MATH 309 - Solution to Homework 2

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1. The rational tangle associated with $[2\ 1\ a_1 a_2 \ldots a_n]$ is equivalent to the rational tangle associated with $[-2\ 2\ a_1 \ldots a_n]$.

Proof. First, let's compute the two continued fractions. We have

$$a_1 + \frac{1}{1+\frac{1}{2}} = a_1 + \frac{1}{2+\frac{1}{-2}},$$

so the continued fraction associated to the sequence $[2\ 1\ a_1 a_2 \ldots a_n]$ is the same as the one associated to $[-2\ 2\ a_1 \ldots a_n]$.

Second, we can relate the two tangles by a sequence of Reidemeister moves:

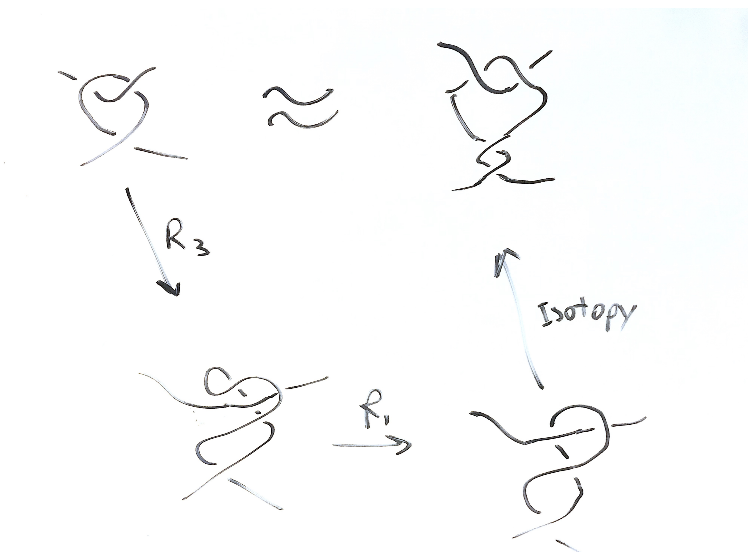


Figure 1: Reidemeister moves to relate $[2\ 1\ a_1 a_2 \ldots a_n]$ to $[-2\ 2\ a_1 \ldots a_n]$

□

2. 2-bridge knots are alternating.

(a) Every rational knot is alternating.

Proof. We can prove this by showing that every rational knot has a description $[a_1 a_2 \ldots a_n]$ where each $a_i \geq 0$ or each $a_i \leq 0$. Check that such rational links are alternating. To see that we can take any continued fraction and turn it into another one where all the integers have the same sign, let's warm up with an example: consider $[4\ -3\ 2]$. The continued fraction is

$$2 + \frac{1}{-3+\frac{1}{4}}.$$

This fractional number is obtained by subtracting something from 2. Let's try to get the same fraction by adding something to 1 instead:

$$\begin{aligned}
2+\frac{1}{-3+\frac{1}{4}} &= 1+\frac{-3+\frac{1}{4}}{-3+\frac{1}{4}}+\frac{1}{-3+\frac{1}{4}} \\
&= 1+\frac{-3+\frac{1}{4}+1}{-3+\frac{1}{4}} \\
&= 1+\frac{1}{\frac{-3+\frac{1}{4}}{-3+1+\frac{1}{4}}} \\
&= 1+\frac{1}{1-\frac{1}{-3+1+\frac{1}{4}}} \\
&= 1+\frac{1}{1+\frac{1}{-(-3+1+\frac{1}{4})}} \\
&= 1+\frac{1}{1+\frac{1}{2-\frac{1}{4}}} \\
&= 1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{3}}}}
\end{aligned}$$

So the tangle labeled $[4\ \ -3\ 2]$ is equivalent to the one labeled $[3\ 1\ 1\ 1\ 1]$. The fact that we were able to go through with the above transformation should already convince you that every rational tangle is equivalent to an alternating rational tangle, assuming you believe Conway's theorem relating continued fractions to rational tangles.

Here is a piece of the full argument: consider the continued fraction associated to $[a_1, \ldots, a_n]$ and suppose that the last two integers (a_{n-1}, a_n) are not $(-1, 1)$ or $(1, -1)$, and that $a_n > 0$. It is not too hard to see what happens if the conditions we assumed are not satisfied. Pick the largest i such that $a_i < 0$, (if no such i exists, then all a_i are nonnegative and we are done). Thus the continued fraction looks something like this in the middle:

$$a_{i+1}+\frac{1}{-a_i'+r_i},$$

where $a_i' = |a_i|$ and $1/r_i$ is the continued fraction associated to $[a_1 a_2 \ldots a_{i-1}]$. We apply the same idea that we used in the example above:

$$\begin{aligned}
a_{i+1}+\frac{1}{-a_i'+r_i} &= (a_{i+1}-1)+\frac{1}{\frac{-a_i'+r_i}{-a_i'+1+r_i}} \\
&= (a_{i+1}-1)+\frac{1}{1+\frac{1}{(a_i'-1)-r_i}}
\end{aligned}$$

In terms of Conway's notation, the transformation is

$$[a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n] \longmapsto [-a_1, \ldots, -a_{i-1}, -a_i-1, 1, a_{i+1}-1, a_{i+2}, \ldots, a_n].$$

Note that the transformation adds a 1 in the $(i+1)^{th}$ position and that if $-a_i - 1 = 0$, then there is an extra step to be done to get rid of the 0 term, namely we have to apply the transformation

$$[c_1, \ldots, c_{i-1}, 0, c_{i+1}, \ldots, c_n] \longmapsto [c_1, \ldots, c_{i-2}, c_{i-1}+c_{i+1}, c_{i+2}, \ldots, a_n].$$

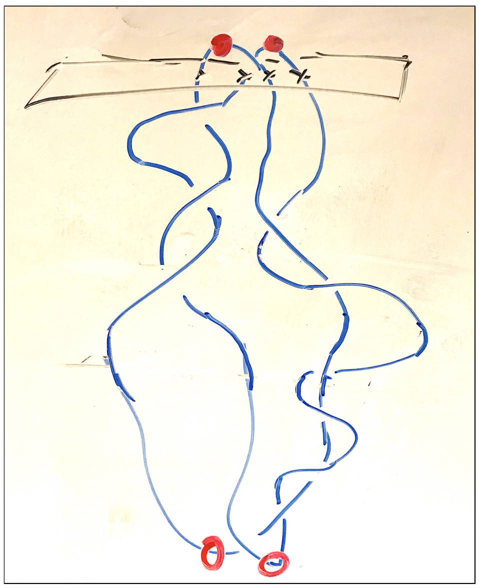
We can keep applying these transformations until we obtain $[\pm a_1, b_2, \ldots, b_m]$, where each $b_i > 0$. If the first term is negative, we can apply it one more time, to get $[|a_1|, 1, b_2 - 1, \ldots, b_n]$. Finally, we obtain a continued fraction where every integer is positive.

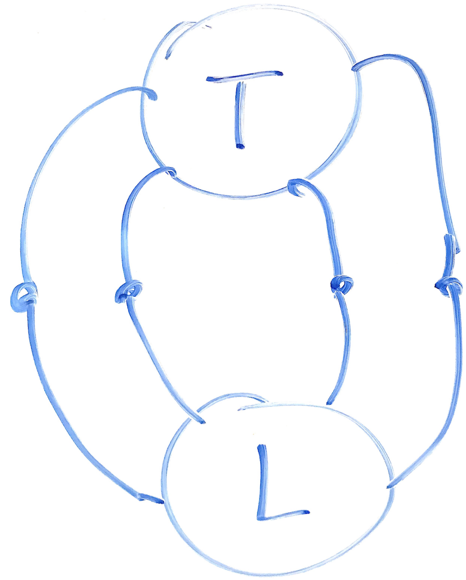
So, given any rational tangle, we can use the above transformation to draw an alternating diagram of an equivalent tangle. □

As mentioned earlier, this process is really the heart of the argument. If you cannot fill in the details of the full proof, let's discuss it during office hours.

(b) Every 2-bridge knot is a rational knot.

Proof. An argument is given in section 3.2 of Adams. Given a 2-bridge knot, we can cut it with a plane in 4 points such that what lies above and below the plane is a rational tangle. This can then be interpreted as the knot obtained from the sum of two tangles:





□

3. The rational knot [5 1 4] has unknotting number at least 2.

Proof. Recall the result due to McCoy mentioned in class: an alternating knot K has unknotting number $= 1$ iff K has an unknotting crossing in every alternating diagram. Since the diagram for [5 1 4] is already alternating, we only need to show that none of the crossings in this diagram are unknotting. Note now that there are essentially only three possible crossing changes, namely one in each integer subtangle. Each of these crossing changes results in the rational knot $[xyz]$, where either $x = 5 - 2$, $y = 1 - 2$ or $z = 4 - 2$ and the other two numbers are the same as in [5 1 4]. See figure 2.

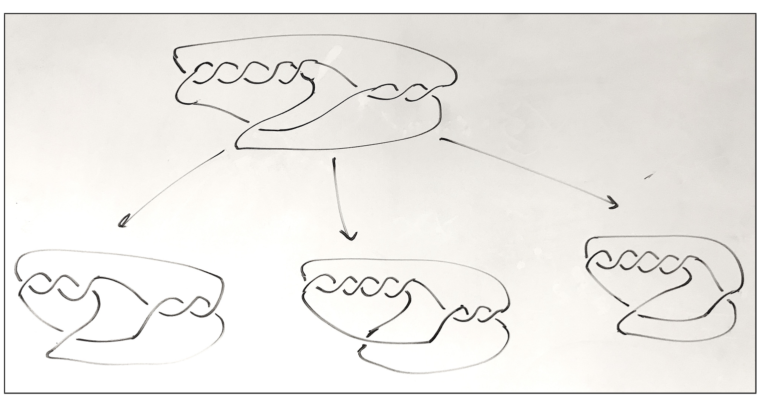


Figure 2: The results of the three possible crossing changes applied to [514]

To see that none of the resulting knots is the unknot, we can see that each of them is either tricolourable or equivalent to a twist knot, which we can identify from the table at the end of Adams' book:

(a) [3 1 4] can be tricoloured:

(b) [5 -1 4] can be easily isotoped into the twist knot 6_1:

(c) [5 1 2] can be isotoped into the twist knot 7_2:

See figure 3. This proves that the unknotting number of [5 1 4] is at least 2.

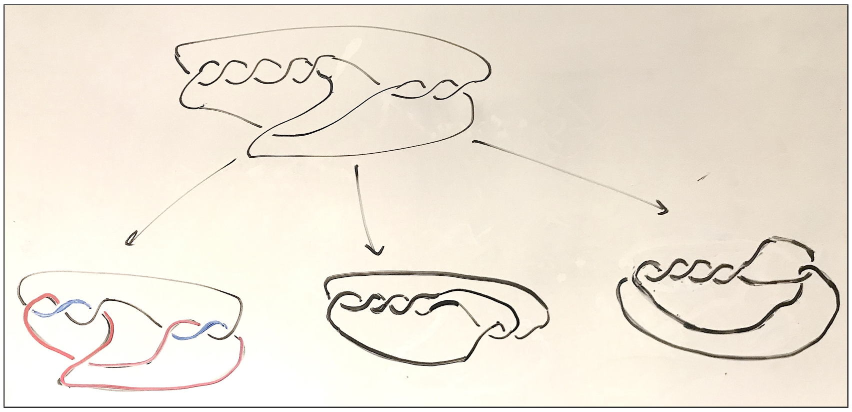


Figure 3: The resulting knots are not trivial.

□

4. Rational knots are prime.

Definition 1. A **composite** knot is a knot that is the connected sum of two nontrivial knots. A knot is **prime** if it is not composite.

Proof. The bridge number of a non-trivial knot is at least 2. Schubert's theorem then implies the bridge number of a composite knot is at least 2+2-1 = 3. If we can show that rational knots are 2-bridge knots, then we are done. But this is easy to see, once the correct picture is drawn. Take a rational tangle and form the corresponding rational knot by adding in two arcs, one connecting the two bottom ends of the tangle and another connecting the upper ends. Imagine that these knot is made of heavy rope. Holding onto these arcs and letting the knot hang down, it is clear that the knot achieves exactly two maxima and two minima. □

5. Adams, exercise 6.8.

Solution. First, the skein relations for the bracket polynomial imply

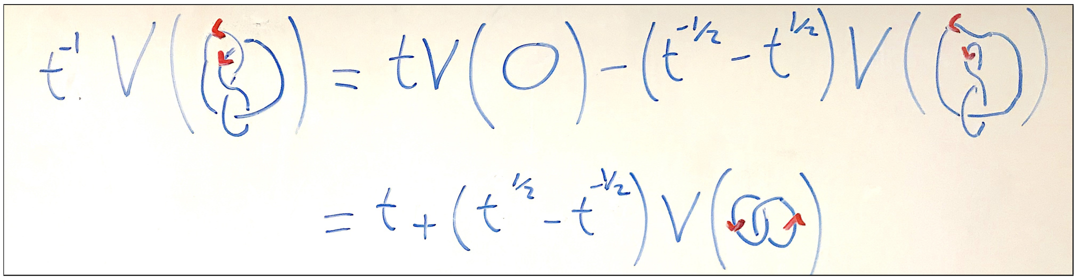
$$A\langle L_+\rangle - A^{-1}\langle L_-\rangle = (A^2 - A^{-2})\langle L_0\rangle.$$

Second, the writhes of L_- and L_+ are equal to $w(L_0)-1$ and $w(L_0)+1$, respectively. Let $w = w(L_0)$, to ease the notation. Using these facts, there are no new ideas required to finish the exercise (recall that $A = t^{-1/4}$):

$$\begin{aligned}
(t^{-1/2} - t^{1/2})V(L_0) &= (A^2 - A^{-2})(-A^3)^{-w}\langle L_0\rangle \\
&= (-A^3)^{-w}\left(A\langle L_+\rangle - A^{-1}\langle L_-\rangle\right) \\
&= (-A^4)(-A^{-3})(-A^3)^{-w}\langle L_+\rangle - (-A^{-4})(-A^3)(-A^3)^{-w}\langle L_-\rangle \\
&= (-A^4)V(L_+) - (-A^{-4})V(L_-) \\
&= -t^{-1}V(L_+) + tV(L_-)
\end{aligned}$$

6. The Jones polynomial of 4_1.

Solution. We make use of the skein relations proved in the previous problem, since the figure-8 knot can be unknotted by a single crossing change and the Jones polynomial is invariant under Reidemeister moves. To do this, we need to orient our knot. See Let 2_1^2 denote the Hopf link obtained in the skein



relation above. So

$$t^{-1}V(4_1) = t + (t^{1/2} - t^{-1/2})V(2_1^2).$$

Note that for this calculation the Hopf link is oriented, and its writhe depends on the orientation. In our case, it is -2. The Kauffman bracket of the Hopf link is $-A^4 - A^{-4}$, so we have

$$\begin{aligned} V(4_1) &= t(t + (t^{1/2} - t^{-1/2})(-t^{-3/4})^2(-t^{-1} - t) \\ &= t^{-2} - t^{-1} + 1 - t + t^2 \end{aligned}$$

This is consistent with the polynomial given in the knot table at the end of Adams.