Math 541 (Harmonic Analysis) Notes

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Lecture 1

First, some notation. Work on the interval $[0,1]$ and consider functions $f:[0,1)\mapsto\mathbb{C}$. Extend this to a periodic function $f:\mathbb{R}\mapsto\mathbb{C}$ which, as can be seen in Figure 1, may make it discontinuous at every integer.

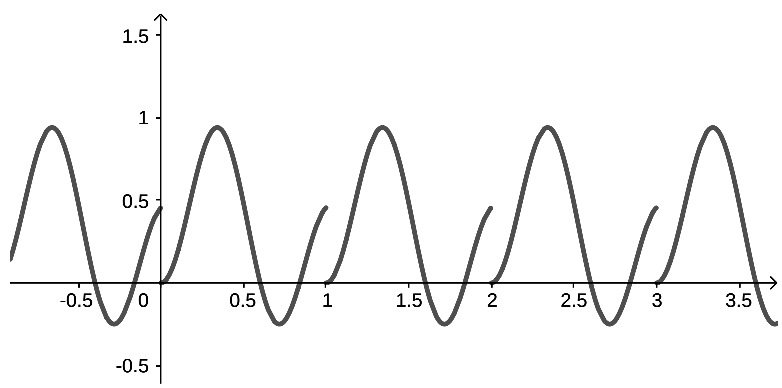


Figure 1: A function f extended into periodicity

1. $\mathbb{T}$ is the Torus. The Torus itself is really just the interval $[0,1]$ but with the endpoints identified, making it a closed circle.
2. Define the special class $\mathcal{C}(\mathbb{T})=\{f\in\mathcal{C}([0,1])\mid f(0)=f(1)\}$.
3. Let $L^p(\mathbb{T})=\left\{f\mid \|f\|_p<\infty\right\}$ where $\|f\|_p=\left(\int_0^1|f|^p\,dx\right)^{1/p}$. Note that L^2 is an inner product space. That is, there is an inner product defined by

$$\langle f,g\rangle=\int_0^1 f\cdot\overline{g}\,dx \tag{1.1}$$

Definition 1.1 (Fourier coefficients). *Let $f\in L^1(\mathbb{T})$. We define*

$$\widehat{f}(n)=\int_0^1 f(x)e^{-2\pi inx}dx \tag{1.2}$$

for all $n \in \mathbb{Z}$. *The* Fourier Series associated with f *is*

$$f \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x} \tag{1.3}$$

Note that since $f \in L^1$,

$$\left| \int_0^1 f(x) e^{-2\pi i n x} dx \right| \leq \int_0^1 |f(x)|\, dx < \infty \tag{1.4}$$

so the coefficients $\widehat{f}(n)$ *are well defined.*

With this, we come to two initial questions about Fourier Series.

Question 1.2. *The first major question to ask is when does the Fourier series converge to* f*? To be more specific, let*

$$S_N f = \sum_{n=-N}^{N} \widehat{f}(n) e^{2\pi i n x} \tag{1.5}$$

Do we have that $S_N f \to f$ *pointwise? How about uniformly? In an* L^p *norm?*

Question 1.3. *Are there other approximations of* f *by trigonometric polynomials* $\sum_{n=-N}^{N} a_n e^{2\pi i n x}$*?*

The following lemma is an answer to these questions in a special case.

Lemma 1.4. *Suppose that* $\sum_{n=-\infty}^{\infty} |a_n| < \infty$*. Then* $\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$ *converges uniformly to a continuous function. Further, in this case* $a_n = \widehat{f}(n)$*.*

Proof. Note that $\{e^{2\pi i n x}\}$ are orthonormal in L^2. That is,

$$\int_0^1 e^{2\pi i n x} e^{-2\pi i m x} dx = \begin{cases} \int_0^1 1 dx = 1 & n = m \\ \int_0^1 e^{2\pi i (n-m) x} = \frac{1}{2\pi(n-m)} e^{2\pi i (n-m) x} \Big|_0^1 = 0 & n \neq m \end{cases} \tag{1.6}$$

Let

$$S_N = \sum_{n=-N}^{N} a_n e^{2\pi i n x} \tag{1.7}$$

and I claim that this is a uniform Cauchy sequence. Indeed, if $M > N$ then

$$|S_M - S_N| = \left| \sum_{N < |n| \leq M} a_n e^{2\pi i n x} \right| \leq \sum_{N < |n| \leq M} |a_n| \to 0 \tag{1.8}$$

as $N \to \infty$. Thus the partial sums converge uniformly and S_N is continuous, so it converges uniformly to f and f is continuous. Finally we show that these coefficients are equal to the Fourier coefficients. Recall

$$\begin{aligned} \widehat{f}(n) &= \int_0^1 f(x) e^{-2\pi i n x} \\ &= \int_0^1 \left(\sum_{m=-\infty}^{\infty} a_m e^{2\pi i m x} \right) e^{-2\pi i n x} dx \\ &= \sum_{m=-\infty}^{\infty} \left(\int_0^1 a_m e^{2\pi i m x} e^{-2\pi i n x} dx \right) \end{aligned} \tag{1.9}$$

which is 0 if $m \neq n$, and $a_m = a_n$ if $m = n$. ☐

Now, some Properties of Fourier coefficients:

1. Linearity: $\widehat{af+bg}(n) = a\widehat{f} + b\widehat{g}(n)$
2. Conjugation: Observe that
$$\widehat{\overline{f}} = \int_0^1 \overline{f}(x)e^{-2\pi inx}dx = \overline{\int_0^1 f(x)e^{2\pi inx}dx} = \overline{\widehat{f}(-n)} \tag{1.10}$$
As a consequence, if $f : \mathbb{T} \to \mathbb{R}$, $\widehat{f}(n) = \overline{\widehat{f}(-n)}$
3. Translation: Let $f_t(x) = f(x-t)$. Then
$$\begin{aligned}\widehat{f_t}(n) &= \int_0^1 f_t(x)e^{-2\pi inx} \\ &= \int_0^1 f(x-t)e^{-2\pi in(x-t+t)}dx \\ &= e^{-2\pi int}\int_0^1 f(x-t)e^{-2\pi in(x-t)} \\ &= e^{-2\pi int}\widehat{f}(n)\end{aligned} \tag{1.11}$$

Lecture 2

Lemma 2.1. *Suppose $f \in L^1(\mathbb{T})$, $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then $f(x) = 0$ for all x such that f is continuous at x.*

Proof. If $\widehat{f}(n) = 0$ for all n, then
$$\int_0^1 f(x)e^{-2\pi inx}dx = 0 \tag{2.1}$$
for all n, and further if $P(x) = \sum_{-N}^{N} a_n e^{-2\pi inx}$,
$$\int_0^1 f(x)P(x) = 0 \tag{2.2}$$
That is, f is orthogonal to all trigonometric polynomials. Suppose that there is an $x_0 \in [0,1]$ where f is continuous and $f(x_0) \neq 0$. We will construct a trigonometric polynomial where (2.2) fails. We may assume that $f(x_0) > 0$, f is real valued, and that $x_0 = 0$. Indeed, if $x_0 \neq 0$, consider $f(x-x_0)$ instead of $f(x)$.

Now, start with $p(x) = \cos(2\pi x) + \varepsilon$ for some $\epsilon > 0$. Let $\delta > 0$ be such that $f(x) > \frac{f(0)}{2}$ for all $|x| < \delta$. We choose ε sufficiently small so that $|p(x)| < 1 - \frac{\varepsilon}{2}$ for $|x| \geq \delta$. Finally, let $P_N(x) = (p(x))^N$. I claim that for large enough N, equation (2.2) fails.
$$\int_{-1/2}^{1/2} f(x)P_N(x)dx = \int_{-\delta}^{\delta} f(x)P_N(x)dx + \int_{\delta<|x|\leq 1/2} f(x)P_N(x)dx \tag{2.3}$$

I claim the first integral goes to infinity and the second goes to 0 as $N \to \infty$. For the second one on $|x| \geq \delta$, we have $|p(x)| < 1 - \frac{\varepsilon}{2}$, $|P_N|(x) < \left(1 - \frac{\varepsilon}{2}\right)^N$. Now

$$\begin{aligned} \left| \int_{\delta < |x| \leq 1/2} f(x) P_N(x) dx \right| &\leq \int ... \left(1 - \frac{\varepsilon}{2}\right)^N \\ &\leq \left(1 - \frac{\varepsilon}{2}\right)^N \to 0 \text{ as } N \to \infty \end{aligned} \tag{2.4}$$

Now, let $0 < \eta < \delta$ be such that $p(x) > 1 + \frac{\varepsilon}{2}$ for all $|x| < \eta$. We can do this because $p(0) = 1 + \varepsilon$.

$$\begin{aligned} \int_{|x|<\delta} P_N(x) f(x) dx &\geq \int_{|x|<\eta} P_N(x) f(x) dx \\ &\geq 2\eta \left(1 + \frac{\varepsilon}{2}\right)^N \frac{f(0)}{2} \to \infty \end{aligned} \tag{2.5}$$

□

Corollary 2.2. *Suppose $f, g \in \mathcal{C}(\mathbb{T})$ and $\widehat{f}(n) = \widehat{g}(n)$ for all $n \in \mathbb{Z}$. Then $f = g$.*

Proof. Just apply lemma 2.1 to the function $f - g$. □

Corollary 2.3. *Assume that $f : \mathbb{T} \mapsto \mathbb{C}$ is continuous, and $\sum_{n=-\infty}^{\infty} \left|\widehat{f}(n)\right| < \infty$. Then $S_N f \to f$ uniformly.*

Proof. Let $a_n = \widehat{f}(n)$. By lemma 1.4, we have $S_N f \to g \in \mathcal{C}(\mathbb{T})$ uniformly and $\widehat{g}(n) = a_n = \widehat{f}(n)$, so $f = g$. □

Lecture 3

Definition 3.1 (Convolution)**.** *If $f,\ g \in L^1(\mathbb{T})$, we define*

$$f * g(x) = \int_0^1 f(x-t) g(t) dt \tag{3.1}$$

which is defined A.E.

A useful way to think of this is a way of averaging the functions. In many ways, it makes functions much more regular. If $g = \begin{cases} 0 & \text{for } |x| > \delta \\ \frac{1}{2\delta} & \text{for } |x| < \delta \end{cases}$, then

$$f * g(x) = \int_{|t|<\delta} \frac{1}{2\delta} f(x-t) dt \tag{3.2}$$

Lemma 3.2. *If $f,\ g \in L^1(\mathbb{T})$, then If $f * g \in L^1(\mathbb{T})$ as well.*

Proof.

$$\begin{aligned}\int_0^1 |f * g(x)|\, dx &= \int_0^1 \left| \int_0^1 f(x-t)g(t)dt \right| dx \\ &\leq \int_0^1 \int_0^1 |f(x-t)|\, |g(t)|\, dtdx \\ &\leq \int_0^1 \int_0^1 |f(u)|\, |g(t)|\, dtdu \\ &= \|f\|_1 \, \|g\|_1 \end{aligned} \tag{3.3}$$

□

Lemma 3.3.

$$\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n) \tag{3.4}$$

Proof.

$$\begin{aligned}\widehat{f * g}(n) &= \int_0^1 \int_0^1 f(x-t)g(t)dte^{-2\pi inx}dx \\ &= \int_0^1 \int_0^1 f(x-t)g(t)e^{-2\pi in(x-t)}e^{2\pi int}dtdx \\ &= \int_0^1 \int_0^1 \left(f(x-t)e^{-2\pi in(x-t)}\right)\left(g(t)e^{2\pi int}\right) dtdx \\ &= \widehat{f}(n)\widehat{g}(n)\end{aligned} \tag{3.5}$$

□

Now, an exercise is to prove the following properties of convolution.

1. $f * g = g * f$
2. $f * (g * h)$
3. $f * (g + h) = f * g + f * h$

Lemma 3.4. *Let $f \in L^1(\mathbb{T})$, and define*

$$K(x) = \sum_{k=-N}^{N} a_k e^{2\pi ikx} \tag{3.6}$$

Then

$$(f * K)(x) = \sum_{k=-N}^{N} a_k \widehat{f}(k) e^{2\pi ikx} \tag{3.7}$$

Proof. By linearity, it suffices to prove this for K having only one term. So, we will assume it is $K(x) = e^{2\pi ikx}$. Thus, using the first property,

$$(f * K)(x) = \int_0^1 K(x-t)f(t)dt = \int_0^1 e^{2\pi ik(x-t)}f(t)dt = e^{2\pi ikx}\int_0^1 f(t)e^{2\pi ikt}dt = e^{2\pi ikx}\widehat{f}(k) \tag{3.8}$$

□

Recall that for $f \in L^1(\mathbb{T})$, we have

$$S_N f = \sum_{n=-N}^{N} \widehat{f}(n) e^{2\pi i n x} \tag{3.9}$$

I claim that $S_N f = D_N * f$, where

$$\begin{aligned} D_N(x) &= \sum_{n=-N}^{N} e^{2\pi i n x} \\ &= e^{s\pi i(-N)x} \times \frac{e^{2\pi i(2N+1)x} - 1}{e^{2\pi i x} - 1} \\ &= \frac{e^{2\pi i(N+1)x} - e^{-2\pi i N x}}{e^{2\pi i x} - 1} \\ &= \frac{e^{\pi i(2N+\frac{1}{2})x} - e^{\pi i(2N-\frac{1}{2})x}}{e^{\pi i x} - e^{-\pi i x}} \\ &= \frac{2i\sin((2N+1)\pi x)}{2i\sin(\pi x)} \\ &= \frac{\sin((2N+1)\pi x)}{\sin(\pi x)} \end{aligned} \tag{3.10}$$

which is called *Dirichlet Kernel*.

Apply the Lemma with $a_k = 1$ for all $-N \leq k \leq N$. Then

$$S_N f(x) = f * \sum_{k=-N}^{N} e^{2\pi i k x} \tag{3.11}$$

Definition 3.5. *A sequence of functions $\{K_n\}_1^\infty \subseteq L^1(\mathbb{T})$ is a* summability kernel *if*

1. $\int_0^1 K_n(x) = 1$
2. $\int_0^1 |K_n(x)|\,dx < M$ *for all* n
3. *For all* $\delta > 0$, $K_n \to 0$ *uniformly on* $(\delta, 1-\delta)$ *as* $n \to \infty$.

Note that the Dirichlet Kernel D_n is not a summability kernel.

Theorem 3.6. *Let $f \in \mathcal{C}(\mathbb{T})$, and let $\{K_n\}$ be a summability kernel. Then $K_n * f \to f$ uniformly as $n \to \infty$.*

Proof. Note that since $f \in \{(\mathbb{T})$, f is uniformly continuous. Let $\varepsilon > 0$, and choose a δ so that $|f(x-y) - f(x)| < \varepsilon$ for all x whenever $|y| < \delta$. Using the fact that the integral of K_n is 1, we have

$$\begin{aligned} |K_n * f(x) - f(x)| &= \left| \int_0^1 K_n(y) f(x-y) dy - \int_0^1 K_n(y) f(x) dy \right| \\ &= \left| \int_0^1 K_n(y) \left(f(x-y) - f(x) \right) dy \right| \\ &= \left| \int_{|y|<\delta} \ldots dy + \int_\delta^{1-\delta} \ldots dy \right| \end{aligned} \tag{3.12}$$

Now, the first integral is

$$\int_{|y|<\delta} ... \, dy \leq \int_0^1 |K_n(y)| \, |f(x-y) - f(y)| \, dy \leq M\varepsilon \tag{3.13}$$

and the second integral is

$$\int_\delta^{1-\delta} ... \, dy \leq \sup_{\delta<y<1-\delta} K_n(y) \int_0^1 |f(x-y) - f(y)| \, dy \to 0 \times 2 \max_y |f(y)| \tag{3.14}$$

as $n \to \infty$. □

Lecture 4

Corollary 4.1.

- *If $f \in L^1(\mathbb{T})$, then $\|K_n * f - f\|_1 \to 0$ as $n \to \infty$.*
- *If $f \in L^1(\mathbb{T})$ and f is continuous at x, then $K_n * f(x) \to f(x)$ as $n \to \infty$.*

Proof. Exercise!

- Approximate f by continuous functions to do the first point, and apply theorem 3.6 and do some approximating.
- For the second point, work through the details in the proof of theorem 3.6, but you fix x. It works very similarly.

□

Recall that

$$S_N f(x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{2\pi i n x} = D_N * f(x) \tag{4.1}$$

is not a summability kernel, because it fails property 2 and 3 from definition 3.5. However, it does satisfy property 1. Now, define some average functions

$$\sigma_N f = \frac{1}{N} \sum_{n=0}^{N-1} S_n f \tag{4.2}$$

and recall that if $S_N f(x) \to L$, then $\sigma_N f(x) \to L$ as well. However, it is possible to have $\sigma_N f(x) \to L$ even if $S_N f(x)$ does not converge. A good example of this is that the sequence $\{(-1)^n\}_{n=0}^{\infty}$ does not converge, but its sequence of averages does.

Definition 4.2. *Rewriting,*

$$\sigma_N f = \frac{1}{N} \sum_{n=0}^{N-1} S_n f = \frac{1}{N} \sum_{n=0}^{N-1} D_n * f = F_n * f \tag{4.3}$$

Where $F_n := \frac{1}{N} \sum_{n=0}^{N-1} D_n$ is the Fejér Kernel. We can compute a closed form expression for these, which is

$$F_N = \frac{1}{N} \left(\frac{\sin(N\pi x)}{\sin(\pi x)} \right)^2 \tag{4.4}$$

Lemma 4.3. *F_n is a summability kernel.*

Proof. We check the properties of summability kernels.

1.
$$\int_0^1 F_N = \frac{1}{N}\sum_{n=0}^{N-1}\int_0^1 D_n = \frac{1}{N}N = 1 \tag{4.5}$$

2.
$$\int_0^1 |F_N| = \int_0^1 F_N = 1 \tag{4.6}$$

3. For $\delta < x < 1-\delta$,
$$F_N(x) \leq \frac{1}{N}\frac{1}{(\sin(\pi x))^2} \leq \frac{1}{N}\frac{1}{(\sin(\pi\delta))^2} \to 0 \tag{4.7}$$

□

Applying Theorem 3.6 and Corollary 4.1 to this summability kernel, we have that

1. If $f \in C(\mathbb{T})$, then $\sigma_N f \to f$ uniformly.
2. If $f \in L^1(\mathbb{T})$, then $\|\sigma_N * f - f\|_1 \to 0$ as $n \to \infty$.
3. If $f \in L^1(\mathbb{T})$ and f is continuous at x, then $\sigma_n * f(x) \to f(x)$ as

Corollary 4.4. *If $f \in L^1(\mathbb{T})$ and $\widehat{f}(n) = 0$ for all n, then $f = 0$ almost everywhere (w.r.t Lebesgue measure).*

Note that this is a discontinuous version of a previous result applying to $f \in C(\mathbb{T})$.

Proof. If $\widehat{f}(n) = 0$ for all n, then $S_N f \equiv 0$ which implies $\sigma_N f \equiv 0$. Thus $\sigma_N f \to 0$ in L^1, so f is the zero function in L^1. □

Corollary 4.5. *If $f \in L^1(\mathbb{T})$ continuous at x and $S_N f(x) \to L$ as $N \to \infty$, then $f(x) = L$.*

Proof. Apply (3) from above to see that $\sigma_N f(x) \to f(x)$ and $S_N f(x) \to L$, and the limits thus must be the same. □

In particular, if f has a jump discontinuity at x, suppose $f \in L^1(\mathbb{T})$ and that
$$\lim_{h\downarrow 0}\frac{1}{2}\left(f(x+h) + f(x-h)\right) = L \tag{4.8}$$
exists. Then $\sigma_N f(x) \to L$ as $N \to \infty$.

To see this, compute
$$\begin{aligned} \sigma_N f(x) - L &= \int_0^1 F_N(y)(f(x-y) - L)dy \\ &= \int_0^1 F_N(y)\left(\frac{f(x-y)+f(x+y)}{2} - L\right)dy \end{aligned} \tag{4.9}$$
Where we used symmetry, and the rest is the same reasoning as in Theorem 3.6.

Lecture 5

Fourier Decay: estimates on $\left|\widehat{f}(n)\right|$ as $n \to \infty$.

Lemma 5.1 (Riemann-Lebesgue)**.** *Suppose $f \in L^1(\mathbb{T})$. Then $\widehat{f}(n) \to 0$ as $n \to \infty$.*

Proof. Let $\varepsilon > 0$. Take N large enough so that $\|\sigma_N f - f\|_{L^1} < \varepsilon$ and take its Fourier coefficients.

$$\left|\widehat{\sigma_N f - f}(k)\right| \leq \|\sigma_N f - f\|_1 < \varepsilon \tag{5.1}$$

But also, $\sigma_N f$ is a trig. polynomial of degree $\leq N$, so $\widehat{\sigma_N f}(k) = 0$ for all $|k| > N$. Thus for all $|K| > N$,

$$\left|\widehat{f}(k)\right| \leq \left|\widehat{\sigma_N f - f}(k)\right| + \left|\widehat{\sigma_N f(x)}\right| < \varepsilon + 0 \tag{5.2}$$

$\square$

When do we have faster decay? (Quantitative estimates) Sometimes we do have it, when f is regular in some way.

Lemma 5.2. *Let F be absolutely continuous on $\mathbb{T}$ (i.e. there is a function $f \in L^1(\mathbb{T})$ so that $F(x) = \int_0^x f(t)dt$, and that $\int_0^1 f(t)dt = 0$). Then $\widehat{F}(n) = \frac{1}{2\pi i n}\widehat{f}(n)$ for all $n \neq 0$. In particular, $\left|\widehat{F}(n)\right| \leq \frac{\|f\|_1}{2\pi n}$. Further, by the R-L lemma, $\left|\widehat{F}(n)\right| = o\left(\frac{1}{n}\right)$.*

Proof. Integrating by parts, we have

$$\begin{aligned} \widehat{F}(n) &= \int_0^1 F(x)e^{-2\pi i n x}dx \\ &= -\int_0^1 f(x)\frac{1}{-2\pi i n}e^{-2\pi i n x}dx \\ &= \frac{1}{2\pi i n}\int_0^1 f(x)e^{-2\pi i n x}dx \\ &= \frac{1}{2\pi i n}\widehat{f}(n) \end{aligned} \tag{5.3}$$

$\square$

Functions with slow Fourier decay

Suppose $\{a_n\}_{-\infty}^{\infty} \subset \mathbb{R}$, $a_n > 0$, $a_n = a_{-n}$, $a_n \to 0$ as $|n| \to \infty$. Also suppose for $n > 0$, $a_{n+1} - 2a_n + a_{n-1} > 0$. Then there is a function $f \in L^1(\mathbb{T})$ such that $\widehat{f}(n) = a_n$. Note that the $a_{n+1} - 2a_n + a_{n-1} > 0$ is really just a condition about concavity, it makes it concave up.

Note also that we cannot just let $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$ because the series may not converge if the sequence of a_n decays slowly. Define instead

$$f(x) = \sum_{n=1}^{\infty} n(a_{n-1} - 2a_n + a_{n+1})F_n(x) \tag{5.4}$$

Where $F_n(x)$ is the Fejér Kernel from before. I claim this converges in L^1. Let $c_n = n(a_{n-1} - 2a_n + a_{n+1})$. Then we need to check

$$\sum_{n=1}^{\infty} \|c_n F_n\|_1 = \sum_{n=1}^{\infty} c_n \|F_n\|_1 = \sum_{n=1}^{\infty} c_n \tag{5.5}$$

Let $A_N = \sum_{n=1}^{N} c_n$, and claim that $A_N = a_0 - a_N - N(a_N - a_{N+1}) = a_0 - (N+1)a_N + Na_{N+1}$.You can prove this relation by induction.

- For $N = 1$, $A_N = A_1 = c_1 = a_0 - 2a_1 + a_2$
- assume it holds for N. Then

$$\begin{aligned} A_{N+1} &= A_N + c_{N+1} \\ &= a_0 - (N+1)a_N + Na_{N+1} + (N+1)(a_N - 2a_{N+1} + a_{N+2}) \\ &= a_0 - (N+2)a_{N+1} + a_{N+2} \end{aligned} \tag{5.6}$$

Thus the relation holds for all N. Now we just need to prove that $\lim_{N\to\infty} A_n$ exists.

$$A_n = a_0 - a_N - N(a_N - a_{N+1}) \tag{5.7}$$

The first two terms are easy. For the last,

$$\begin{aligned} N(a_N - a_{N+1}) &= (a_N - a_{N+1}) + ... + (a_N - a_{N+1}) \\ &= 2\left((a_N - a_{N+1}) + (a_{N-1} - a_N) + ... + (a_{\lfloor N/2 \rfloor} - a_{\lfloor N/2 \rfloor + 1})\right) \\ &= 2\left(a_{\lfloor N/2 \rfloor} - a_{N+1}\right) \\ &< 2a_{\lfloor N/2 \rfloor} \to 0 \end{aligned} \tag{5.8}$$

Lecture 6

Theorem 6.1 (From last time)**.** *Suppose* $\{a_n\}_{-\infty}^{\infty} \subset \mathbb{R}$*,* $a_n > 0$*,* $a_n = a_{-n}$*,* $a_n \to 0$ *as* $|n| \to \infty$*. Also suppose for* $n > 0$*,* $a_{n+1} - 2a_n + a_{n-1} > 0$*. Then there is a function* $f \in L^1(\mathbb{T})$ *such that* $\widehat{f}(n) = a_n$*.*

Cont. So $f \in L^1$, so we just need to find the Fourier coefficients of f. First, recall the Fourier coefficients of F_n.

- $\widehat{D_N}(k) = \begin{cases} 1 & \text{if } |k| \le n \\ 0 & \text{if } |k| > n \end{cases}$
- $\widehat{F_N}(k) = \frac{1}{N}\sum_{n=0}^{N-1} \widehat{D_n}(k) = \begin{cases} 0 & \text{if } |k| \ge N \\ 1 - \frac{|k|}{N} & \text{if } |k| < N \end{cases}$

Note that $\|f - f_n\| \to 0$ in L^1, where

$$f_n = \sum_{m=1}^{n} m(...)F_m(x)$$

Indeed

$$\begin{aligned}\left|\widehat{f_n}(k) - \widehat{f}(k)\right| &= \left|\int_0^1 (f_n - f)\, e^{2\pi ikx} dx\right| \\ &\leq \int_0^1 |f_n - f| \to 0\end{aligned} \tag{6.1}$$

Hence

$$\begin{aligned}\widehat{f}(k) &= \sum_{n=1}^{\infty} n\,(a_{n-1} - 2a_n + a_{n+1})\, \widehat{F_n}(k) \\ &= \sum_{n=|k|+1}^{\infty} n\,(a_{n-1} - 2a_n + a_{n+1}) \left(\frac{n-|k|}{n}\right) \\ &= \sum_{n=|k|+1}^{\infty} (a_{n-1} - 2a_n + a_{n+1})\,(n - |k|) \\ &= a_{|k|} + (\text{terms that go to } 0)\end{aligned} \tag{6.2}$$

by computations very similar to last time.

□

6.1 Fourier coefficients of measures

If μ is a measure on $\mathbb{T}$, define

$$\widehat{\mu(k)} = \int e^{-2\pi ixk} d\mu(x) \tag{6.3}$$

For example, the delta function where $\int_{\mathbb{R}} f d\delta_x = f(x)$. In this case, $\widehat{\delta_0 = 1}$ for all $k \in \mathbb{Z}$, since $e^{-2\pi ixk} = 1$ for $x = 0$.

6.2 Pointwise convergence

Let $f \in L^1(\mathbb{T}) \to \widehat{f}(k)$. When does $\sum_{-\infty}^{\infty} \widehat{f}(k) e^{2\pi ikx} \to f(x)$ pointwise?

Theorem 6.2. *If $f \in L^1(\mathbb{T})$ and f is differentiable at x, then $S_N f(x)$ converges to $f(x)$ as $N \to \infty$.*

Proof. Let

$$F(t) = \begin{cases} \frac{f(x-t)-f(x)}{t} & \text{if } t \neq 0 \\ -f'(x) & \text{if } t = 0 \end{cases} \tag{6.4}$$

which is continuous at x. Is $F \in L^1(\mathbb{T})$? Well, for $|t| > \delta$, $|F(t)| \leq \frac{1}{\delta}\left(|F(x-t)| + |F(x)|\right)$ and it's also bounded inside the δ interval if you choose it sufficiently small. Using the fact that $\int D_N = 1$,

$$
\begin{aligned}
S_N f(x) - f(x) &= f * D_N(x) - f(x) \\
&= \int_{-1/2}^{1/2} f(x-t) D_N(t) dt - \int_{-1/2}^{1/2} f(x) D_N(t) dt \\
&= \int_{-1/2}^{1/2} \left(f(x-t) - f(x)\right) D_N(t) dt \\
&= \int_{-1/2}^{1/2} \frac{f(x-t) - f(x)}{t} t D_N(t) dt \\
&= \int_{-1/2}^{1/2} F(t) t \left(\frac{\sin((2N+1)\pi t)}{\sin(\pi t)}\right) dt \\
&= \int_{-1/2}^{1/2} G(t) \left(\sin((2N+1)\pi t)\right) dt
\end{aligned}
\tag{6.5}
$$

since $\frac{t}{\sin(\pi t)}$ is bounded, and thus $G(t) = F(t)C_1$ is an L^1 function, which means that by Riemann Lebesgue,

$$\int G(t) e^{-2\pi i k t} dt \to 0 \tag{6.6}$$

and

$\square$

Lecture 7

Start with a function

$$g(x) = \begin{cases} \pi i(1-2x) & 0 \leq x < 1 \\ \text{Extend periodically} \end{cases} \tag{7.1}$$

$$
\begin{aligned}
\widehat{g}(0) &= \int_0^1 g = 0 \\
\widehat{g}(n) &= \int_0^1 \pi i (1-2x) e^{-2\pi i n x} dx \\
&= -\int_0^1 2\pi i x e^{-2\pi i n x} dx \\
&= \frac{x e^{-2\pi i n x}}{n}\Big|_0^1 - \int_0^1 \frac{e^{-2\pi i n x}}{n} dx \\
&= 1/n
\end{aligned}
$$

Now define

$$
\begin{aligned}
f_N &= S_N g = \sum_{0<|n|<N} \frac{e^{2\pi i n x}}{n} \\
\widetilde{f}_N &= \sum_{n=-N}^{-1} \frac{e^{2\pi i n x}}{n}
\end{aligned}
\tag{7.2}
$$

two lemmas are that f_N is uniformly bounded, and that $\left|\widetilde{f}_N(0)\right| \geq c \log N$. Finally,

$$\begin{aligned} P_N(x) &= e^{4\pi i N x} f_N(x) \\ \widetilde{P}_N(x) &= e^{4\pi i N x} \widetilde{f}_N(x) \\ \widehat{P_N}(n) &= \int_0^1 e^{4\pi i N x} f_N(x) e^{-2\pi i n x} dx \\ &= \widehat{f_N}(n - 2N) \end{aligned}$$

$$\widehat{f_N}(n) = \begin{cases} 0 & \text{if } n = 0 \text{ or } |n| > N \\ \frac{1}{n} & \text{if } 1 \leq |n| \leq N \end{cases}$$

Now observe that

$$S_m P_N = \begin{cases} P_N & m \geq 3N \\ \widetilde{P}_N & m = 2N \\ 0 & m < N \end{cases} \tag{7.3}$$

More defining. Let $\{a_k\}_{k=1}^{\infty}$ be so that $a_k \geq 0$, $\sum_{a_k} < \infty$. ALSO DEFINE $\{N_k\}$ so that $N_{k+1} > 3N_k$, and $a_k \log N_k \to \infty$ as $k \to \infty$. For example, you could let $N_k = 3^{2^k}$, $a_k = \frac{1}{k^2}$.

Finally, define

$$f(x) = \sum_{k=1}^{\infty} a_k P_{N_k}(x) \tag{7.4}$$

which is uniformly convergence, because hw.

Lecture 8: L^2 theory

L^2 is a space. $L^2(\mathbb{T}) = \left\{f : \mathbb{T} \to \mathbb{C} \mid \int_{\mathbb{T}} |f|^2 < \infty\right\}$. We have a metric. $\|f\|_2 = \left(\int_0^1 |2|^2 \, dx\right)^{1/2}$, and we have an inner product. $\langle f, g\rangle = \int_0^1 f\overline{g} dx$

- $\mathcal{C}(\mathbb{T})$ is dense in L^2, that is for all $f \in L^2(\mathbb{T})$ and $\varepsilon > 0$, there is a $g \in \mathcal{C}(\mathbb{T})$ so that $\|f - g\|_2 < \varepsilon$
- Holder inequality holds:

$$\int_0^1 |f| \, dx \leq \left(\int_0^1 |f|^2 \, dx\right)^{1/2} \tag{8.1}$$

  which means $\|f\|_1 \leq \|f\|_2$, and hence $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$. The containment is proper; just consider $f(x) = x^{-\alpha}$ for $0 < x < 1$, and $\frac{1}{2} < \alpha < 1$.

Now, let $e_n := e^{2\pi i n x}$. We proved that

$$\langle e_n, e_m\rangle = \int_0^1 e^{2\pi i (n-m) x} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \tag{8.2}$$

This means that $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal set. For $f \in L^2$,

$$\widehat{f}(n) = \int_0^1 f(x)e^{-2\pi i n x} = \langle f, e_n\rangle \tag{8.3}$$

Also,

$$S_N f = \sum_{|n|\le N} \widehat{f}(n)e^{2\pi i n x} = \sum_{|n|\le N} \langle f, e_n\rangle e_n \tag{8.4}$$

Lemma 8.1. *The linear span of $\{e_n\}_{n\in\mathbb{Z}}$ is dense in $L^2(\mathbb{T})$.*

Proof. Let $f \in L^2(\mathbb{T})$, $\varepsilon > 0$. Let $g \in \mathcal{C}(\mathbb{T})$ such that $\|f-g\|_2 < \frac{\varepsilon}{2}$. Now, if I take N large enough so that $|g(x) - \sigma_N g(x)| < \frac{\varepsilon}{2}$ for all $x \in \mathbb{T}$. Then $\|g - \sigma_N g\|_2 < \frac{\varepsilon}{2}$, so $\|f - \sigma_N g\| < \varepsilon$. Since $\sigma_N g$ is a finite combination of $\{e_n\}$s, we are done. □

Lemma 8.2. *if $p(x) =$ a trig polynomial of degree at most N, then*

$$\|S_N f - f\|_2 \le \|P - f\|_2 \tag{8.5}$$

Proof. Consider $P - f = (P - S_N f) + (S_N f - f)$. I claim that these functions are orthogonal, so that

$$\langle P - S_N f, S_N f - f\rangle = 0 \tag{8.6}$$

So, let $|n| \le N$. Then

$$\begin{aligned}\langle S_N f - f, e_n\rangle &= \langle S_N f, e_n\rangle - \langle f, e_n\rangle \\ &= \langle \sum_{|m|\le N} \widehat{f}(m)e_m, e_n\rangle - \widehat{f}(n) \\ &= \widehat{f}(n) - \widehat{f}(n) = 0\end{aligned} \tag{8.7}$$

If $Q(x) = \sum_{|n|\le N} b_n e_n$, then $\langle S_N f - f, Q\rangle = 0$ by linearity. Apply this with $Q = P - S_N f$ to prove the claim. We can now take the norm and get

$$\|P - f\|_2^2 = \|P - S_N f\|^2 + \|S_N f - f\|_2^2 \ge \|S_N f - f\|_2^2 \tag{8.8}$$

□

Now combine the lemmas to get

Theorem 8.3. *If $f \in L^2(\mathbb{T})$, then $\|S_N f - f\|_2 \to 0$ as $N \to \infty$. Also,*

$$\sum_{-\infty}^{\infty} \left|\widehat{f}(n)\right|^2 = \|f\|_2^2 \tag{8.9}$$

Corollary 8.4. *If $f, g \in L^2(\mathbb{T})$ then*

$$\langle f, g\rangle = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{g}(n)} \tag{8.10}$$

8.3 Equidistributed sequences

Definition 8.5. $\{\zeta_n\}_{n=1}^{\infty}$ *for* $\zeta_n \in \mathbb{T}$ *is* equidistributed *if for any open interval* $(a,b) \subset \mathbb{T}$*, we have*

$$\lim_{N\to\infty} \frac{1}{N} \# \{i : 1 \le i \le N,\, \zeta_i \in (a,b)\} = b - a \tag{8.11}$$

For example, taking

$$\left\{0, \frac{1}{2}, 0, \frac{1}{3}, \frac{2}{3}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 0, \frac{1}{5}, ...\right\} \tag{8.12}$$

These essentially form a "uniform sampling" of the interval. One thing we would like to have is that $\int_0^1 f(x)dx$ could be evaluated using these, like we might have

$$\int_0^1 f(x)dx \leftarrow \frac{1}{N}\sum_{n=1}^{N} f(\zeta_n) \tag{8.13}$$

The real question is, what functions satisfy this?

Lecture 9

An equivalent definition of equidistant sequences is that we can replace equation (8.11) with

$$\int \chi_{(a,b)}(x)dx = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \chi_{(a,b)}(\zeta_i) \tag{9.1}$$

and this is also the case for step functions.

Theorem 9.1 (Weyl's Criterion)**.** *The following are equivalent for a* $\{\zeta_n\}_{n=1}^{\infty}$ *with* $\zeta_n \in \mathbb{T}$*,*

1. $\{\zeta_n\}_{n=1}^{\infty}$ *is equidistributed*
2. *For any* $f \in \mathcal{C}(\mathbb{T})$*, we have*

$$\int_0^1 f(x)dx = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} f(\zeta_i) \tag{9.2}$$

3. *For all* $k \in \mathbb{Z}$*,* $k \ne 0$*, we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} e^{2\pi i k \zeta_i} = 0 \tag{9.3}$$

Proof.

1. $\implies$ 3: by passing to the special case $f(x) = e^{2\pi i k x}$.
3. $\implies$ 2: Let $f \in \mathcal{C}(\mathbb{T})$ and let $\varepsilon > 0$. Let P be a trig polynomial such that $|f(x) - P(x)| < \varepsilon$ for all $x \in \mathbb{T}$. Then we have

$$\left| \frac{1}{N}\sum_{i=1}^{N} P(\zeta_n) - \int_0^1 P(x)dx \right| < \varepsilon \tag{9.4}$$

if N is large enough. The contribution of the integral here is really just to eliminate the constant term from the sum! So we have

$$\begin{aligned}\left|\frac{1}{N}\sum_{i=1}^{N} f(\zeta_n) - \int_0^1 f(x)dx\right| &\leq \frac{1}{N}\left|\sum_{i=1}^{N}(P(\zeta_n) - P(\zeta_n))\right| \\ &\quad + \left|\frac{1}{N}\sum_{i=1}^{N} P(\zeta_n) - \int_0^1 P(x)dx\right| \\ &\quad + \int_0^1 |f(x) - P(x)|\, dx \\ &< 3\varepsilon\end{aligned} \tag{9.5}$$

2. $\implies$ 1: Let $g = \chi_{(a,b)}$, and we will approximate g by continuous functions. Given $\varepsilon > 0$, let f_+ and f_- be

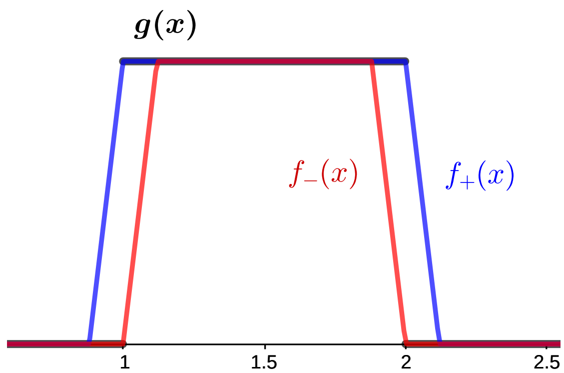


Figure 2: f_- and f_+, assuming $(a,b) = (1,2)$.

Then we have $f_- \leq g \leq f_+$, and choose them so that

$$\int f_- \leq \int g \leq \int f_+$$
$$-\varepsilon + \int f_+ \leq \int g \leq \varepsilon + \int f_-$$

Then

$$\frac{1}{N}\sum_{n=1}^{N} f_-(\zeta_n) \leq \frac{1}{N}\sum_{n=1}^{N} g(\zeta_n) \leq \frac{1}{N}\sum_{n=1}^{N} f_+(\zeta_n) \tag{9.6}$$

and taking the limit, we see that if N is large enough

$$-2\varepsilon \int g - \varepsilon + \int f_-(\zeta_n) \leq \frac{1}{N}\sum_{n=1}^{N} g(\zeta_n) \leq \varepsilon + f_+(\zeta_n) < 2\varepsilon + \int g \tag{9.7}$$

$\square$

Corollary 9.2. *If ζ is irrational, then the sequence $\{n\zeta\}_{n=1}^{\infty}$ is equidistributional*

Proof. Use (3.) $\square$

Lecture 10

What about $\{n^d\gamma\}_{n=1}^{\infty}$ for $d = 2, 3$? What about $\{P(n)\gamma\}_{n=1}^{\infty}$? What about just $\{P(n)\}_{n=1}^{\infty}$ when $P(n)$ has an irrational coefficient? ETC...

Let $P(x) = c_d x^d + c_{d-1}x^{d-1} + ...$ and fix an $h \in \mathbb{R}$. Consider

$$P(x+h) - P(x) = c_d(d+h)^d + ... \quad - c_d x^d - .... \tag{10.1}$$

which is a polynomial of degree one less than d. Based on this, we set up an inductive scheme. If we know some statement $(*)$ for polynomials of degree 1, and we know that "$(*)$ for degree $d \implies (*)$ for degree $d+1$", then we know it for all polynomials.

Lemma 10.1. *Let $a_n \in \mathbb{C}$, $|a_n| \leq 1$, then for $1 \leq H \leq N$,*

$$\left|\frac{1}{N}\sum_{n=1}^{N} a_n\right| \leq C\left(\frac{1}{H}\sum_{h=0}^{H-1}\left|\frac{1}{N}\sum_{n=1}^{N} a_{n+h}\overline{a_n}\right|\right)^{1/2} + O\left(\frac{H}{N}\right) \tag{10.2}$$

for some $C \in \mathbb{R}$.

Proof. For $0 \leq h < H$

$$\frac{1}{N}\sum_{n=1}^{N} a_n = \frac{1}{N}\sum_{n=1}^{N} a_{n+h} + O\left(\frac{H}{N}\right) \tag{10.3}$$

To see this, call these sums S_1 and S_2. Then

$$|S_1 - S_2| \leq \frac{1}{N}\left(|a_1| + ... + |a_h| + |a_{N+1}| + ... + |a_{N+h}|\right) \leq \frac{1}{N}2h \leq \frac{2H}{N} \tag{10.4}$$

Now average equation (10.3) for $0 \leq h \leq H-1$. We get

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N} a_n &= \frac{1}{H}\sum_{h=0}^{H-1}\left(\frac{1}{N}\sum_{n=1}^{N} a_{n+h}\right) + O\left(\frac{H}{N}\right) \\ &= \frac{1}{H}\sum_{n=1}^{N}\left(\frac{1}{N}\sum_{h=0}^{H-1} a_{n+h}\right) + O\left(\frac{H}{N}\right)\end{aligned} \tag{10.5}$$

And applying Cauchy Schwartz, we have

$$\begin{aligned}\left|\frac{1}{N}\sum_{n=1}^{N} a_n\right| &\leq \frac{1}{H}\sum_{n=1}^{N}\left(\frac{1}{N}\left|\sum_{h=0}^{H-1} a_{n+h}\right|\right) + O\left(\frac{H}{N}\right) \\ &\leq \left(\frac{1}{N}\sum_{n=1}^{N}\left|\frac{1}{H}\sum_{h=0}^{H-1} a_{n+h}\right|^2\right)^{1/2} + O\left(\frac{H}{N}\right) \\ &= \left(\frac{1}{N}\sum_{n=1}^{N}\frac{1}{H^2}\sum_{h,h'}^{H-1} a_{n+h}\overline{a_{n+h'}}\right)^{1/2} + O\left(\frac{H}{N}\right) \\ &= \left(\frac{1}{N}\sum_{n=1}^{N}\frac{2}{H}\sum_{h=0}^{H-1} a_{n+h}\overline{a_n}\right)^{1/2} + O\left(\frac{H}{N}\right)\end{aligned} \tag{10.6}$$

At the last step: if $h' \leq h$, relabel $n + h' \to n$, and $h - h' \to h$. You get an additional $O\left(\frac{H}{N}\right)$ error terms from this. This is apparently very confusing to see. Note that the $a_i\overline{a_i}$ terms occur twice in the second sum (see the factor of 2), but once in the first. So we need to check that these are an acceptable error. Apparently we can't figure out how to do this atm, but the equation should hold so we power through. Changing the order of summation now completes the lemma. □

Corollary 10.2. *If we have a sequence $\{\zeta_n\} \subset \mathbb{T}$, and $\{\zeta_{n+h} - \zeta_n\}_{n=1}^{\infty}$ is equidistributed for each fixed $h \in \mathbb{N}$, then $\{\zeta_n\}$ is equidistributed as well.*

Proof. We use Lemma 10.1 with $H \approx \sqrt{N}$. Then

$$\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k\zeta_n}\right| \leq C\left(\frac{1}{H}\sum_{h=0}^{H-1}\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k(\zeta_{n+h}-\zeta_n)}\right|\right)^{1/2} + O\left(\frac{H}{N}\right) \tag{10.7}$$

□

Lecture 11

Apply this corollary to polynomials.

Corollary 11.1. *If η is irrational, then $\{n^2\eta\}_{n=1}^{\infty}$ is equidistributed.*

Proof. Let $h \in \mathbb{N}$. Consider

$$(n+h)^2\eta - n^2\eta = n^2\eta + 2nh\eta + h^2\eta - n^2\eta = n(2h\eta) + h^2\eta = n\xi + c \tag{11.1}$$

is equidistributed. □

More generally, any polynomial with at least one irrational coefficient, you get an equidistributed sequence. Other examples: {fractional parts of n^σ, for a fixed $0 < \sigma < 1$}. The sequence {fractional parts of $\log n$} is not equidistributed.

11.4 Fourier transform on $\mathbb{R}^n$

If $f \in L^1(\mathbb{R}^n)$ (that is, $\int_{\mathbb{R}^n} |f|\,dx < \infty$), we define the Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i(x\cdot\xi)} f(x)dx \tag{11.2}$$

and note that x, $\xi \in \mathbb{R}^n$ and the $\cdot$ between them is the dot product here. We will also need the inverse Fourier transform

$$\check{g}(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} g(\xi)d\xi \tag{11.3}$$

we would expect that

$$\begin{aligned}
\check{\widehat{f}}(x) &= \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\widehat{f}(\xi)d\xi \\
&= \int e^{2\pi i x\cdot\xi}\int e^{-2\pi i y\cdot\xi}f(y)dyd\xi \\
&= \int f(y)\left(\int e^{2\pi i(x-y)\cdot\xi}d\xi\right) \\
&= \int f(y)\delta_{x=y}dy \\
&= f(x)
\end{aligned} \tag{11.4}$$

However, this equation is all wrong. Line 3 onward makes no sense! However, we can use some approximation identities to make something like this rigorous.

If $f \in L^1$, then $\left|\widehat{f}(\xi)\right| \leq \int |f| = \|f\|_1$. You can also extend the definition of the Fourier transform to measures. We can define

$$\widehat{\mu}(\xi) = \int e^{-2\pi i x\cdot\xi}d\mu \tag{11.5}$$

If $f \in L^1$, then $\widehat{f}$ is uniformly continuous. To see this, consider

$$\begin{aligned}
\left|\widehat{f}(\xi-\eta) - \widehat{f}(\xi)\right| &= \int \left(e^{-2\pi i x\cdot(x+\eta)} - e^{-2\pi i x\cdot\eta}\right)f(x)dx \\
&= \int e^{-2\pi i x\cdot(x)}\left(e^{-2\pi i x\cdot\eta} - 1\right)f(x)dx \\
&\leq \int \left|e^{-2\pi i x\cdot\eta} - 1\right||f(x)|\,dx
\end{aligned} \tag{11.6}$$

Apply the dominated convergence theorem. Each of the integrals are bounded $\leq \int 2|f|$, so the whole integral goes to 0 as $\eta \to 0$ uniformly in ξ, since there is no dependence on ξ.

11.5 Translation/modulation

For a fixed $a \in \mathbb{R}^n$, define $f_a(x) = f(x-a)$. Then

$$\begin{aligned}
\widehat{f_a}(\xi) &= \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}f_a(x)dx \\
&= \int_{\mathbb{R}^n} f(y)e^{-2\pi i(y+a)\cdot\xi}dx \\
&= e^{-2\pi i a\cdot\xi}\widehat{f}(\xi)
\end{aligned} \tag{11.7}$$

And another identity is that if $e_a(x) = e^{-2\pi i a\cdot x}$, then $\widehat{e_a f}(\xi) = \widehat{f}(\xi+a)$.

Now, let $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ be invertible (so it corresponds to a non-singular $n \times n$ matrix). Then let $f_T(x) = f(Tx)$. Then

$$\begin{aligned}
\widehat{f_T}(\xi) &= \int f(Tx)e^{-2\pi i x\cdot\xi}dx \\
&= \int f(y)e^{-2\pi i (T^{-1}y)\cdot\xi}\frac{1}{|\det T|}dy \\
&= \int f(y)e^{-2\pi i y\cdot (T^{-1})^\intercal\xi}\frac{1}{|\det T|}dy \\
&= \frac{1}{|\det T|}\widehat{f}\left((T^{-1})^\intercal\xi\right)
\end{aligned} \tag{11.8}$$

Lecture 12

Some special cases:

1. If $T = \lambda I$, then $\widehat{f_T} = \frac{1}{|\lambda|^n}\widehat{f}(\lambda^{-1}\xi)$. For example, consider a bump function.

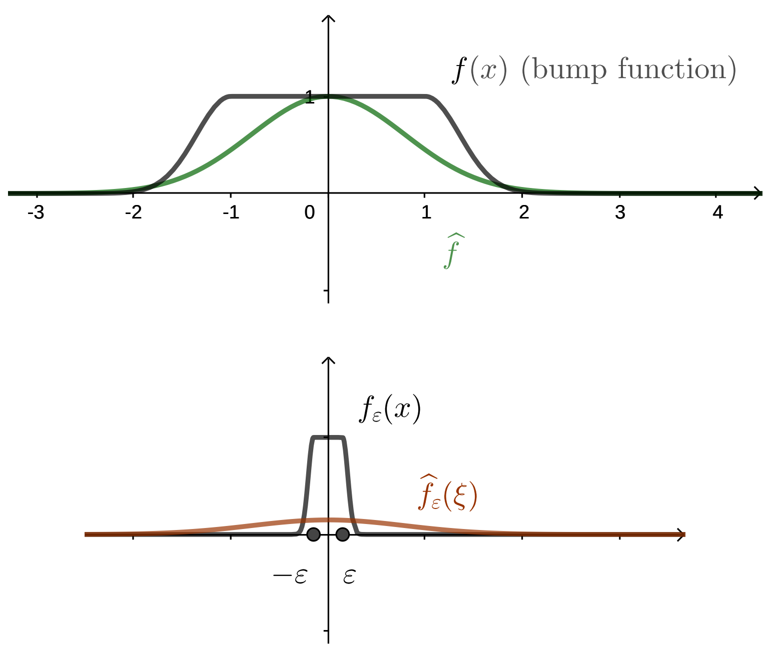


Figure 3: A narrower bump function has a smaller bump for its transform

2. If T is orthogonal (a rotation or reflection), then $T^{-1} = T^\intercal$ and $|\det T| = 1$, so $\widehat{f_T}(\xi) = \widehat{f}(T\xi)$.

3. If f is radially symmetric (i.e. $f(x) = h(|x|)$ for some function h), then $\widehat{f}$ is also radially symmetric. This is because radially symmetric implies $f_T = f$ for all orthogonal T, which by the previous property means $\widehat{f}(T\xi) = \widehat{f_T}(\xi) = \widehat{f}(\xi)$ giving $\widehat{f}$ the same invariance property.

12.6 Schwartz functions on $\mathbb{R}^n$

Define

$$\mathcal{J} = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^n) \;\middle|\; \text{for any multiindices } \alpha, \beta, \text{ we have } \sup_{x\in\mathbb{R}^n}\left|x^\alpha D^\beta f(x)\right| < \infty \right\} \tag{12.1}$$

Notation: for $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ for $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \{0, 1, 2, \dots\}$,

$$x^\alpha = x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n} \qquad D^\beta f = \frac{\partial \beta_1 + \dots + \beta_n}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}} f \tag{12.2}$$

Examples:

1. $f \in C_c^\infty(\mathbb{R}^n) \subset \mathcal{J}$ (the subscript c means compactly supported), then $D^\beta f \in C_c^\infty(\mathbb{R}^n)$ for all β. But $\widehat{f} \notin C_c^\infty$.
2. $f(x) = -e^{|x|^2} \in \mathcal{J}$. This is because e^{-x} goes to 0 faster than any polynomial equation.

Lemma 12.1. *If $f \in \mathcal{J}$, then*

$$\begin{aligned} \widehat{\frac{\partial f}{\partial x_j}}(\chi) &= \int \frac{\partial f}{\partial x_j}(x) e^{-2\pi i x \cdot \xi} dx \\ &= -\int f(x)(-2\pi i \xi_j) e^{-2\pi i x \cdot \xi} dx \\ &= (2\pi i \xi_j) \widehat{f}(\xi) \end{aligned} \tag{12.3}$$

Similarly, we can see

$$(\widehat{-2\pi i x_j})\, f(\xi) = \frac{\partial \widehat{f}}{\partial \xi_j}\, (\xi) \tag{12.4}$$

Corollary 12.2. *If $f \in \mathcal{J}$, then $\widehat{f} \in \mathcal{J}$.*

Proof. Iterate the formulas in the lemma.

$$\widehat{D_\beta f}(\xi) = (2\pi i \xi)^\beta \widehat{f}(\xi) \qquad (\widehat{-2\pi i x})^\alpha f(\xi) = D^\alpha \widehat{f}(\xi) \tag{12.5}$$

So let $f \in \mathcal{J}$. Then $\left| x^\alpha D^\beta f \right| \leq C_{\alpha,\beta}$ for all α, β. Now, $\widehat{f} \in \mathcal{C}^\infty$ and

$$\left| \xi^\alpha D^\beta \widehat{f} \right| = \left| \xi^\alpha (\widehat{-2\pi i x})^\beta f(\xi) \right| = \left| \frac{D^\alpha}{(2\pi i)^{abs\alpha}} \widehat{(-2\pi i x)^\beta f(\xi)} \right| \tag{12.6}$$

$\square$

Lecture 13

Definition 13.1. *A Gaussian is a function $e^{-\pi a |x|^2}$ for $a > 0$ and $x \in \mathbb{R}^n$. These functions are $\mathcal{C}^\infty$ and Schwartz.*

Lemma 13.2. *Let $f(x) = e^{-\pi x^2}$ for $x \in \mathbb{R}$. Then $\widehat{f}(\xi) = f(\xi)$.*

Proof. Note that f satisfies $f(0) = 1$ and $f'(x) = -2\pi x f(x)$. I claim that $\widehat{f}$ also satisfies this. Let $F = \widehat{f}$ for notation's sake.

$$F(0) = \widehat{f}(0) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x \cdot 0} dx = 1 \tag{13.1}$$

Recall, the way to solve this is to do $\int_0^\infty e^{-\pi x^2}dx$ and set up a 2d integral and evaluate using polar coordinates. We now use the fact that f is Schwarts to differentiate under the integral, and then integrate by parts from line 3 to 4 (note: no boundary terms).

$$\begin{aligned} F'(\xi) &= \frac{d}{d\xi}\int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx \\ &= \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}(-2\pi i x)dx \\ &= i\int_{-\infty}^{\infty} f'(x)e^{-2\pi i x\xi}dx \\ &= -i\int_{-\infty}^{\infty} f(x)(-2\pi i\xi)e^{-2\pi i x\xi}dx \\ &= -i(-2\pi i\xi)\int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx \\ &= -2\pi\xi F(\xi) \end{aligned} \tag{13.2}$$

so by uniqueness of ODEs, $f = F$. $\square$

Proposition 13.3. *Let $f \in L^1(\mathbb{R}^n)$, and* $\operatorname{supp} f \subset \{|x| \leq R\}$. *Then $\widehat{f} \in \mathcal{C}^\infty$ and*

$$\left|D^\alpha \widehat{f}\right| \leq (2\pi R)^{|\alpha|}\left\|f\right\|_1 \tag{13.3}$$

(So the decay of f implies regularity of $\widehat{f}$.)

Proposition 13.4. *Assume $f \in \mathcal{C}^N(\mathbb{R}^n)$, and $D^\alpha f \in L^1(\mathbb{R}^n)$ for $0 \leq |\alpha| \leq N$. Then*

$$\widehat{D^\alpha f}(\xi) = (2\pi i\xi)^\alpha \widehat{f}(\xi) \qquad \left|\widehat{f}(\xi)\right| \leq C(1+|\xi|)^{-N} \tag{13.4}$$

Note that we already had proved the first formula for Schwarts functions, it followed from integration by parts. The same proof actually works here too if $f \in \mathcal{C}_c^N(\mathbb{R}^n)$, and you might be able to work though it if $f \to 0$ at ∞, but this proposition does not assume that! We will use some kind of approximation argument.

Proof. (1) $\implies$ (2): For $0 < |\alpha| \leq N$, consider

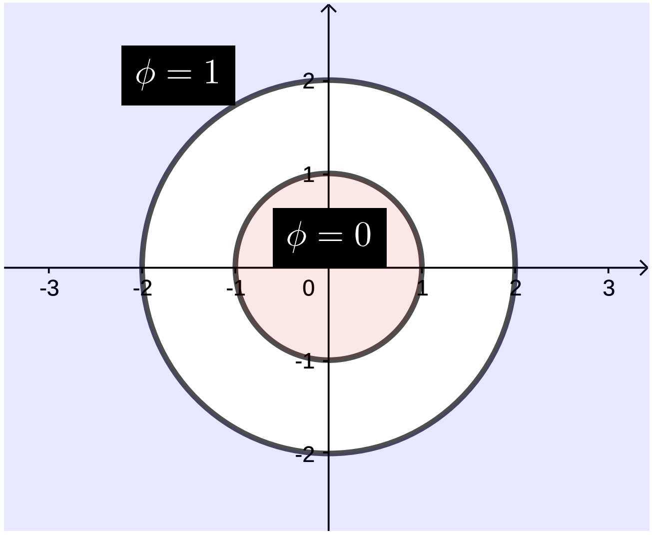
$$\begin{aligned} \left|\xi^\alpha \widehat{f}(\xi)\right| &= \left|\left(\frac{1}{2\pi i}\right)^{|\alpha|}\widehat{D^\alpha f}(\xi)\right| \\ &\leq \left(\frac{1}{2\pi}\right)^{|\alpha|}\left\|D^\alpha f\right\|_1 \end{aligned} \tag{13.5}$$

Thus $\left|P(\xi)\widehat{f}(\xi)\right|$ is bounded for all P polynomials of degree $\geq N$. $\square$

Lecture 14

Define a continuous bump function $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ so that

$$\phi = \begin{cases} 1 & |x| \leq 1 \\ \text{something nice} & |x| \in (0,1)\,, \qquad 0 \leq \phi \leq 1 \\ 0 & |x| \geq 2 \end{cases} \tag{14.1}$$



and also let $\phi_k(x) = \phi(x/k)$ for $k \in \mathbb{N}$. We use $f_k := \phi_k f$ to approximate f. We do have the conclusion of the prop for each f_k, so consider

1. $$\left|\frac{\partial}{\partial x_j}\phi_k(x)\right| = \left|\frac{1}{k}\frac{\partial \phi}{\partial x_j}\left(\frac{x}{k}\right)\right| \leq \frac{1}{k}\left\|\nabla \phi\right\|_\infty \tag{14.2}$$

   Iterating this, we can get

   $$|D^\alpha \phi_k(x)| \leq \frac{1}{k^{|\alpha|}}\left\|D^\alpha \phi\right\|_\infty \tag{14.3}$$

2. For $|\alpha| \geq 1$, $D^\alpha \phi_k$ is supported in $\{x : k \leq |x| \leq 2k\}$.

3. We have $\lim_{k\to\infty}\left\|D^\alpha f_k - D^\alpha f\right\|_1 = 0$ for $|\alpha| \leq N$. To see this, consider

   $$\left\|D^\alpha f_k - D^\alpha f\right\|_1 \leq \left\|D^\alpha f_k - \phi_k D^\alpha f\right\|_1 + \left\|\phi_k D^\alpha f - D^\alpha f\right\|_1 \tag{14.4}$$

   By the dominated convergence theorem, the second term on the RHS goes to 0 where we use $2D^\alpha f$ as the dominating function (or, “majorant”). Further, if $|\alpha| = 0$, then $f_k = \phi_k f$, so the first term is 0. So now we suppose $|\alpha| \geq 1$. Then we have

   $$D^\alpha f_k = D^\alpha\left(\phi_k f\right) = \phi_k D^\alpha f + \sum_{|\beta|\geq 1} c_\beta \left(D^\beta \phi_k\right)\left(D^{\alpha-\beta} f\right) \tag{14.5}$$

   and so

   $$\left\|D^\alpha f_k - \phi_k D^\alpha f\right\|_1 \leq \sum_{|\beta|\geq 1} c_\beta \left\|\left(D^\beta \phi_k\right)\left(D^{\alpha-\beta} f\right)\right\|_1 \tag{14.6}$$

   so now let’s look at

   $$\left\|\left(D^\beta \phi_k\right)\left(D^{\alpha-\beta} f\right)\right\|_1 \leq \left\|D^\beta \phi_k\right\|_\infty \left\|D^{\alpha-\beta} f\right\|_1 \leq \frac{1}{k^{|\beta|}} C(\phi,\beta)\left\|D^{\alpha-\beta} f\right\|_1 \to 0 \tag{14.7}$$

   as $k \to \infty$.

So recall we have

$$\widehat{D^\alpha f_k}(\xi) = (2\pi i \xi)^\alpha \, \widehat{f}_k(\xi) \tag{14.8}$$

for each f_k. As we let $k \to \infty$, from property 3, we have

$$\left| \widehat{D^\alpha f_k(\xi)} - \widehat{D^\alpha} f(\xi) \right| \leq \|...\|_1 \to 0 \tag{14.9}$$

so the right hand side of equation (14.8) converges uniformly to $\widehat{D^\alpha} f(\xi)$, and $\widehat{f}_k(\xi)$ goes uniformly to $\widehat{f}(\xi)$, so the whole left hand side of equation (14.8) goes pointwise to $(2\pi i\xi)^\alpha \, \widehat{f}(\xi)$. This is only possible if the equation holds for f.

14.7 Convolution

If $f, g : \mathbb{R}^n \mapsto \mathbb{C}$, define

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y) dy = \int_{\mathbb{R}^n} f(y) g(x-y) dy \tag{14.10}$$

If f, $g \in L^1$ we claim that $f * g$ is well defined as an L^1 function. This is because we can write

$$\int |f * g(x)| \, dx \leq \int \int |f(x-y)| \, |g(y)| \, dy dx = \int \int |f(u)| \, |g(y)| \, du dy = \|f\|_1 \, \|g\|_1 \tag{14.11}$$

where we changed variables $u = x - y$.

Now, if we take $f \in L^1$ and $g \in L^\infty$, then

$$|f * g(x)| \leq \int |f(y)| \, |g(x-y)| \, dy \leq \|g\|_\infty \, \|f\|_1 \tag{14.12}$$

so $f * g$ is in L^∞ as well. More generally, if $f \in L^p$ and $g \in L^q$ with $1 \leq p, q \leq \infty$ dual to each other ($\frac{1}{p} + \frac{1}{q} = 1$), then $|f * g(x)| \leq \|f\|_p \, \|g\|_q$. To see this you must apply Hölder's inequality.

MORE generally, Young's inequality says that

$$\|f * g\|_r \leq \|f\|_p \, \|g\|_q \tag{14.13}$$

if $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

It is useful to think of convolution as averaging. If $g = \frac{1}{|B_\varepsilon|} \chi_{B_\varepsilon}$, where $B_\varepsilon = \{x : |x| < \varepsilon\}$, then

$$f * g(x) = \int f(x-y) g(y) dy = \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} f(x-y) dy = \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon(x)} f(y) dy \tag{14.14}$$

Lecture 15

Lemma 15.1. *Suppose $\phi \in \mathcal{J}$ and $f \in L^1$. Then $f * \phi \in C^\infty$ and $D^\alpha(f * \phi) = f * D^\alpha \phi$.*

Proof.

$$\begin{aligned} \frac{\partial}{\partial x_j}(f * \phi) &= \frac{\partial}{\partial x_j} \int \phi(x-y) f(y) dy \\ &= \lim_{h \to 0} \int \frac{1}{h} \left(\phi(x + he_j - y) - \phi(x-y) \right) f(y) dy \\ &= \int \frac{\partial}{\partial x_j} \phi(x-y) f(y) dy \end{aligned} \tag{15.1}$$

To justify differentiating under the integral, not that $\left|\frac{1}{h}(\phi(x+he_j-y)-\phi(x-y))\right| \leq \left\|\frac{\partial \phi}{\partial x_j}\right\|_\infty$ by the mean value theorem. So the integrand is bounded by $\left\|\frac{\partial \phi}{\partial x_j}\right\|_\infty |f(y)| \in L^1$, so by the dominated convergence theorem, ye. We can now iterate this to get all the derivs since ϕ is schwartz. $\square$

Corollary 15.2. *If $f, g \in \mathcal{J}$, then $f * g \in \mathcal{J}$.*

Proof. By the lemma, $f * g \in C^\infty$ and $D^\alpha(f * g) \in C^\infty$, so we just need to show that $(1+|x|)^N D^\alpha(f*g)$ is bounded for all N. It suffices to show that $(1+|x|)^N f * g$ bounded for all N, $f, g \in \mathcal{J}$. This is because $D^\alpha f \in \mathcal{J}$ as well so.

$$\begin{aligned} \left|(1+|x|)^N f * g(x)\right| &\leq (1+|x|)^N \int |f(x-y)g(y)|\, dy \\ &\leq \int (1+|y|)^N (1+|x-y|)^N |f(x-y)|\, |g(y)|\, dy \\ &= \left\|(1+|x-y|)^N f(x-y)\right\|_\infty \int (1+|y|)^N |g(y)|\, dy \end{aligned} \tag{15.2}$$

The first term in the product is bounded because $f \in \mathcal{J}$. For the integral term, note that $|g(y)|(1+|y|)^M \leq C_M$ for all $M \in \mathbb{N}$. So we can take $M = N+n+1$ where n is the dimension of the space. Then we get $(1+|y|)^N |g(y)| \leq C_M (1+|y|)^{-n-1}$ and the right hand side is in $L^1(\mathbb{R}^n)$.

Also, you might see that moving to the second line here was non trivial. We check this:

$$\begin{aligned} (1+|y|)(1+|x-y|) &= 1+|y|+|x-y|+|x-y|\,|y| \\ &\geq 1+|y|+|x-y| \\ &\geq 1+|x| \end{aligned} \tag{15.3}$$

by the triangle inequality on $x = (x-y)+y$. $\square$

Let $\phi \in \mathcal{J}$, $\int \phi = 1$, $\varepsilon > 0$, and define

$$\phi^\varepsilon(x) = \frac{1}{\varepsilon^n} \phi\left(\frac{x}{\varepsilon}\right) \tag{15.4}$$

We want to approximate f by $f * \phi^\varepsilon$ as $\varepsilon \to 0$ (should have decay same or better than f at infinity).

Lemma 15.3.

1. *$\int \phi^\varepsilon(x) dx = 1$*
2. *$\int |\phi^\varepsilon(x)|\, dx = \int |\phi(x)|\, dx < M$ uniformly in ε.*
3. *$\int_{|x|>\eta} |\phi^\varepsilon(x)|\, dx \to 0$ as $\varepsilon \to 0$ for any fixed $\eta > 0$.*

Lemma 15.4.

1. *If $f \in C(\mathbb{R}^n)$, $f \to 0$ at ∞, then $\phi^\varepsilon * f \to f$ uniformly as $\varepsilon \to 0$.*
2. *If $f \in L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$, then $\|\phi^\varepsilon * f - f\|_p \to 0$ as $\varepsilon \to 0$.*

Proof.

1. Assume that $f \in C(\mathbb{R}^n)$, $f \to 0$ at ∞ (hence, uniformly continuous). Let $\varepsilon_1 > 0$. Then by uniform continuity, there is an $\eta > 0$ so that $|f(x-y) - f(x)| \leq \frac{\varepsilon_1}{M}$ for all $|y| \leq \eta$ (the M here is the same one in the previous lemma).

$$\begin{aligned} |\phi^\varepsilon * f(x) - f(x)| &= \left| \int \phi^\varepsilon(y) f(x-y) dy - f(x) \right| \\ &= \left| \int \phi^\varepsilon(y) \left(f(x-y) - f(x) \right) dy \right| \\ &\leq \int |\phi^\varepsilon(y)| \, |f(x-y) - f(x)| \, dy \\ &= \int_{|y|<\eta} ...dy + \int_{|y|>\eta} ...dy \end{aligned} \tag{15.5}$$

The first equation is

$$\int_{|y|<\eta} ...dy \leq \frac{\varepsilon_1}{M} \int |\phi^\varepsilon(y)| \, dy \leq \varepsilon_1$$

We also have

$$\int_{|y|>\eta} ...dy \leq 2 \|f\|_\infty \int_{|y|>\eta} |\phi^\varepsilon(y)| \, dy \to 0$$

as $\varepsilon \to 0$, so we can make it less than ε_1 if ε is small enough.

2. If (2) holds for $f \in L^p \cap C_c(\mathbb{R}^n)$, then it holds for all $f \in L^p(\mathbb{R}^n)$. To see this let $f \in L^p(\mathbb{R}^n)$, $\delta > 0$, $g \in C_c(\mathbb{R}^n)$, $\|g - f\|_p < \delta$. Then

$$\|f * \phi^\varepsilon - f\|_p \leq \|(f-g) * \phi^\varepsilon\|_p + \|g * \phi^\varepsilon - g\|_p + \|f - g\|_p \tag{15.6}$$

so by some estimates yea.

□

Lecture 16

Theorem 16.1 (Fourier Duality)**.** *If $f, g \in L^1(\mathbb{R}^n)$, then*

$$\int \widehat{f}(x) g(x) dx = \int f(y) \widehat{g}(y) dy \tag{16.1}$$

Note that if $f \in L^1$, $\widehat{f} \in L^\infty$ and $g \in L^1$ so $\widehat{f} g \in L^1$.

Proof.

$$\begin{aligned} \int \widehat{f}(x) g(x) dx &= \int \int f(y) e^{-2\pi i x \cdot y} dy g(x) dx \\ &= \int \int f(y) g(x) e^{-2\pi i x \cdot y} dy dx \\ &= \int \int g(x) e^{-2\pi i x \cdot y} dx f(y) dy \\ &= \int f(y) \widehat{g}(y) dy \end{aligned} \tag{16.2}$$

□

Theorem 16.2 (Inversion formula)**.** *Assume that f, $\widehat{f} \in L^1$ (which is true for example for schwartz functions). Then*

$$f(x) = \int \widehat{f}(y) e^{2\pi i x \cdot y} dy \tag{16.3}$$

for almost every x.

Proof. Let $\Gamma_\varepsilon(x) = e^{-\pi\varepsilon^2|x|^2}$. Then

$$\widehat{\Gamma_\varepsilon}(\zeta) = \Gamma^\varepsilon(\zeta) = \frac{1}{\varepsilon^n} e^{-\pi|x|^2/\varepsilon^2} \tag{16.4}$$

Let $I_\varepsilon(x) = \int \widehat{f}(\zeta) g_\varepsilon(\zeta) d\zeta$, and $g_\varepsilon(\zeta) + \Gamma_\varepsilon(\zeta) e^{2\pi i \zeta \cdot x}$. Now, by Fourier Duality,

$$\begin{aligned} I_\varepsilon(x) &= \int \widehat{f}(\zeta) g_\varepsilon(\zeta) d\zeta \\ &= \int f(y) \widehat{g_\varepsilon}(y) dy \\ &= \int f(y) \widehat{\Gamma_\varepsilon}(y - x) dy \\ &= \int f(y) \Gamma^\varepsilon(x - y) dy \\ &= f * \Gamma^\varepsilon(x) \end{aligned} \tag{16.5}$$

we used in here that Γ^ε is an even function. Hence, $I_\varepsilon \to f$ in L^1, but also

$$I_\varepsilon = \int \widehat{f}(\zeta) \Gamma_\varepsilon(\zeta) e^{2\pi i \zeta \cdot x} d\zeta \tag{16.6}$$

Now, $\Gamma_\varepsilon(x)$ goes to 1 pointwise, so by dominated convergence theorem,

$$I_\varepsilon(x) \to \int \widehat{f}(\zeta) e^{2\pi i \zeta \cdot x} d\zeta \tag{16.7}$$

□

Lecture 17

Observe that if $\widehat{f} \notin L^1$, the integral in the inversion formula may not converge. But what about

$$\lim_{R \to \infty} \int_{|y| \le R} \widehat{f}(y) e^{2\pi i x \cdot y} \tag{17.1}$$

is this defined pointwise a.e.? Is it defined in some L^p space? In L^2, these functions converge to f, but otherwise this subject is an open area of research. Consider

$$\lim_{R \to \infty} \int \chi\left(\frac{y}{R}\right) \widehat{f}(y) e^{2\pi i x \cdot y} dy \tag{17.2}$$

where χ is a cutoff function, i.e. an indicator function for the unit ball $|y| \le 1$. We had $\chi = e^{-\pi|y|^2}$ in the proof of the inversion formula. There are complications when χ is a non-continuous thing though.

Corollary 17.1.

1. *If $f \in L^1(\mathbb{R}^n)$, $\widehat{f} \equiv 0$ a.e., then $f = 0$ a.e..*

2. *$\mathcal{F} : \mathcal{J} \to \mathcal{J}$ bijectively. This is because if $g \in \mathcal{J}$, take $f(x) = \int g(y)e^{2\pi i x\cdot y}dy$ which is a Schwartz function, then by inversion we have $f(x) = \int \widehat{f}(y)...$ so this holds by the first corollary.*

3. *If $f, g \in L^1(\mathbb{R}^n)$, then $\widehat{f * g} = \widehat{f}\widehat{g}$.*

4. *If $f, g \in \mathcal{J}$, then $\widehat{fg} = \widehat{f} * \widehat{g}$.*

Proof of 3. $f * g$ is in L^1, and

$$\begin{aligned}
\widehat{f * g}(\xi) &= \int f * g(x)e^{-2\pi i x\cdot\xi}d\xi \\
&= \int\int f(x-y)g(y)e^{-2\pi i(x-y)\cdot\xi}e^{-2\pi i y\cdot\xi}dyd\xi \\
&= \int\int \left(f(x-y)e^{-2\pi i(x-y)\cdot\xi}\right)\left(g(y)e^{-2\pi i y\cdot\xi}\right)dyd\xi \\
&= \widehat{f}(\xi)\widehat{g}(\xi)
\end{aligned} \tag{17.3}$$

□

Proof of 4. Take Fourier transform of both sides. Then $\widehat{\widehat{fg}}(x) = f(-x)g(-x)$ using the fact[1] that $F(x) = \int \widehat{F}(y)e^{2\pi i x\cdot y}dy$ with $F(x) = f(x)g(x)$. Now, applying the corollary part (3.), we get

$$\widehat{\widehat{f} * \widehat{g}} = \widehat{\widehat{f}}(x)\widehat{\widehat{g}}(x) = f(-x)g(-x) \tag{17.4}$$

again applying fact[1]. □

Theorem 17.2 (Plancherel)**.** *If f, $g \in \mathcal{J}$, then*

$$\int f(x)\overline{g}(x)dx = \int \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi \tag{17.5}$$

Proof. By Fourier duality,

$$\begin{aligned}
\int f(x)\overline{g}(x)dx &= \int \widehat{\widehat{f}}(-x)\overline{g}(x)dx \\
&= \int \widehat{\widehat{f}}(x)\overline{g}(-x)dx \\
&= \int \widehat{f}(x)\widehat{\overline{g}}(-x)dx \\
&= \int \widehat{f}(x)\overline{\widehat{g}}(x)dx
\end{aligned} \tag{17.6}$$

where we got to the last line by change of variables. The variable change was this,

$$\begin{aligned}\widehat{\overline{g(-x)}}(\xi) &= \int \overline{g(-x)} w^{-2\pi i x\cdot\xi} dx \\ &= \overline{\int g(-x) e^{2\pi i x\cdot\xi} dx} \\ &= \overline{\int g(x) e^{-2\pi i x\cdot\xi} dx} \\ &= \overline{\widehat{g}(\xi)}\end{aligned} \tag{17.7}$$

□

17.1 L^2 theory

$\|f\|_2 = \left(\int |f|^2\, dx\right)^2$ and $\langle f, g\rangle = \int f\overline{g}dx$. By Plancherel, if f and $g \in \mathcal{J}$, then

$$\langle \widehat{f}, \widehat{g}\rangle = \langle f, g\rangle \qquad \left\|\widehat{f}\right\|_2 = \|f\|_2 \tag{17.8}$$

This allows us to extend the F.T. to L^2. For $f \in L^2$, let $f_n \in L^1$ (or $\in \mathcal{J}$) so that $\|f_n - f\|_2 \to 0$ as $n \to \infty$. Now define

$$\widehat{f} = \lim_{n\to\infty} \widehat{f}_n \tag{17.9}$$

where the limit is taken in L^2. This limit exists because L^2 is complete, and f_n is a cauchy sequence in L^2;

$$\left\|\widehat{f}_m - \widehat{f}_n\right\|_2 = \|f_m - f_n\|_2 \to 0 \tag{17.10}$$

as $m, n \to \infty$. This definition of F.T. does not depend on the choice of f_n, and it is also consistent with the definition for L^1 functions. If $f \in L^1 \cap L^2$...

Lecture 18

18.1 L^p theory, $1 \le p \le 2$

Let $f \in L^p(\mathbb{R}^n)$, $1 \le p \le 2$. Write $f = f_1 + f_2$, where $f_1 = f \cdot \chi_{\{x:|f|>1\}}$, $f_2 = f \cdot \chi_{\{x:|f|\le 1\}}$. Then $f_1 \in L^1$ and $f_2 \in L^2$. We define $\widehat{f} = \widehat{f_1} + \widehat{f_2}$. Now if we have a different decomposition $f = g_1 + g_2$, $g_1 \in L^1$ and $g_2 \in L^2$, then $f_1 - g_1 = g_2 - f_2 \in L^1 \cap L^2$. So then, do we have $\widehat{f} = \widehat{g_1} + \widehat{g_2} = \widehat{f_1} + \widehat{f_2}$? Since the functions are in both spaces, we have $\widehat{f_1} - \widehat{g_1} = \widehat{g_2} - \widehat{f_2}$ so yea.

We have $\left\|\widehat{f}\right\|_\infty \le \left\|\widehat{f}\right\|_1$, $\left\|\widehat{f}\right\|_2 = \|f\|_2$. By interpolation, we have the Hausdorff-Young inequality $\left\|\widehat{f}\right\|_{p'} \le \|f\|_p$ for $1 \le p \le 2$, $\frac{1}{p} + \frac{1}{p'} = 1$. For $1 < p < 2$, there is a constant $C_p < 1$ so that $\left\|\widehat{f}\right\|_{p'} \le C_p \|f\|_p$

For $p > 2$, nothing nice happens. $f \to \widehat{f}$ is not bounded. That is, the H.Y. inequality does not hold.

Lecture 19

Lemma 19.1. *The Fourier transform is not bounded $L^p \to L^q$ for any $q < p$. That is, there is no estimate $\left\|\widehat{f}\right\|_q \leq C\|f\|_p$ with C independent of f (in particular, the H.Y. inequality does not hold for $p > 2$).*

The idea is to suppose we could find $\phi_1, \phi_2, \phi_3, \ldots \in \mathcal{J}$ such that $\|\phi_1\|_p = \|\phi_2\|_p = \ldots$ and $\|\phi_1\|_q = \|\phi_2\|_q = \ldots$ and the supports of ϕ_j are pairwise disjoint, and supports of $\widehat{\phi_j}$ pairwise disjoint too. If this was possible, then let $\Phi_N = \sum_{j=1}^N \phi_j$. Then $\|\Phi_N\|_p = \left(\int (\sum |\phi_j|)^p\right)^{1/p}$ which by the disjoint supports, equals

$$\left(\sum \int |\phi_j|^p\right)^{1/p} = \left(\sum_1^N \|\phi_j\|_p^p\right)^{1/p} = (NA^p)^{1/p} = N^{1/p}A$$

and for the fourier serieses,

$$\left\|\widehat{\Phi_N}\right\|_q = \left(\int \left(\sum \left|\widehat{\phi_j}\right|\right)^q\right)^{1/q} = \ldots = N^{1/q}B \tag{19.1}$$

So if we had $\left\|\widehat{f}\right\|_q \leq C\|f\|_p$, then $N^{1/q}B \leq CN^{1/p}A$ for all N, but $N^{\frac{1}{q}-\frac{1}{p}}B \leq CA$, and $\frac{1}{q} - \frac{1}{p} > 0$, a contradiction. This is dependent on finding that crazy sequence of ϕ though, which isn't true.

Proof of 19.1. Take $\phi \in C_c^\infty(-\frac{1}{2}, \frac{1}{2})$ in dimension 1, and let $\phi_k(x) = e^{2\pi i x a_k}\phi(x-k)$. We compute

$$\begin{aligned} \widehat{\phi_k}(\xi) &= \int e^{2\pi i x a_k}\phi(x-k)e^{-2\pi i x\xi}dx \\ &= \int e^{-2\pi i x(\xi - a_k)}\phi(x-k)dx \\ &= \int \phi(u)e^{-2\pi i(u+k)(\xi-a_k)}du \\ &= e^{-2\pi i(\xi-a_k)k}\widehat{\phi}(\xi - a_k) \end{aligned} \tag{19.2}$$

with the change of variables $x - k = u$. The claim now is that if $a_k \to \infty$ fast enough, then $\left\|\widehat{\Phi_N}\right\|_q^q \geq \frac{N}{2}\left\|\widehat{\phi}\right\|_q^q$. ☐

Lecture 20

We will talk about Localization and the uncertainty principle for awhile. Basically, this is about two things: First, $\widehat{f}$ is supported in $\{\|\xi\| \leq R\} \iff f$ is approximately constant; on the scale of $\frac{1}{R}$. Second, it's not usually possible for f and $\widehat{f}$ to be compactly supported.

Proposition 20.1. *Let $f \in \mathcal{C}(\mathbb{R})$, $f \not\equiv 0$. Then f, $\widehat{f}$ cannot both be compactly supported.*

Proof. Suppose that $f \in [a,b]$ compact, and supp $\widehat{f}$ is compact as well. By scaling and translation, we may assume that $0 < a < b < 1$. ☐

Example 20.2. *If F is a trig polynomial on $\mathbb{R}$, $F(x) = 0$ for all $x \in E$ for some $|E| > 0$, then $f \equiv 0$.*

Theorem 20.3. *Assume $f \in \mathcal{J}$ with $\|f\|_2 = 1$. Then*

$$\left(\int |x|^2 |f(x)|^2 \, dx\right) \left(\int |\xi|^2 \left|\widehat{f}(\xi)\right|^2 d\xi\right) \geq \frac{1}{16\pi^2} \tag{20.1}$$

Corollary 20.4. *If ξ_0, $x_0 \in \mathbb{R}$, $f \in \mathcal{J}$, $\|f\|_2 = 1$, then*

$$\left(\int |x - x_0|^2 |f(x)|^2 \, dx\right) \left(\int |\xi - \xi_0|^2 \left|\widehat{f}(\xi)\right|^2 d\xi\right) \geq \frac{1}{16\pi^2} \tag{20.2}$$

We also have an $\mathbb{R}^n$ version of this, but the lower bound is $\frac{n^2}{16\pi^2}$.