MATH 309 - Solution to Homework 3
By Mihai Marian
March 11, 2019

1. Midterm #3.

   Proof. See the midterm solution set. $\square$

2. Mutation preserves the bracket polynomial.

   Proof. See class notes from lecture 10. $\square$

3. Let K be a knot, $\langle K \rangle = \sum_{i=0}^{n} c_i A^i$ its bracket polynomial, and $V_K(t)$ its Jones polynomial. The $\mathrm{Span}(\langle K \rangle)$ is a knot invariant and $\mathrm{Span}(\langle K \rangle) = 4\mathrm{Span}(V_K(t))$

   Proof. The proof that $\mathrm{Span}(\langle K \rangle)$ is a knot invariant appears in Adams' book right after the definition of span, on page 160. The key idea is that $\langle K \rangle$ is the same for two different knot projections up to scaling (multiplication by A^m for some $m \in \mathbb{Z}$), and scaling a polynomial does not affect its span.

   Now, according to the definition at the end of Adams 6.1, $V_K(t)$ is obtained from the bracket polynomial by first scaling it by $(-A^3)^{-w(K)}$ and then replacing the polynomial variable by $t^{-1/4}$. Scaling does not affect the span, but replacing the variable by $t^{-1/4}$ gives a polynomial with a span 4 times as small. $\square$

4. The calculation of $\langle K \rangle$ from class.

   Solution. Note that this calculation is done for the left-handed trefoil knot on page 158 of Adams' book. The states of the right-handed trefoil are obtained from the ones of the left-handed trefoil by switching the labels A and B; see Figure 1 below, and be wary of the slight abuse of notation: the regions in the plane are labeled A and B, but this labeling has nothing to do with the variable A appearing in the Kauffman bracket polynomial.
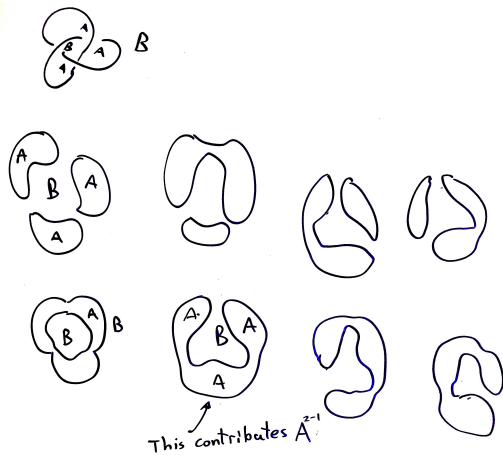

Figure 1: The 8 states of the right-handed trefoil

   Thus we can obtain $\langle K \rangle$ either from the computation on page 158 of Adams by replacing $a(S)$ with $b(S)$ and $b(S)$ with $a(S)$, or by reading the contribution of each state directly off of Figure 1. Let's use the second method: in the first row of states in Figure 1 the first state is the trivial unlink of 3 components, and it was obtained by 3 "B-splits" (i.e. by resolving 3 crossings such that B-regions got connected), therefore its contribution to the Kauffman bracket is $A^{0-3}(A^2 - A^{-2})^{3-1}$. The other 3

states have 2 components and were obtained by 1 A-split and 2 B-splits, thus each of them contributes $A^{1-2}(A^2 - A^{-2})^{2-1}$. And so on. We get

$$\langle K \rangle = A^{0-3}(-A^2 - A^{-2})^{3-1} + 3\left(A^{1-2}(-A^2 - A^{-2})^{2-1}\right) + 3\left(A^{2-1}(-A^2 - A^{-2})^{1-1}\right) + A^{3-0}(-A^2 - A^{-2})^{2-1}$$
$$= A^{-3}(-A^2 - A^{-2})^2 + (3A^{-1} + A^3)(-A^2 - A^{-2}) + 3A$$
$$= A^{-7} - A^{-3} - A^5$$

Note that $\langle K \rangle$ is obtained from the bracket polynomial of the left-handed trefoil on p. 158 of Adams by replacing A with A^{-1}.

5. The Alexander and Jones polynomials of 4_1 and their spans.

Solution. In homework 2 we computed the Jones polynomial: $V_{4_1}(t) = t^{-2} - t^{-1} + 1 - t + t^2$, so $\mathrm{Span}(V_{4_1}(t)) = 4$. To compute the Alexander polynomial, let us use the Skein relation at the beginning of 6.3. We first pick an orientation on the knot (a direction to travel along the knot) and a crossing, as in Figure 2 below. Note that the end result will be independent of the choice of orientation and crossing. So
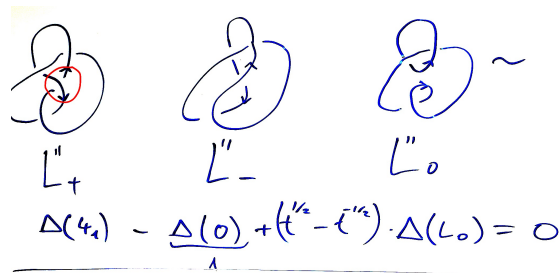

Figure 2: The Alexander polynomial of 4_1

$$\Delta(4_1) = 1 - (t^{1/2} - t^{-1/2})\Delta(L_0'')$$

and we just need to compute the Alexander polynomial of L_0, the Hopf link with a chosen orientation. Again, pick a crossing on L_0''. It is negative. Turning it into a positive crossing gives a disjoint union of two unknots and resolving the crossing as in the Skein relation gives us an unknot. Therefore we have

$$\Delta(\bigcirc\ \bigcirc) - \Delta(L_0'') + (t^{1/2} - t^{-1/2})\Delta(\bigcirc) = 0.$$

By exercise 6.15, $\Delta(\bigcirc\ \bigcirc) = 0$, so $\Delta(L_0'') = (t^{1/2} - t^{-1/2})$. In conclusion,

$$\Delta(4_1) = 1 - (t^{1/2} - t^{-1/2})^2,$$

so $\mathrm{Span}(\Delta(4_1)) = 2$, which strictly less than the span of the Jones polynomial for the same knot.