} \subset Y$ *is closed.*

For example, consider our bad prevariety

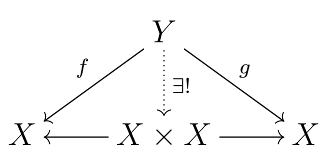
$$\begin{array}{ccccc} \mathbb{P}^1 & = & A^1 & \cup & \mathbb{A}^1 \\ (x_0, x_1) & & z = \frac{x_0}{x_1} & & w = \frac{1}{z} = \frac{x_1}{x_0} \end{array} \tag{11.1}$$

We can end up with $\mathbb{A}^1$ with a doubled origin. Let $f, g : \mathbb{A}^1 \mapsto X$ be the two inclusions. The set $\{f(p) = g(p)\} = \mathbb{A} - \{0\} \subset \mathbb{A}^1$ which is open!

Lemma 11.2. *A prevariety* X *is a variety* $\iff$ *the diagonal* $\Delta \subset X \times X$ *is closed.*

Proof. "$\implies$" Let $Y = X \times X$ and let f, g be the projections. Then $\Delta = \{f(p) = g(p)\}$, so it is closed.

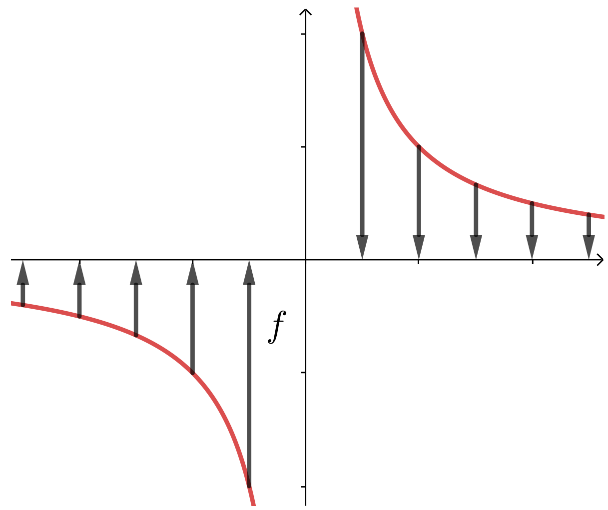
"$\Leftarrow$" Let f, $g : Y \mapsto X$ be any morphisms. by the universal property of products



(11.2)

we have $Y \mapsto X$ via (f, g) and $\{f(p) = g(p)\} = (f, g)^{-1}(\Delta)$. □

We want a notion of "compactness", e.g. in the analytic topology, projective varieties are compact, whereas affine varieties are non-compact (or zero dimensional). But in the Zariski topology, everything is compact, so use instead that the image of a compact set is compact. We will show that the image of a projective variety under a morphism is closed.



Note that this is false for affine varieties; consider $Z = \{xy = 1\} \subset \mathbb{A}^2$ and project it onto a coordinate by f, then $f(Z) = \mathbb{A} - \{0\}$ is not closed.

Theorem 11.3. *The projection $\pi : \mathbb{P}^n \times \mathbb{P}^m \mapsto \mathbb{P}^n$ is closed, i.e. if $X \subset \mathbb{P}^n \times \mathbb{P}^m$ is closed, then so is $\pi(X) \subset \mathbb{P}^n$.*

Proof. Let $X = Z(f_1(x, y), \dots, f_r(x, y))$ where f_i are bihomogeneous. W.L.O.G. we may assume the bi-degree is the same for all $f_i's$.

To see this, consider $Z(f(z_0, \dots, z_n)) = Z(z_0^d f, \dots, z_n^d f) \subset \mathbb{P}^n$ so we can make things to a specific degree at the expense of adding more generators.

Let $p \in \mathbb{P}^n$, then $p \in \pi(X) \iff Z(f_i(p, y)) \neq \emptyset$. In other words,

$$p \in \pi(X) \iff (y_0, \dots, y_m)^s \not\subset (f_1(p, y), \dots, f_r(p, y)) = \star_s \tag{11.3}$$

for all $s \geq 0$, $\deg f_i = d$. This is true when $s < d$, so we show that for each $s \geq d$, $\star_s$ is given by polynomials.

$(y_0, \dots, y_m)^s$ is generated by $\binom{m+s}{m}$ monomials of degree s, which we call $\{M_\ell(y)\}_{1 \leq \ell \leq \binom{m+s}{m}}$. Now, $\star_s$ is not true iff there are $g_{\ell,k}(y)$ such that

$$M_\ell(y) = \sum_k g_{\ell,k} f_k, \qquad \deg g_{\ell,k} = s - d \tag{11.4}$$

Let $\{N_\ell(y)\}$ be the monomials of degree $s - d$, then $\star_s$ is *not* true iff the collection

$$\{N_\ell(y) f_k\}_{1 \leq \ell \leq \binom{m+s-d}{m}, 1 \leq k \leq r}$$

spans the vector space of degree s polynomials in Y, which has dimension $\binom{m+s}{m}$. We write the coefficients of $N_\ell(y) f_k(y)$ in the basis $\{M_j(y)\}$ to get a $\binom{m+s}{m} \times r\binom{m+s-d}{m}$ matrix. $\star_s$ is not true if this matrix has full rank, $\star_s$ *is* true if all the $\binom{m+s}{m} \times \binom{m+s}{m}$ minors have determinant 0. But these determinants are a bunch of homogeneous polynomials in the x's. $\square$

Corollary 11.4. *The projection map $\mathbb{P}^n \times Y \mapsto Y$ is closed for any (pre-?)variety Y.*

Proof. First assume $Y \subseteq \mathbb{A}^m$ is affine, so $Y \subset \mathbb{A}^m \subset \mathbb{P}^m$. Let $X \subset \mathbb{P}^n \times Y$ be closed, so let $\bar{Z}$ be its closure in $\mathbb{P}^n \times \mathbb{P}^m$. By theorem 11.3, $\pi(\bar{Z})$ is closed in $\mathbb{P}^m$. Then

$$\pi(Z) = \pi(\bar{Z} \cap \mathbb{P}^n \times Y) = \pi(\bar{Z} \cap Y)$$

is closed in Y. For general varieties Y, to check that $\pi(Z) \subset Y$ is closed, we need only check that its restriction is closed in each affine cover. $\square$

Definition 11.5. *A variety X is complete if $\pi : X \times Y \mapsto Y$ is closed for all (pre-?)varieties Y.*

This means that any projective variety is complete. However, the converse is false; not all complete sets are projective varieties (although counterexamples are hard to construct).

Jargon: a variety which is an open set in a projective variety is called quasi-projective. All affine varieties are quasi-projective (again, almost all varieties you could think of are quasi-projective; non-quasi-projective varieties are hard to construct).

Lecture 12

12.5 Important projective varieties

Just as projective space $\mathbb{P}^{n-1} = \{\mathbb{C} \subset \mathbb{C}^n\}$ is the space of 1-dim linear subspaces of $\mathbb{C}^n$, We can define

Definition 12.1. *As a set,* $\mathrm{Gr}(k,n)$ *is the set of k-dim linear subspaces $L \subset \mathbb{C}^n$, $L \cong \mathbb{C}^k$. We can also view this as the set of $\mathbb{P}^{k-1} \subset \mathbb{P}^{n-1}$. Sometimes you will see the notation $G(k-1, n-1)$ so be careful.*

We want to exhibit $\mathrm{Gr}(k,n)$ as a projective variety. We could say a set of k independent vectors $v_1, \dots, v_k \in \mathbb{C}^n$ determine $L = \mathrm{span}\,\{v_1, \dots, v_k\}$, but these generators are ambiguous; the coordinates coming from this basis could be many different things. If we want to describe $L \in \mathrm{Gr}(k, n-k)$ in terms of coordinates, we want something like collections $\{v_1, \dots, v_k\}$, but 'up to' the ambiguity. Should identify equivalent coordinates somehow.

The notion of exterior product is an efficient way to handle this issue.

Definition 12.2. *Let $(e_1, \dots, e_n)$ be the standard basis for $\mathbb{C}^n$, and let $\bigwedge^k \mathbb{C}^n$ be the vector space spanned by $\binom{n}{k}$ vectors, written formally as*

$$e_{i_1} \wedge \dots \wedge e_{i_k} \qquad 1 \leq i_1 < \dots < i_k \leq n \tag{12.1}$$

with $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$. By convention, we have $\bigwedge^0 \mathbb{C}^n \cong \mathbb{C}$, $\bigwedge^n \mathbb{C}^n \cong \mathbb{C}$ with basis vectors $e_1 \wedge \dots \wedge e_n$, and $\bigwedge^k \mathbb{C}^n = \{0\}$ for $k > n$.

If we extend the product $\wedge$ to any sequence $i_1, \dots, i_k$ (not necessarily increasing) by

$$\begin{aligned} &e_{i_1} \wedge \dots \wedge e_{i_k} = 0 \text{ if } i_j = i_\ell \text{ for some } i \neq \ell \\ &e_{i_1} \wedge \dots \wedge e_{i_k} = (-1)^\sigma e_{i_{\sigma(1)}} \wedge \dots \wedge e_{i_{\sigma(n)}} \text{ for } \sigma \in \mathrm{perm}(1, \dots, n) \end{aligned} \tag{12.2}$$

(like for example, $e_1 \wedge e_3 \wedge e_5 = -e_3 \wedge e_1 \wedge e_5 = e_3 \wedge e_5 \wedge e_1$) where we extend this linearly

$$
\begin{array}{ccc}
\overbrace{\mathbb{C}^n \otimes \ldots \otimes \mathbb{C}^n}^{k} & \mapsto & \bigwedge^k \mathbb{C}^n \\
v_1 \otimes \ldots \otimes v_k & \mapsto & v_1 \wedge \ldots \wedge v_k
\end{array}
\tag{12.3}
$$

where if $v_j = \sum_i a_{ji} e_i$,

$$
\left(\sum_{i_1} a_{1i_1} e_{i_1}\right) \wedge \ldots \wedge \left(\sum_{i_k} a_{ki_k} e_{i_k}\right) = \sum_{i_1,\ldots,i_k} (a_{1i_1} \cdot \ldots. \cdot a_{ki_k}) e_{i_1} \wedge \ldots \wedge e_{i_k}
\tag{12.4}
$$

Key observation: if $V_1, \ldots, V_n \in \mathbb{C}^n$, $v_i = \sum_j a_{ij} e_j$, then $v_1 \wedge \ldots \wedge v_n = \det(a_{ij}) e_1 \wedge e_n$. Thus, we have

$$
v_1 \wedge \ldots \wedge v_n = \sum_{i_1,\ldots,i_n} (a_{1i_1} \cdot \ldots. \cdot a_{ni_n}) e_{i_1} \wedge \ldots \wedge e_{i_n}
\tag{12.5}
$$

whcih is only non-zero when $i_1, \ldots, i_n$ are distinct, so that equals

$$
\begin{aligned}
&= \sum_{\sigma \in \mathrm{perm}\{1,\ldots,n\}} a_{1\sigma(1)} \cdot \ldots \cdot a_{n\sigma(n)} e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(n)} \\
&= \left(\sum_{\sigma \in \mathrm{perm}\{1,\ldots,n\}} (-1)^\sigma a_{1\sigma(1)} \cdot \ldots \cdot a_{n\sigma(n)} \right) e_1 \wedge \ldots \wedge e_n
\end{aligned}
\tag{12.6}
$$

More generally, if $v_1, \ldots, v_k \in \mathbb{C}^n$, for $v_i = \sum_j a_{ij} e_j$,

$$
a:\ k\left\{\left(\overbrace{\underset{i_1}{\Big|}\quad\underset{i_k}{\Big|}}^{n}\right)\right.
\tag{12.7}
$$

Lemma 12.3. $\{v_1, \ldots, v_k\}$ *spans a* k*-dim subspace iff* $0 \neq v_1 \wedge \ldots \wedge v_k \in \bigwedge^k \mathbb{C}^n$.

If $L = \mathrm{span}\{v_1, \ldots, v_k\}$, then $v_1 \wedge \ldots \wedge v_k \in \bigwedge^k \mathbb{C}^n$ are called the "plucker coordinates" of L.

Lemma 12.4. *Let* $\{v_1, \ldots, v_k\}$ *and* $\{w_1, \ldots, w_k\}$ *be two sets of linearly independent vectors in* $\mathbb{C}^n$. *Then*

$$
\mathrm{span}\{v_1, \ldots, v_k\} = \mathrm{span}\{w_1, \ldots, w_k\}
\tag{12.8}
$$

if and only if

$$
v_1 \wedge \ldots \wedge v_k = \lambda w_1 \wedge \ldots \wedge w_k \textit{ for some } \lambda \in \mathbb{C}^\times
\tag{12.9}
$$

Proof. Suppose $\mathrm{span}\{v_1, \ldots, v_k\} = \mathrm{span}\{w_1, \ldots, w_k\}$. Then $v_i = \sum_{a_{ij} w_j}$ where (a_{ij}) is invertible. Then

$$
v_1 \wedge \ldots \wedge v_k = \det(a_{ij}) w_1 \wedge \ldots \wedge w_k
\tag{12.10}
$$

Now assume $v_1 \wedge \ldots \wedge v_k = \lambda w_1 \wedge \ldots \wedge w_k$ and suppose $\mathrm{span}\{v_1, \ldots, v_k\} \neq \mathrm{span}\{w_1, \ldots, w_k\}$. Then there is some $w_i \notin \mathrm{span}\, v_i$. Then $\{w_i, v_1, \ldots, v)k\}$ is linearly independent, so $w_i \wedge v_1 \wedge \ldots \wedge v_k \neq 0$. But then

$$
w_i \wedge v_1 \wedge \ldots \wedge v_k = \lambda w_i \wedge w_1 \wedge \ldots \wedge w_k = 0
\tag{12.11}
$$

since the w_i factor repeats. □

This lemma then gives a map

$$\begin{aligned} \mathrm{Gr}(k,n) &\mapsto \mathbb{P}^{\binom{n}{k}-1} \\ \mathrm{span}\,\{v_1, \dots, v_k\} &\mapsto \left\{\text{line through } v_1 \wedge \dots \wedge v_k \text{ in } \wedge^k \mathbb{C}^n\right\} \end{aligned} \tag{12.12}$$

For example, the image of $\mathrm{Gr}(k,n)$ in $\mathbb{P}^{\binom{n}{k}-1}$ are given by "pure tensors" in $\bigwedge^k \mathbb{C}^n$, i.e. vectors of the form $v_1 \wedge \dots \wedge v_k$.

Example 12.5. *Consider $w = e_1 \wedge e_2 + e_3 \wedge e_4$. Is it possible that $w = v_1 \wedge v_2$ in $\bigwedge^2 \mathbb{C}^4$? Consider*

$$\begin{aligned} &\mathbb{C}^4 \mapsto \wedge^3\mathbb{C}^4 \cong \mathbb{C}^4 \text{ with basis } e_1 \wedge e_2 \wedge e_3,\ e_1 \wedge e_2 \wedge e_4,\ e_1 \wedge e_3 \wedge e_4,\ e_2 \wedge e_3 \wedge e_4 \\ &\quad u \mapsto u \wedge w \end{aligned} \tag{12.13}$$

If $w = v_1 \wedge v_2$, then both v_1 and v_2 would be in the kernel of the map, so the map $u \mapsto u \wedge w$ would be at most rank 2. however, we see

$$\begin{aligned} e_1 &\mapsto e_1 \wedge e_3 \wedge e_4 \\ e_2 &\mapsto e_2 \wedge e_3 \wedge e_4 \\ e_3 &\mapsto e_3 \wedge e_1 \wedge e_2 = e_1 \wedge e_2 \wedge e_3 \\ e_3 &\mapsto e_4 \wedge e_1 \wedge e_2 = e_1 \wedge e_2 \wedge e_4 \end{aligned} \tag{12.14}$$

so the map has full rank.

Lemma 12.6. *For $w \in \bigwedge^k \mathbb{C}^n$, $w \neq 0$, let*

$$\begin{aligned} f : \mathbb{C}^n &\mapsto \bigwedge^{k+1} \mathbb{C}^n \\ v &\mapsto v \wedge w \end{aligned} \tag{12.15}$$

Then $\operatorname{rank} f \geq n - k$ with $\operatorname{rank} f = n - k$ iff $w = \lambda v_1 \wedge \dots \wedge v_k$ for some $v_1 \dots, v_k \in \mathbb{C}^n$, $\lambda \in \mathbb{C}^\times$.

Proof. Let $r = n - \operatorname{rank} f$ so that $\dim \ker f = r$. Let $v_1 \dots, v_r$ be a basis for $\ker f$, and extend it to a basis $v_1 \dots, v_r, v_{r+1}, \dots, v_n$. Then $\{v_{i_1} \wedge \dots \wedge v_{i_k}\}_{i_1,\dots,i_k}$ is a basis for $\bigwedge^k \mathbb{C}^n$. So

$$w = \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k} v_{i_1} \wedge \dots \wedge v_{i_k} \tag{12.16}$$

where $v_i \in \ker f$ iff $1 \leq i \leq r$. For these, we have

$$0 = v_i \wedge w = \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k} v_i \wedge v_{i_1} \wedge \dots \wedge v_{i_k} \tag{12.17}$$

which is 0 iff $i \in \{i_1, \dots, i_k\}$. This means that $a_{i_1 \dots i_k}$ if $i \notin \{i_1, \dots, i_k\}$ for $1 \leq i \leq r$.

Thus $w \neq 0$ implies $a_{i_1 \dots i_k}$ can be non-zero only if

$$\{1, \dots, r\} \subset \{i_1, \dots, i_k\} \implies r \leq k \tag{12.18}$$

so then $n - \operatorname{rank} f \leq k$. For the equality statement, if $n - \operatorname{rank} f = k$, then we get equality in equation (12.18). □

Corollary 12.7. $\mathrm{Gr}(k,n) \subset \mathbb{P}^{\binom{n}{k}-1}$ *is a closed set.*

Proof. $\mathrm{Gr}(n,n)$ is a point, so assume $k<n$. Then $w \in \mathrm{Gr}(k,n) \subset \mathbb{P}^{\binom{n}{k}-1}$ iff $w = [v_1 \wedge ... \wedge v_k]$ for some $v_1 ..., v_k$, which happens iff

$$f_w : \mathbb{C}^n \mapsto \bigwedge^{k+1} \mathbb{C}^n \qquad v \mapsto v \wedge w \tag{12.19}$$

has rank $f = n-k$, which happens iff rank $f \leq n-k$, which happens iff all $n-k$ minors of f_w have determinant 0. ☐

Lecture 13

Last time we talked about $\mathrm{Gr}(k,n) \subseteq \mathbb{P}^{\binom{n}{k}-1}$, and if a line $L_k \subset \mathbb{C}^n$ is the span of $\{v_1, ..., v_k\}$, then $\bigwedge^k L_k \subset \bigwedge^k \mathbb{C}^k$ which is spanned by $v_1 \wedge ... \wedge v_k$, and we can write $\sum_{\{i_1,...,i_k\}} a_{i_1,...,i_k} e_{i_1} \wedge ... \wedge e_{i_k}$, the $a_{i_1,...,i_k}$ are the Plucker coordinates. $\{i_1, ..., i_k\}$ determines a $k \times k$ minor of the matrix with v_i as rows.

Cover $\mathrm{Gr}(n,k)$ by affine spaces. Let $U_0 \subset \mathrm{Gr}(k,n) \subset \mathbb{P}^{\binom{n}{k}-1}$ be the open set where the coefficient of $e_1 \wedge ... \wedge e_k$ is non-zero.

Proposition 13.1.

$$\mathrm{Gr}(k,n) \equiv \mathrm{Gr}(n-k,n) \tag{13.1}$$

under $L \mapsto L^\perp$, *where* $L^\perp = \{x \in \mathbb{C}^n \mid \langle x,y\rangle = 0$ *for all* $y \in L\}$.

Definition 13.2. *Let X and Y be varieties. a rational map $f : X \dashrightarrow Y$ is a morphism $f : U \mapsto Y$ where U is an open set. f_1¡ $f_2 : U_i \mapsto Y$ define the same rational map if on any $V \subseteq U_i$ open $f_1\Big|_V = f_2\Big|V$*

A rational map $f : X \dashrightarrow Y$ is dominant *if there is an open set in the image of f, if $f : X \dashrightarrow Y$, $g : Y \dashrightarrow Z$ and f is dominant, then $g \circ f : X \dashrightarrow Z$ makes sense.*

$f : X \dashrightarrow Y$ is a birational map if there is a rational inverse $g : Y \dashrightarrow X$ such that $g \circ f = id_X$ and $f \circ g = id_Y$. In this case we say that X is birational to Y ($X \simeq Y$), i.e. X and Y contain isomorphic open sets.

Example 13.3. *THe varieties $\mathbb{P}^{kk-n}$, $\mathrm{Gr}(k,n)$, $\mathbb{P}^1 \times ... \times \mathbb{P}^1$ ($k(n-k)$ terms), and $\mathbb{A}^{k(k-n)}$ are all birationally equivalent. Jargon: a variety X with $X \simeq \mathbb{A}^N$ is called a rational variety. Note that if $f : X \dashrightarrow Y$ is a birational map, then we get an isomorphism $f^* : K(Y) \mapsto K(X)$.*

13.1 Blowups

These are a very important class of birational morphisms. They are honest morphisms $f : \widetilde{X} \mapsto X$ with rational inverses $g : X \dashrightarrow \widetilde{X}$, i.e. f is a morphism which is an isomorphism on an open set.

Let $X \subset \mathbb{A}^n$ affine variety. Let $f_1, ..., f_r \in A[X]$ and $U = X - Z(f_1, ..., f_r)$. We have a map $f : U \mapsto \mathbb{P}^{r-1}$ by sending $x \mapsto (f_1(x), ..., f_r(X))$ which is a well defined morphism. Let $\Gamma_f = \{(x, f(x))\} \subset U \times \mathbb{P}^{r-1}$ be the graph, then $\Gamma_f \subset U \times \mathbb{P}^{r-1}$ is a closed subset. (Why?

$\Gamma_f = (id, f)^{-1}(\Delta)$.) Morover, $\Gamma_f \cong U$ by projection onto U. Let $\widetilde{X}$ be the closure $\overline{\Gamma_f} \subset X \times \mathbb{P}^{r-1}$. Then $\widetilde{X} \mapsto X$ by the map π is an isomorphism on U. So this construction is of $\widetilde{X} = Bl_{(f_1,\dots,f_r)}$ is the blowup of X along the ideal $(f_1, \dots, f_r)$. Notice

$$\widetilde{X} \subset \left\{(x, y) \in X \times \mathbb{P}^{r-1} \mid y_1 f_j(x) = y_j f_i(x) \text{ for all } i, j\right\} \tag{13.2}$$

Example 13.4. *Let's blow up $\mathbb{A}^n$ at the origin, using the functions $(x_1, \dots x_n)$. Then*

$$\widetilde{\mathbb{A}}^n \subseteq \left\{(x, y) \in \mathbb{A}^n \times \mathbb{P}^{n-1} \mid y_i x_j = y_j x_i\right\}$$

and we claim that this inclusion is actually equality.

Lecture 14

Lemma 14.1. $\mathcal{B}\ell_p(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathcal{B}\ell_{a,b}\,\mathbb{P}^2$

Proof. Note by automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$, the assertion is independent of the choice of points. Consider $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ as the Veronese surface given by $\{x_0 x_3 = x_1 x_2\}$ with the point $p = (0 : 0 : 0 : 1)$. This has the ideal (x_0, x_1, x_2). (Veronese is the map $\mathbb{P}^1 \times \mathbb{P}^1 \mapsto \mathbb{P}^3$, taking $(y_0 : y_1), (z_0 : z_1) \mapsto (y_0 z_0 : y_0 z_1 : y_1 z_0 : y_1 z_1)$.)

Consider $\widetilde{X} = \mathcal{B}\ell_p(\mathbb{P}^1 \times \mathbb{P}^1) \subseteq \mathbb{P}^3 \times \mathbb{P}^2$, then

$$\widetilde{X} = \overline{\{(x_0 : x_1 : x_2 : x_3)(x_0 : x_1 : x_2) \mid x_0 x_3 = x_1 x_2\}} \tag{14.1}$$

Let $a = (0 : 1 : 0)$, $b = (0 : 0 : 1) \in \mathbb{P}^2$, and $\widetilde{\mathbb{P}}^2 = \mathcal{B}\ell_{\{a,b\}}\,\mathbb{P}^2 = \mathcal{B}\ell_I\,\mathbb{P}^2$ where

$$I = (y_0, y_2) \cdot (y_0, y_1) = \left(y_0^2, y_0 y_1, y_0 y_2, y_1 y_2\right)$$

so now we blow up,

$$\widetilde{\mathbb{P}}^2 = \left\{(y_0 : y_1 : y_2)\left(y_0^2, y_0 y_1, y_0 y_2, y_1 y_2\right)\right\} \subset \mathbb{P}^2 \times \mathbb{P}^3 \tag{14.2}$$

so the claim is that $\widetilde{X} \cong \widetilde{\mathbb{P}}^2$ under the obvious isomorphism $\mathbb{P}^2 \times \mathbb{P}^3 \cong \mathbb{P}^3 \times \mathbb{P}^2$. To prove this claim, we must find explicit open sets where $\square$

Recall

- For $a \in X \subset \mathbb{A}^N$, $a = (0, \dots, 0)$ then the tangent cone of X at a $C_a X$ is the affine cone over $X = Z(f_1, \dots, f_r)$, $\pi^{-1}(a) \subset \mathbb{P}^r$, $\mathcal{B}\ell_a X \xrightarrow{\pi} X$

We saw that $C_a X$ is ddefined by keeping the lowest homogeneous terms of $f_i's$, $T_a X = Z((f_1)_{(1)}, \dots, (f_r)_{(1)})$ and $C_a X \subseteq T_a X$.

Definition 14.2. *$a \in X$ is smooth if $C_a X = T_a X$.*

Lemma 14.3. *$X \subset \mathbb{A}^n$, $a = (0, \dots, 0) \in X$, $m = (x_1, \dots, x_n)$, $m/m^2 \cong (T_a X)^v$.*

if we define $T_a X = (m/m^2)^v$ where $m = I(a) \subset A(X)$ is the ideal of the point a, then we see $T_a X$ is independent of embedding.