MATH 309 - Solution to Homework 4

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1. Splittable links have 0 Alexander polynomial.

   Proof. If L is a splittable link, then it has a projection as L_0 in figure 6.36 (in Adams). Applying the Skein relation of the Alexander polynomial to the sequence of links in figure 6.36, we obtain

$$\Delta(L_+) - \Delta(L_-) + (t^{1/2} - t^{-1/2})\Delta(L_0) = 0.$$

   L_+ and L_- are projections of the same knot, so their polynomials are equal and cancel in the above equation. We get $(t^{1/2} - t^{-1/2})\Delta(L_0) = 0$, therefore $\Delta(L) = 0$. □

2. Twist knots have genus 1.

   Proof. First of all, twist knots are nontrivial knots (a proof of this: a twist knot with n twists can be obtained as the closure of the rational link $[n\ 2]$. Since the fraction associated to this link is not 0 or $1/n$, it is a nontrivial knot). Thus the genus of a twist knot is ≥ 1.

   So we need to prove that every twist knot has genus $\leq$1. The most obvious Seifert surface for a twist knot is the one in figure 1 below. Note that this procedure always gives a surface, but it only gives

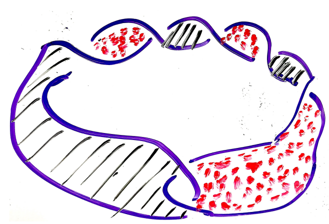

Figure 1: A Seifert surface for the twist knot $[-5\ 2]$.

an orientable surface if the twist knot has an odd number of twists. However, even if the number of twists is odd, we can add a crossing as in figure 2, by moving the indicated strand. Then we can apply

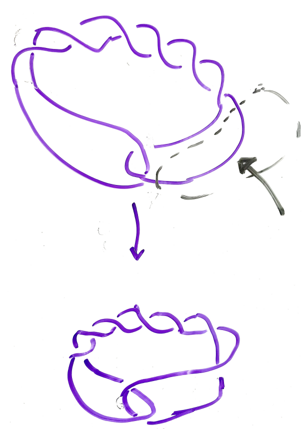

Figure 2: Two projections of the knot $[6\ 2]$.

the same procedure as for twist knots with an odd number of twists to get an orientable surface; see figure 3. In either case, we can triangulate the surface by adding 4 vertices and 2 edges, as in figure 4.

Figure 3: A Seifert surface for the knot [6 2].

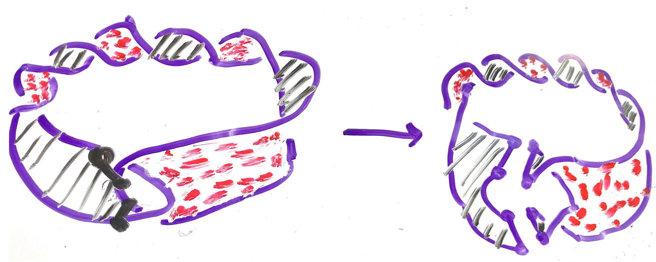

Figure 4: A triangulation on the Seifert surface of a twist knot

Cutting along the black edges makes it clear that this is indeed a triangulation, with 1 face, 6 edges and 4 vertices. The Euler characteristic is therefore

$$\chi = 4 - 6 + 1 = -1,$$

which is the Euler characteristic of a torus with one boundary component. By the classification of surfaces (explained in Adams p. 92, without proof), the Seifert surface is indeed a torus with one boundary component, therefore the knot has genus ≤ 1. $\square$

3. The Alexander polynomial of a projection of the figure 8 via the linking matrix.

Solution. It is not hard to see that the figure in the homework is a projection of the figure 8. This particular projection suggests we consider the Seifert surface drawn in figure 5, where the different colour schemes indicate the two sides of the surface and the red curves c_1 and c_2 are the cores of the handles. Pushing c_1 and c_2 in the unit normal direction, we obtain two new curves, c_1^+ and c_2^+. Their

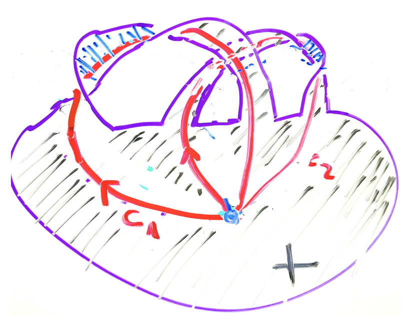

Figure 5: A Seifert surface for the figure-eight knot.

position is given in figure 6. In the same figure, the curves c_1, c_2, c_1^+, c_2^+ are also drawn by themselves, without the Seifert surface. It is easy to see now that the linking matrix is

$$A = \begin{pmatrix} lk(c_1^+, c_1) & lk(c_1^+, c_2) \\ lk(c_2^+, c_1) & lk(c_2^+, c_2) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$$

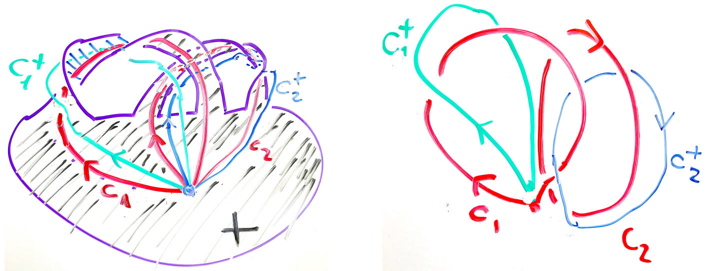

Figure 6: The Seifert surface with c_1^+ and c_2^+, and the link formed by c_1, c_2, c_1^+, c_2^+.

Therefore

$$\det\left(A - tA^t\right) = \det\begin{pmatrix} -1+t & 0-t \\ 1-0 & 1-t \end{pmatrix} = (-1+t)(1-t)+t = t(-t+3-t^{-1}),$$

which is the Alexander polynomial of the figure 8, up to scaling by a power of t.

4. Let us compute the HOMFLY polynomial of 6_3. The algorithm applies in the same way to 6_1 and 6_2.

   Solution. First, put an orientation on the projection of 6_3 and select a crossing to resolve, as in figure 7. The links we obtain by changing the crossing are a right-handed trefoil and a link known as the Whitehead link. Let us use T and Wh to denote their HOMFLY polynomials. We further need to resolve a crossing on each of these links, which we do in the same figure. This lets us see that the HOMFLY polynomial of 6_3 can be expressed in terms of T and the HOMFLY polynomial of the Hopf link oriented so that its linking number is $+1$. Let X denote the HOMFLY polynomial of this oriented Hopf link. Since we will need this later, in figure 8 we show the resolution of the Hopf link, as well as

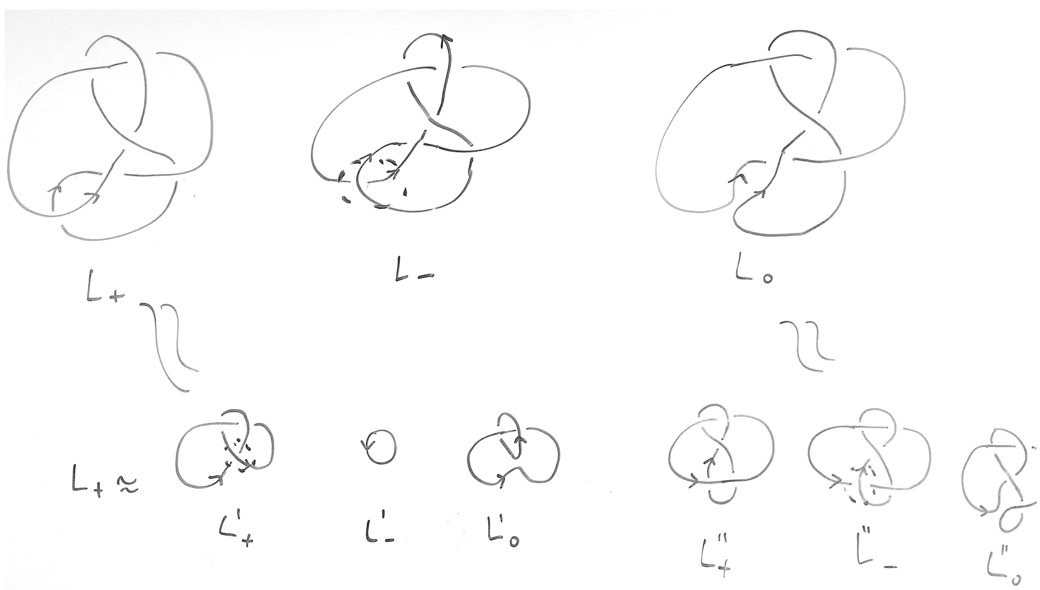

Figure 7: Resolution of a crossing of 6_3 and of the resulting knots.

that of the disjoint union of two unknots. Explicitly, the Skein relations in figure 8 let us conclude

$$P(\circ\circ) = -m^{-1}(l+l^{-1})$$

and thus

$$X = P(L'_0) = -l^{-1}(l^{-1}P(\circ\circ) + mP(\circ)) = -l^{-1}(l^{-1}(-m^{-1}(l+l^{-1})) + m).$$

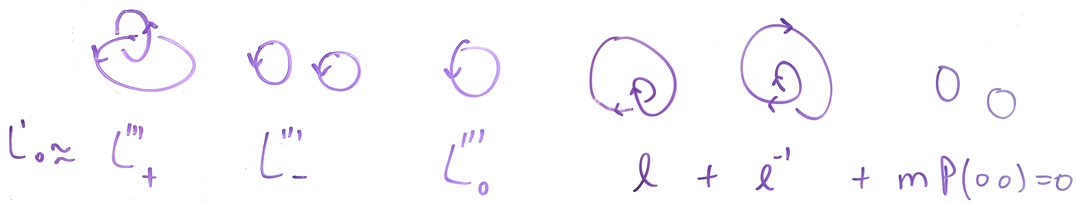

Figure 8: Resolution of the Hopf link with a given orientation, and computation of $P(\circ\circ)$.

We can now compute the polynomial of our right-handed trefoil with the given orientation, but let us write everything in terms of X so that the polynomial remains somewhat readable and we can do as little algebra as possible:

$$\begin{aligned} T = P(L_+) &= -l^{-1}\left(l^{-1}P(\circ) + mP(L_0')\right) \\ &= -l^{-2} - l^{-1}mX \end{aligned}$$

In terms of T, the polynomial of our (oriented) Whitehead link:

$$P(L_0) = -l(lX + mT)$$

This lets us compute the polynomial of 6_3 in terms of T and X:

$$\begin{aligned} P(L_-) &= -l(lP(L_+) + mP(L_0)) \\ &= -l^2T - lm(-l(lX + mT)) \\ &= (-l^2 + m^2l^2)T + ml^3X \end{aligned}$$

In terms of X, this is

$$\begin{aligned} P(L_-) &= (-l^2 + m^2l^2)(-l^{-2} - l^{-1}mX) + ml^3X \\ &= (1 - m^2) + lm(1 - m^2 + l^2)X \end{aligned}$$

Finally, plugging in the value of $X = P(L_0')$ obtained above, we get the HOMFLY polynomial of 6_3:

$$\begin{aligned} P(L_-) &= 1 - m^2 + m(1 - m^2 + l^2)(-l^{-1}P(\circ\circ) - m) \\ &= 1 - m^2 + m(1 - m^2 + l^2)(-l^{-1}(-m^{-1}(l + l^{-1})) - m) \\ &= 1 - m^2 + (1 - m^2 + l^2)(1 + l^{-2} - m^2) \\ &= (1 - m^2) + (1 - m^2)(2 + l^{-1} + l^2) + m^4 \\ &= (1 - m^2)(l^{-2} + 3 + l^2) + m^4 \end{aligned}$$

If we replace the variables by $l \mapsto it^{-1}$ and $m \mapsto i(t^{-1/2} - t^{1/2})$, we obtain the Jones polynomial:

$$\begin{aligned} V(6_3) &= (1 - (i(t^{-1/2} - t^{1/2}))^2)((it^{-1})^{-2} + 3 + (it^{-1})^2) + (i(t^{-1/2} - t^{1/2}))^4 \\ &= (t^{-1} - 1 + t)(-t^2 + 3 - t^{-2}) + t^{-2} - 4t^{-1} + 6 - 4t + t^2 \\ &= -t^{-3} + 2t^{-2} - 2t^{-1} + 3 - 2t + 2t^2 - t^3 \end{aligned}$$

If we replace the variables by $l \mapsto i$ and $m \mapsto i(t^{1/2} - t^{-1/2})$, we obtain the Alexander polynomial:

$$\begin{aligned} \Delta(6_3) &= (1 - (i(t^{1/2} - t^{-1/2}))^2)((i)^{-2} + 3 + (i)^2) + (i(t^{1/2} - t^{-1/2}))^4 \\ &= (t^{-1} - 1 + t) + t^{-2} - 4t^{-1} + 6 - 4t + t^2 \\ &= t^{-2} - 3t^{-1} + 5 - 3t + t^2 \end{aligned}$$

5. Applying Seifert's algorithm to 6_3.

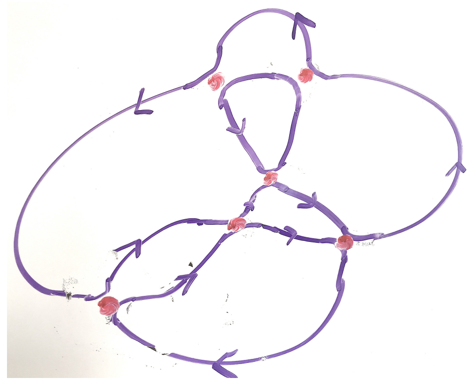

Figure 9: Seifert circles obtained from 6_3. Red dots indicate places where we need to attach bands

Solution. First, given an orientation on 6_3 (for example the one from figure 7) we form Seifert circles. There are three of them: see figure 9. The genus of this surface is given by the formula

$$g = \frac{\#\text{crossings} - \#\text{Seifert circles} + 1}{2} = 2.$$

The Alexander polynomial has span 4, and this is a lower bound on twice the genus of 6_3, i.e. the genus of 6_3 is at least 2. Since the Seifert surface we found achieves this lower bound, we can conclude that the genus of 6_3 is in fact 2.