Solutions / Comments to selected problems in Assignments I to IX

§ 2.3, (P.86), #46

$$\lim_{x\to 0^-} \frac{x}{x-|x|} = ?$$

Solution That $x \to 0^-$ implies that x is negative.
If so, |x| = -x, just like |-2| = - (-2) .

Hence $? = \lim_{x\to 0^-} \frac{x}{x-(-x)} = \lim_{x\to 0^-} \frac{x}{2x} = \frac{1}{2}$ (ANSWER)

§ 2.4, (P.97), #33

Continuity for $\frac{1}{\sin 2x}$ fails when sin 2x = 0,

which means $2x = \pm m\pi$, or $x = \frac{m}{2}\pi$,

where m = 0, 1, 2, 3, 4 …. These however, are also the points where the function is undefined. Thus : $\frac{1}{\sin 2x}$ is continuous whenever it is defined.

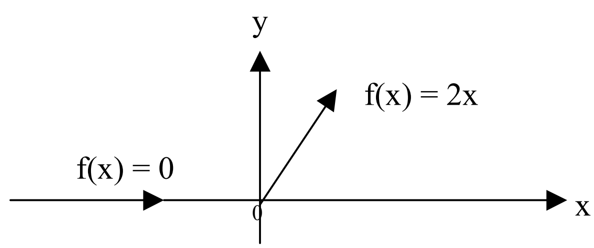
§ 3.1, (P.114) #52 (b)

The average rate of change of population from 1983 to 1988 is
$\frac{P(8)-P(3)}{8-3}$ where t = 3 means 1983 and t = 8 means 1988.
Now get this average by substitution. The important point here is to understand what "average rate" means.

§ 3.2, (P.115), #59

Note that $f(x) = x + |x| = \begin{cases} x+x=2x & if \quad x \ge 0 \\ x-x=0 & if \quad x<0 \end{cases}$

Hence its graph is



At all $x \neq 0$, f ′(x) exists, and is given by $f'(x) = \begin{cases} (2x)' = 2 & when \quad x>0 \\ (0)'=0 & when \quad x<0 \end{cases}$

It is also clear from the graph that x=0 is a point of nondifferentiability.
There, left derivative = 0, right derivative = 2 and no overall derivative is defined.

§3.3, P.124 #53

Note that this problem asks for the value of $\frac{dV}{dh}$ when h = 600cm. It is not asking for the value of $\frac{dV}{dt}$.

§ 3.3, P.133 #55

The purpose of this exercise is to practise with the chain rule, using the notation
$G(t) = f(h(t))$. In this notation $G'(t) = f(h(t))\ h'(t)$
Substituting $t = 1$ gives $G'(1) = f'(h(1)) \times h'(1) = f'(4)\ h'(1)$
$= 3 \times (-6) = -18$ (ANSWER) .

§ 3.3, P.135, # 58

$V = \frac{4}{3}\pi r^3$. Differentiating with respect to time t gives

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

When $r = 5$, $\frac{dV}{dt} = 200\pi$;

One has $\frac{dr}{dt} = \frac{dv/dt}{4\pi r^2} = \frac{200\pi}{4\cdot\pi\cdot 5^2} = 2$ cm/s .

§ 3.4, P.140, # 69

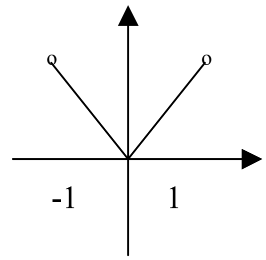
Let the normal line through (0, 2.5) to $y = x^{2/3}$ meet the curve at $(x, x^{2/3})$.
Since $y' = \frac{2}{3}x^{-1/3}$, the slope of the normal line is $\frac{-1}{y'} = \frac{-3}{2}x^{1/3}$.

Thus $\frac{x^{2/3} - 2.5}{x - 0} = -\frac{3}{2}x^{1/3}$, or, clearing denominator, $x^{2/3} - \frac{5}{2} = -\frac{3}{2}x^{4/3}$.

Write this as $3x^{4/3} + 2x^{2/3} - 5 = 0$ and factor it as $(3x^{2/3} + 5)(x^{2/3} - 1) = 0$.
Since $x^{2/3} = (x^{1/3})^2$ is always positive, we get $x^{2/3} - 1 = 0$ or $x = \pm 1$.

ANSWER : There are 2 normal lines through (0, 2.5).
They intersect the curve $y = x^{2/3}$ at (1, 1) and (-1, 1) respectively.

§ 3.5, P.148, # 3



The function f (x) = |x| on the open interval (-1, 1) does not attain a maximum value. From the graph, it is clear that it attains a minimum value 0 at x = 0 .

§3.6, P.172, # 48

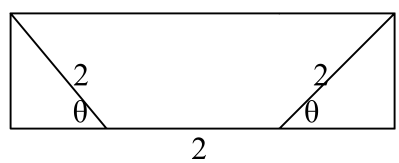
$x = (\sec\sqrt{t}) + \tan\sqrt{t}$.

One must get $\frac{dx}{dt}$ carefully with chain rule and product rule.

$$\frac{dx}{dt} = \left[\left(\sec\sqrt{t}\ \ \tan\sqrt{t}\right)\ \frac{1}{2}t^{-1/2}\right]\ \tan\sqrt{t} + \sec\sqrt{t}\ \left[\left(\sec^2\sqrt{t}\right)\frac{1}{2}t^{-1/2}\right]$$

$$= \frac{1}{2\sqrt{t}} \cdot \sec\sqrt{t}\ \left(\tan^2\sqrt{t} + \sec^2\sqrt{t}\right) .$$ (ANSWER)

§ 3.7 P. 173, #79



The trapezoidal area A is obtained from the rectangular area minus the areas of the two triangles. These triangles have horizontal side 2 cos θ and vertical side 2 sinθ . Thus

$$A = (2 + 2 \times 2\cos\theta) \times 2\sin\theta - 2 \times \frac{1}{2}(2\cos\theta)(2\sin\theta) .$$

or $A = 4(1 + 2\cos\theta) \times \sin\theta - 4\cos\theta\sin\theta$
$= 4\sin\theta + 4\cos\theta\sin\theta$.

To maximize A, set $\frac{dA}{d\theta} = 4\cos\theta - 4\sin^2\theta + 4\cos^2\theta = 0$

Dividing by 4 and using $\sin^2\theta = 1 - \cos^2\theta$, we get $2\cos^2\theta + \cos\theta - 1 = 0$. Thus

$(2\cos\theta - 1)(\cos\theta + 1) = 0$, giving $\cos\theta = \frac{1}{2}$ or $\theta = \frac{\pi}{3}$.

This clearly is the value of θ to giver A_{max} .

§ 3.7, (P.173) , # 87

This is a highly recommended exercise for those who want more practices with optimization problems. Please try it.

§ 3.7, (P.173) #87

To differentiate the function f (x) at x = 0 means to try to get the following limit :

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h\ \sin\frac{1}{h} - 0}{h}$$

$$= \lim_{h\to 0} \left(\sin \frac{1}{h} \right).$$

Referring to Figure 2.3.5 of P.76 we see that this limit does not exist.
Hence f(x) is not differentiable at x = 0.

§ 3.8, (P.187), #57

Use logarithmic differentiation :

$y = (\sqrt{x})^{\sqrt{x}}$. Take *l*n of both sides :

$$\ln y = \sqrt{x}\ \ln\sqrt{x} = \sqrt{x}\ \left(\frac{1}{2}\ln x\right) = \frac{1}{2}\sqrt{x}\ \ln x\ .$$

Taking derivative with respecti to x yields :

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{2\sqrt{x}}\ln x + \frac{1}{2}\sqrt{x}\cdot\frac{1}{x} = \frac{\ln x + 2}{4\sqrt{x}}$$

So $$\frac{dy}{dx} = y\cdot\frac{\ln x + 2}{4\sqrt{x}} = (\sqrt{x})^{\sqrt{x}}\cdot\frac{\ln x + 2}{\sqrt{x}} \quad \text{(ANSWER)}$$

§ 8.1, (P.557) #1

$\frac{dy}{dx} = 2y$, $y(1) = 3$ has solution $y = C e^{2x}$.

Substitution by $x = 1$, $y = 3$ gives $3 = C e^2$, so $C = \frac{3}{e^2}$.

The final solution is $y = \frac{3}{e^2} e^{2x} = 3 e^{2x-2}$ (ANSWER)

§ 8.1, (P.557) #2

$\frac{dy}{dx} = -3y$, $y(5) = 10$ has solution $y = C e^{-3x}$.

Substitution by $x = 5$, $y = -10$, gives $-10 = C e^{-15}$,
So $C = -10 e^{15}$.
The final solution is $y = -10 \times e^{15} e^{-3x} = -10 e^{-3x+15}$ (ANSWER)

§ 8.1, (P.557) # 7

Write the D.E. as $\frac{d}{dx}(y +1) = y + 1$, $y (0) = 5$.
This makes the (y+1) on the left match precisely with the (y+1) on the right.
Note here that $\frac{dy}{dx} = \frac{d}{dx}(y +1)$ because y and y+1 have the same derivative.
The solution is $y+1 = C e^x$.
Substitution by $x = 0$, $y = 5$ gives $5+1 = C \times e^o$, or $C = 6$

So the solution is $y = 6 e^x - 1$ (ANSWER)

§ 8.3, (P.576) #23

To solve $\frac{dy}{dx} = 2y - 3$, $y(0) = 2$.

Note that $2y - 3 = 2(y - 3/2)$. So we write the $\frac{dy}{dx}$ on the left side as $\frac{d}{dx}(y - 3/2)$,
Noting that y and (y - 3/2) have the same derivative.
The D.E. now reads $\frac{d}{dx}(y - 3/2) = 2 (y - 3/2)$, with perfect match of $y - 3/2$ on both the left and the right side. From this we can immediately get the solution $(y - 3/2) = C e^{2x}$

Substitution by $x = 0$, $y = 2$ gives $C = \frac{1}{2}$.

Hence the final solution is $y = \frac{3}{2} + \frac{1}{2} e^{2x}$ (ANSWER)

§ 8.3, (P.576), #29

$\frac{dv}{dt}$ = 10 (10 - v). Rewrite as

$\frac{dv}{dt}$ = -10 (v - 10), and then as

$\frac{d}{dt}$ (v - 10) = -10 (v - 10) .

Hence v - 10 = C e^{-10t} .

Substitution by t = 0, v = 0 gives -10 = C.
The final solution is therefore v = 10 -10 e^{-10t} .

§ 8.3, (P.576) #31

According to the question, Zembla's population P, in millions, satisfies

$\frac{dP}{dt}$ = 0.04 P + 0.05, P (0) = 1.5

Where the year 1990 corresponds to time t = 0.

Rewrite this equation as $\frac{dP}{dt}$ = 0.04 (P + 1.25)

and then as $\frac{d}{dt}$ (P + 1.25) = 0.04 (P + 1.25) .

We have P + 1.25 = C $e^{0.04t}$ and C = 2.75
by substitution of the initial conditions. Thus the population in t years will be

P = -1.25 + 2.75 $e^{0.04t}$ million.

For the year 2010 substitute t = 20 to get
P (20) = -1.25 + 2.75$e^{0.8}$ = 4.87 million (ANSWER)

§ 8.3, (P.576), #32

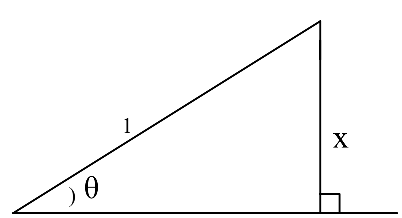
This is entirely parallel to Question #5, in our quiz I, Oct. 29, 2004. Please do it yourself.

§ 8.3, (P.576), # 33, 34

Use these as further practice ! How would you rewrite the equation $\frac{dP}{dt}$ = rP - C ?

§8.3, (P.477), #72

To show $\sin^{-1}$ x = $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$, let θ = $\sin^{-1}$ x .

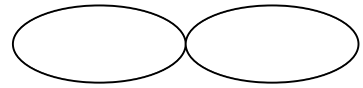


This means sin θ = x = x/1, as in the diagram shown .
By Pythagoras theorem the horizontal side is $\sqrt{1-x^2}$,

and so tan θ = $\frac{x}{\sqrt{1-x^2}}$. This means θ = $\tan^{-1} \frac{x}{\sqrt{1-x^2}}$

and the desired equality is established.

§ 3.9, (P.195) #5



The equation of the lemniscate curve L is

$$(x^2 + y^2)^2 = x^2 - y^2 \qquad (I)$$

We take the view point that at points (x_o, y_o) on L it is theoretically possible , with few exceptions, to solve y as a function of x from (I), satisfying $y(x_o) = y_o$. However, the slope $y'(x_0)$ of the tangent line at (x_o, y_o) can be much more conveniently obtained via the method of implicit differentiation. Thus we apply D_x to the terms of (I) and use chain rule to get

$$2(x^2 + y^2)\left[2x + 2y\frac{dy}{dx}\right] = 2x - 2y\frac{dy}{dx},$$

which leads to $\dfrac{dy}{dx} = \dfrac{2x - 4x\ (x^2 + y^2)}{2y\ [1 + 2\ (x^2 + y^2)]}$.

At points with horizontal tangent line $\dfrac{dy}{dx} = 0$, so

$$2x - 4x(x^2 + y^2) = 0 \qquad \text{or} \qquad 2x[1 - 2(x^2 + y^2)] = 0 .$$

The case x = 0 is discarded because it leads to the point (0,0) on L for which slope is not defined. We are then left with $1 - 2(x^2 + y^2) = 0$, or

$$\boxed{x^2 + y^2 = \frac{1}{2}}$$

Substitution back into (I) gives

$$\boxed{x^2 - y^2 = \frac{1}{4}}$$

and the two boxed equations yield four solutions

$$(x, y) = \left(\sqrt{\frac{3}{8}}, \sqrt{\frac{1}{8}}\right), \left(\sqrt{\frac{3}{8}}, -\sqrt{\frac{1}{8}}\right), \left(-\sqrt{\frac{3}{8}}, \sqrt{\frac{1}{8}}\right), \left(-\sqrt{\frac{3}{8}}, -\sqrt{\frac{1}{8}}\right).$$

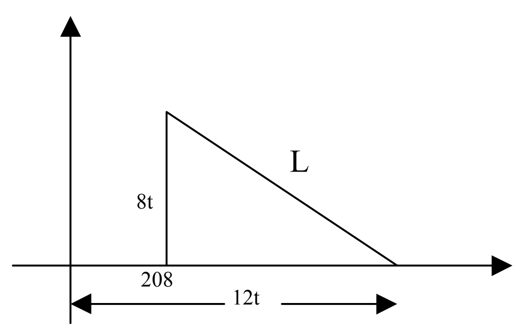
These correspond to 4 points on L with horizontal tangent line, as can be visually checked from the graph.

For points on L with vertical tangent line we need $\dfrac{dy}{dx} = \pm\infty$. This requires

$$2y[1 + 2(x^2 + y^2\}] = 0 \qquad \text{and} \qquad 2x - 4x(x^2 + y^2) \neq 0 .$$

Since $1 + 2(x^2 + y^2)$ is always positive, the first requirement leads to y = 0 whose substitution back into (I) gives $x^4 = x^2$, or x = 1, -1, 0. We exclude x = 0 because it violates the second requirement. That leaves (x, y) = (1,0) or (-1,0), two points on L where the tangent lines are vertical.

§3.9, (P.198) #59



The diagram shows the situation of the 2 planes at a <u>general time moment t</u>. (Note, it is very essential to produce this diagram). At this time the distance between the two airplanes is

$$L = \sqrt{(8t)^2 + (12t - 208)^2}\ ,$$

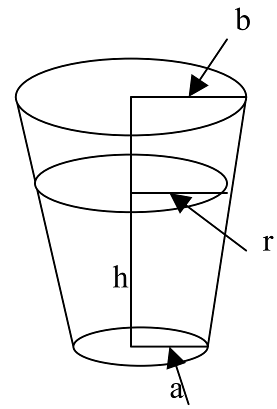
The question asks for the absolute minimum value of L on the interval [0, ∞). This can be done by the usual critical point method :

$$\frac{dL}{dt} = \frac{128t + 2(12t - 208)\times 12}{2\sqrt{(8t)^2 + (12t - 208)^2}} \quad .$$

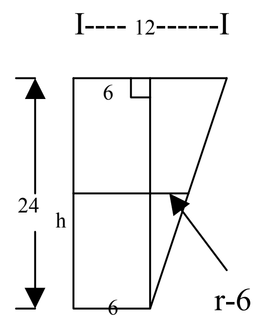
Setting this derivative equal 0 gives $416t - 2 \times 208 \times 12 = 0$,
or t = 12 minutes, for which $L = 32\sqrt{13}$ or roughly 115.38 miles.
One must not fail to check end point , and point out that $32\sqrt{13}$ is less than 208, the value of L at the end point t = 0. Physically then , $32\sqrt{13}$ must be the minimum distance between the 2 planes, at the critical time of 12 minutes after flight started.

§ 3.9 (P.198) #63

For this kind of rate problem about leaking water from a bucket, the important step, again, is to draw a diagram showing the situation, at a general time t. The substitution h = 12 inches should be done only at the end, after all differentiations with respect to time has been accomplished.
At time t let the water level be h inches with water surface radius equal to r inches. Here h and r are functions of time, a, b and the height of the bucket are constants, with water volume at time t given by



$$V = \frac{\pi h}{3}\ (36 + 6r + r^2)\ . \qquad (\text{I})$$



The important point now is to find the relationship between h and r. Use the auxiliary diagram shown.
By similar triangles

$$\frac{r-6}{12-6} = \frac{h}{24}\ ,$$

or

$$r = 6 + \frac{1}{4}h. \qquad (\text{II})$$

Applying D_t to both (II) and (I) , and using the chain rule correctly, we get

$$\frac{dr}{dt} = \frac{1}{4}\frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{\pi}{3}\frac{dh}{dt}\ (36 + 6r + r^2) + \frac{\pi h}{3}\left(6\frac{dr}{dt} + 2r\frac{dr}{dt}\right)$$

$$= \frac{\pi}{3}\left[\frac{dh}{dt}\left(36 + 6r + r^2\right) + h\left(6 + 2r\right)\frac{dr}{dt}\right]$$

At the specific moment when h = 12 inches, we deduce from (II) that r = 9 . Substitution into the last equation using h = 12, r = 9 , $\frac{dV}{dt} = -10$, and $\frac{dr}{dt} = \frac{1}{4}\frac{dh}{dt}$ will give

$$-10 = \frac{\pi}{3}\left[\frac{dh}{dt}\times 171 + \left(12\times 24\right)\times\frac{1}{4}\frac{dh}{dt}\right]$$

or $-10 = \frac{\pi}{3} \times 243 \frac{dh}{dt}$, i.e. $\frac{dh}{dt} = \frac{-10}{81\pi}$

ANSWER At the moment when water is exactly up to 12 inches, the water level is dropping at the rate of $\frac{10}{81\pi}$ inches per minute.

§ 4.2 (P.225) #18

$f(x) = (1+x)^{-1/2}$, $f'(x) = -\frac{1}{2}(1+x)^{-3/2}$, $f(0) = 1$, $f'(0) = -\frac{1}{2}$.

Thus the linear approximation $f(x) \approx f(0) + f'(0)(x-0)$ is simply $(1+x)^{-1/2} \approx 1 - \frac{1}{2}x$.

§4.2 (P.225) #27

To estimate $\sqrt[4]{15}$ without any calculator help, observe that 15 is close to 16 and

$\sqrt[4]{16} = 2$ is easy to get. So, introduce the function $f(x) = x^{1/4}$ with

$a = 16$, $x = 15$, $f'(x) = \frac{1}{4}x^{-3/4}$ and use tangent line approximation

$$f(x) \approx f(a) + f'(a)(x-a).$$

This gives

$$\sqrt[4]{15} \approx \sqrt[4]{16} + \frac{1}{4}(16)^{-3/4} \cdot (15-16)$$

or

$$\sqrt[4]{15} \approx 2 + \frac{1}{32} \times (-1) = 2 - \frac{1}{32} = \frac{63}{32} ,$$

so that $\frac{63}{32}$ is an approximate value of $\sqrt[4]{15}$.

[Check : $\left(\frac{63}{32}\right)^4 = 15.023038$, quite close to 15] .

Suggestion: Try to estimate the error in this approximation using the second derivative f'' of f.

§ 4.2 (P225) #31

We want to get an approximate value of $\cos 43^0$ by hand, and to estimate the error of this approximation.

Note that $\cos 45^0 = \frac{\sqrt{2}}{2}$ is known and that 43^0 is quite close to 45^0. We must, however, convert all degrees into radians before using calculus.

So, introduce $f(x) = \cos x$, with $a = \frac{\pi}{4}$, $x = \frac{43\pi}{180}$ (which corresponds to 43^0) .

Now , $f'(x) = -\sin x$ and tangent line approximation $f(x) \approx f(a) + f'(a)(x-a)$

gives $\cos\frac{43\pi}{180} \approx \cos\frac{\pi}{4} - \sin\frac{\pi}{4}\left(\frac{43\pi}{180} - \frac{\pi}{4}\right)$,

or $\cos\frac{43\pi}{180} \approx \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \times \frac{(-2\pi)}{180} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \times \frac{\pi}{90}$

or $\cos\frac{43\pi}{180} \approx 0.73179$

To estimate the error committed, note f''(x) = - cosx and that throughout the interval

$[\frac{43\pi}{180}, \frac{\pi}{4}]$ one has $|f''| \le K$ for the constant value K = 1 .
Hence, in using 0.73179 to approximate $\cos 43^0$, the error satisfies

$$|\text{error}| \le \frac{K}{2}(x-a)^2 = \frac{1}{2}\left(\frac{43\pi}{180} - \frac{\pi}{4}\right)^2 = 0.00061$$

We summarize the situation into a diagram,

I----------- .00061 --------------I ------------- .00061 ----------- I

.73118 .73179 .73240

and claim that the true value of $\cos 43^0$ lies in the interval [.73118 , .73240] .

Suggestion Use a pocket calculator to obtain $\cos 43^0$ and see if it indeed lies inside this interval.

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Selected solutions to Assignment I

Section 2.3, p.85, #27

Start with the fact $-1 \le \cos\frac{1}{\sqrt[3]{x}} \le 1$ which is true for all $x \ne 0$.

Multiply by the positive quantity x^2 to get $-x^2 \le \cos\frac{1}{\sqrt[3]{x}} \le x^2$

Since $\lim_{x\to 0}(-x^2) = 0 = \lim_{x\to 0}(x^2)$,
the Squeeze Principle implies immediately
that $\lim_{x\to 0} x^2\cos\frac{1}{\sqrt[3]{x}} = 0$ (Answer !)

Discussion : $\cos \frac{1}{\sqrt[3]{x}}$ is undefined at x = 0.

Why is this fact irrelevant as far as our problem is concerned ?

[Suggested further practice : p.85, #26]

Section 2.4, p.97, #55

Consider the continuous function $f(x) = x^3 - 3x^2 + 1$ on the closed interval [0,1] . By substitution f(0) = 1, f(1) = -1 . Obviously the value 0 is intermediate between f(0) and f(1), namely f(0) > 0 > f(1) . By the Intermediate Value Theorem there exists at least one real number r between 0 and 1 such that f (r) = 0. In other words the equation $x^3 - 3x^2 + 1 = 0$ has a solution x = r in the interval [0,1] .

[Suggested further practice : p.98, #71]

§ 4.2, p.225

The last sentence in this question should read : What, approximately, is the resulting error in the calculated surface area ?

The solution is as follows : $S = 2\pi r^2$, $\Delta r = 0.01$ meter.

Therefore $\Delta S \approx S'(\Delta r)$ { This is equivalent to "tangent line approximation", see formula (4), p.219 }. Here $S' = \frac{dS}{dr} = 4\pi r$, and so the error in the calculation of hemi-spherical area is approximately

$$\Delta S \approx 4\pi \times 100 \times 0.01 = 4\pi \text{ m}^2 \qquad \text{(Answer)!}$$

§ 4.3, p .235, #33

The function $f(x) = 3x^2 + 6x - 5$ is clearly continuous and differentiable throughout the closed interval [a , b] = [-2, 1] , so the hypotheses of the mean value theorem is satisfied.

According to that theorem, there is a number c between -2 and 1 such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} .$$

Since $f'(x) = 6x + 6$ this translates into $6c + 6 = \frac{f(1) - f(-2)}{1 - (-2)}$.

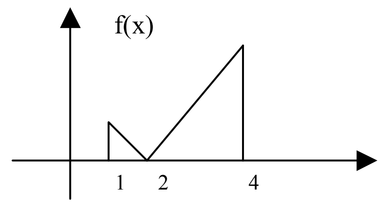
The right hand side is $4 - (-5)/\ 3 = 3$, So $6c + 6 = 3$ and the answer is c = - ½

§ 4.3, p.235, #37

The function $f(x) = |x - 2|$ is not differentiable at the point $x = 2$, which is inside the interval $[1,4]$. Hence the hypotheses of the mean value theorem are NOT satisfied.

If we try to find a number c between 1 and 4

such that $f'(c) = \dfrac{f(4) - f(1)}{4 - 1}$ (#)



we will fail because the right side equals

$[\,|4 - 2| - |1 - 2|\,]/3 = \frac{1}{3}$,

while the left side, being the slope of the graph of f (x) at the point x = c, is either -1, +1 or undefined (see diagram) .

Hence there does not exist a number c between 1 and 4 to make the left side and right side of (#) match. [End of proof.]