MATH 309 - Solution to Homework 5

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1. (a) The Jones polynomial of the Kanenobu knot $K(n, -n)$ is independent of n.

Proof. Note that the writhe $w(K(n, -n))$ is 0 for all n, so the bracket polynomial of $K(n, -n)$ is identical to its X polynomial, from section 6.1 in Adams, which becomes the Jones polynomial after changing the variable. So we just need to compute the bracket $\langle K(n, -n)\rangle$. Do this by resolving the two crossings shown in figure 1 and applying the idea of 6.2 (in Adams) to record that each A-split adds a factor of A and each B-split adds a factor of A^{-1}: □

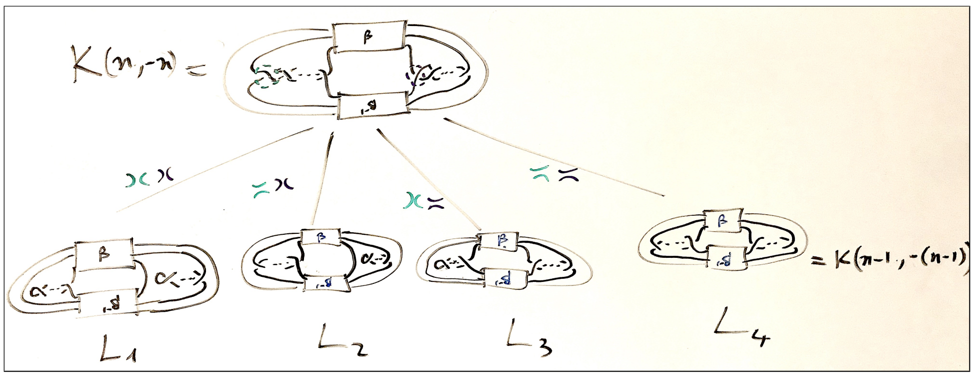


Figure 1: A depth 2 resolution tree for $K(n, -n)$

Note that L_1 is obtained from $K(n, -n)$ by a B-split in the left twists (indicated by the green tangle along the line connecting L_1 to $K(n, -n)$) and an A-split in the right twists. Smilarly, note the number of A- and B-splits to get each L_i. So we have

$$\langle K(n, -n)\rangle = A^{-1} \cdot A\langle L_1\rangle + A^2\langle L_2\rangle + A^{-2}\langle L_3\rangle + A \cdot A^{-1}\langle L_4\rangle.$$

In all cases, the resulting knot has two fewer crossings, so $L_4 = K(n - 1, -(n - 1)$. Recall that Reidemeister 1 (R1) moves affect the bracket polynomial as in figure 2. Now we can simplify

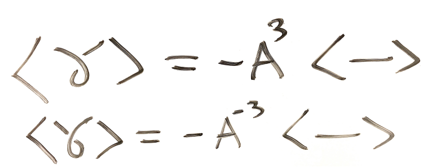
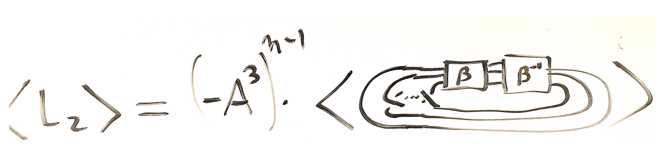
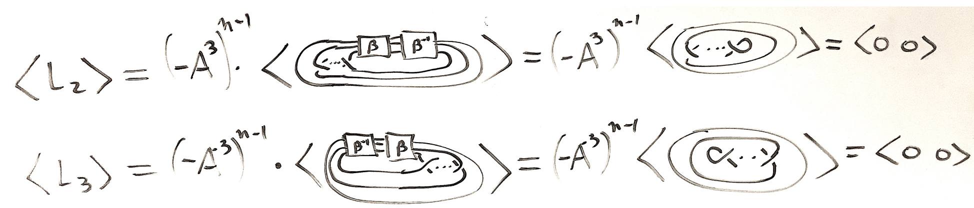


Figure 2: The effect of R1 moves on the Kauffman bracket

L_1, L_2 and L_3 by applying a certain number of $R1$ moves, each of which will add a factor of $-A^3$ or $-A^{-3}$ to the bracket polynomial. For example, we can apply $n - 1$ $R1$ moves to L_2 and then slide β^{-1} next to β to obtain the following:



Recall also that $R2$ and $R3$ moves do not affect the bracket polynomial, so we can annihilate the braid β and its inverse Do this, apply some more $R1$ moves and do the same thing to L_3 to obtain the following:



Similarly, $\langle L_1 \rangle$ is the bracket polynomial of three nested circles. Finally, recall that, given any link L and a separated unlink $\bigcirc$, we have

$$\langle L \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle L \rangle.$$

This applies to the computation of L_1, L_2 and L_3 (it is used twice to compute $\langle L_1 \rangle$). We have

$$\begin{aligned}
\langle K(n,-n) \rangle &= \langle L_1 \rangle + A^2\langle \bigcirc \cup \bigcirc \rangle + A^{-2}\langle \bigcirc \cup \bigcirc \rangle + \langle K(n-1,-(n-1)) \rangle \\
&= (-A^2 - A^{-2})^2 + A^2(-A^2 - A^{-2}) + A^{-2}(-A^2 - A^{-2}) + \langle K(n-1,-(n-1)) \rangle \\
&= \langle K(n-1,-(n-1) \rangle
\end{aligned}$$

Thus $\langle K(n,-n) \rangle$ is the same for all values of $n \in \mathbb{Z}$ and the Jones polynomial of $K(n,-n)$ is also the same for all values of n, by the argument given in the first paragraph of this proof. To compute it, we should simply compute it for $n = 0$. In this case the Kanenobu knot is a connected sum of two figure-8 knots and the Jones polynomial is multiplicative over connected sums, so

$$V(K(n,-n)) = (V(4_1))^2 = (t^{-2} - t^{-1} + 1 - t + t^2)^2.$$

(b) The Alexander polynomial of $K(2n, 0)$ is the same for all values of n.

Proof. We can see this by a direct application of the Alexander polynomial skein relation. In this case we need to know what the braid is in order to put a consistent orientation on every strand of the knot. In figure 3, we only indicated the orientation. Note that the Alexander polynomial is

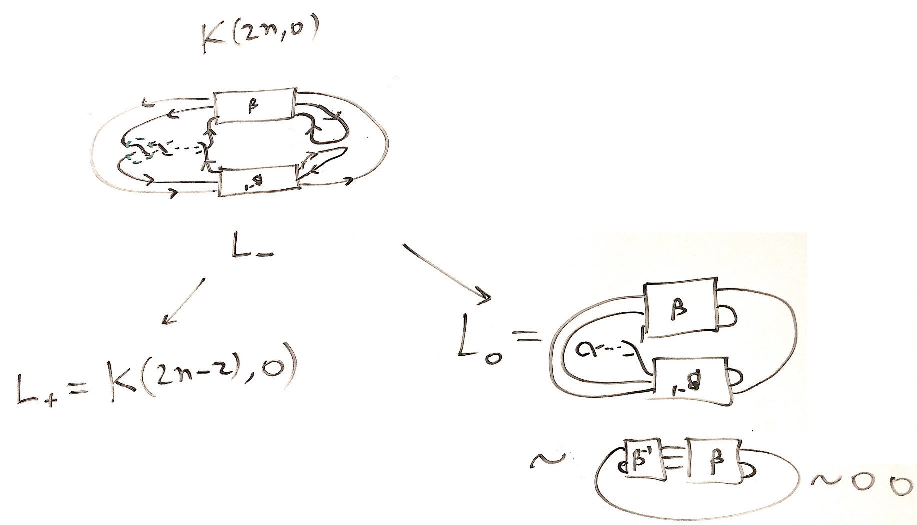


Figure 3: Skein relation applied to a crossing in $K(2n, 0)$

invariant under all Reidemeister moves, so there is no need to worry about what types of moves we are applying to the knots obtained from resolving a crossing. We have

$$\Delta(K(2n-2,0)) - \Delta(K(2n,0)) + (t^{1/2} - t^{-1/2})\Delta(\bigcirc \cup \bigcirc) = 0,$$

but separable links have trivial Alexander polynomial, so we have

$$\Delta(K(2(n-1),0) = \Delta(K(2n,0)),$$

proving our claim. To compute this polynomial, we compute it for $K(0,0) = 4_1 \# 4_1$:

$$\Delta(K(0,0)) = (\Delta(4_1))^2 = (-t^{-1} + 3 - t)^2.$$

□

2. We show how to find a braid representing 6_1, which is the hardest one. The idea is explained in Adams' Exercise 5.16. In particular, the idea does not involve applying the algorithm presented in the book.

   Solution. Note that we can obtain a braid representation of a knot by finding an oriented projection of the knot such that walking in the positive direction, we are always turning counterclockwise around some point, which we call the centre. This because once we have such a projection, we can cut it along a line segment emanating from the centre and the result is necessarily a braid. The following sequence of pictures should be more enlightening than any explanation. Remark: the red dot is the centre, and the orientation was chosen arbitrarily.

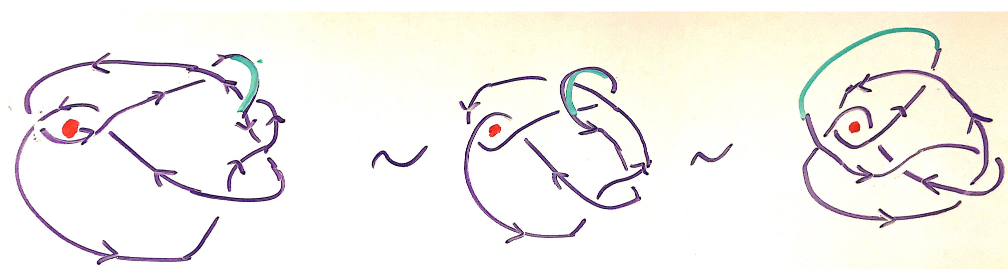


Figure 4: First move: take a strand running clockwise and push it over the centre to make it run counterclockwise

Let us record the following useful move:

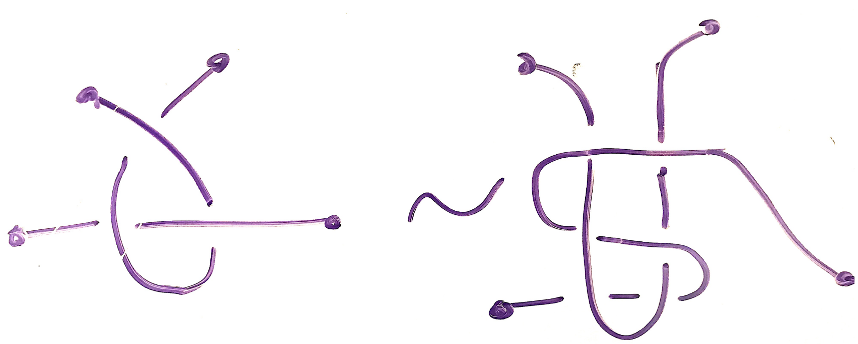


Figure 5: A useful move to change a common tangle

Applying the move to figure 4, we find a projection with all strands running counterclockwise in figure 6. The resulting braid is drawn in figure 7.

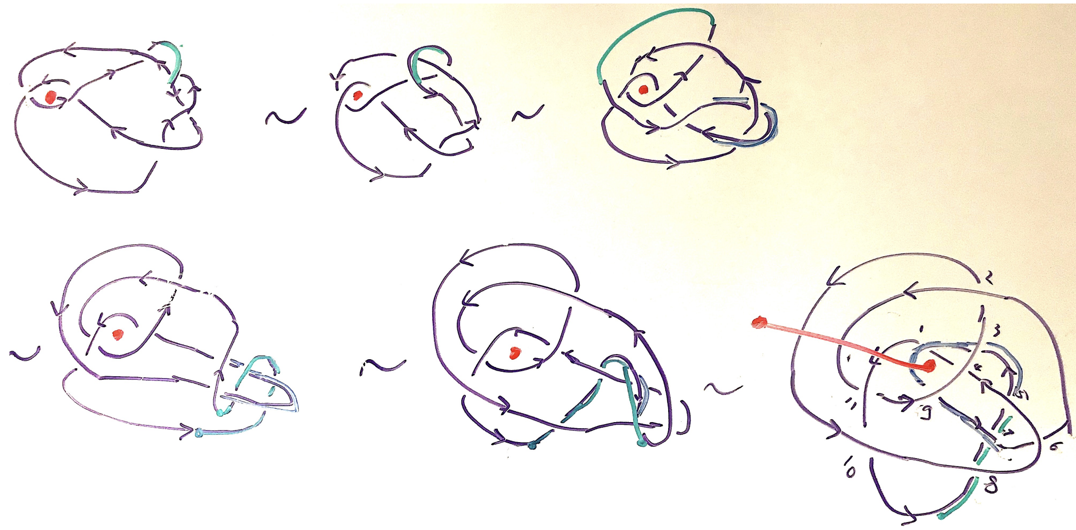


Figure 6: An oriented projection of 6_1 with all strands running counterclockwise

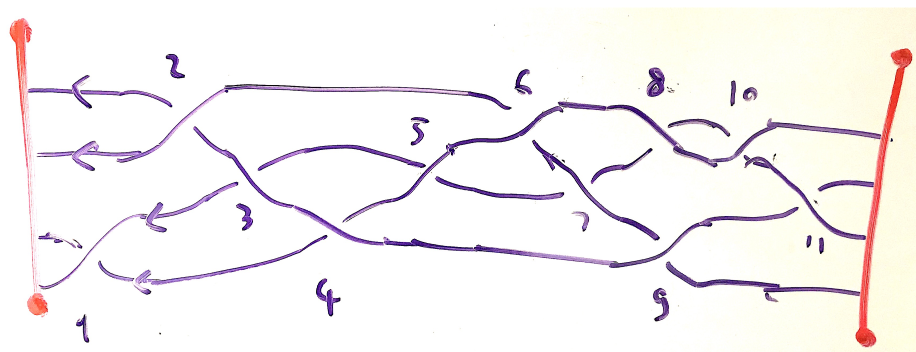


Figure 7: A braid whose closure is the knot 6_1. The labels on the crossings correspond to those in figure 6

3. The knots with braid index 2.

   Solution. These are the knots which can be obtained from a 2-strand braid, minus the knots which can also be obtained as the closure of a 1-strand braid (i.e. minus the unknot). Braid closures of 2-strand braids are exactly the same as closures of rational tangles $[n]$. Out of these, the rational tangles $[-1]$ and $[1]$ close up to be unknots, which have braid index 1, so we must exclude them from our list of knots with braid index 2. Note that the closure of $[n]$ is a knot if n is odd and a 2-component link if n is even.

   Equivalently, knots with braid index 2 are $(n, 2)$-torus links, for $n \neq \pm 1$.

4. The closure of the n-strand braid $(\sigma_1 \ldots \sigma_{n-1})^m$ is a knot iff n and m are coprime.

   Proof. Note: to see why this is true and even get a hint on how to prove it, you should look at some examples with different values of n and m. In particular, try to see why it is that $(\sigma_1\sigma_2 \ldots \sigma_5)^2$ forms a link that is not a knot.

   To figure out whether the closure of a braid is a knot or not, we only need to keep track of the endpoints of the strings forming the braid. For example, if the n strings start and end at the same height, then we will get an n-component link, independently of how the strings of the braid are interlocking.

In our case, labeling the heights from top to bottom by $1, 2, \ldots, n$, the following happens: the string starting at height 1 goes down to height n, the string starting at height 2 ends up at height 1, etc. See figure 8 below.

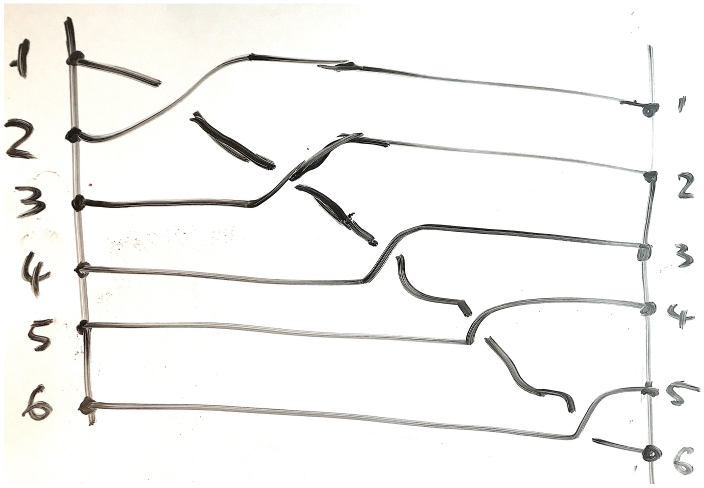


Figure 8: The braid $\sigma_1\sigma_2 \ldots \sigma_5$

In the standard notation for permutations, this switching of heights is denoted by: $(1\ n\ n-1\ \ldots\ 3\ 2)$. In your head, you can read this as "1 goes to n, n goes to $n-1$, …, and 3 goes to 2", to indicate how the strings in the braid change heights.

Remark. (A review of permutations) We can apply the above permutation twice to get $(1\ n\ n-1\ \ldots\ 3\ 2)^2$, which is "1 goes to $n-1$, $n-1$ goes to $n-3$, etc." Note that this permutation is not necessarily of the form $(1\ n-1\ n-3\ \ldots\ 5\ 3)$, as can be seen in the example:

Example 1. $(1\ 4\ 3\ 2)^2 = (1\ 3)(2\ 4)$; in other words, in the braid $(\sigma_1\sigma_2\sigma_3)^2$, the string at height 1 ends up at height 3 and the string at height 3 end up at height 1 and similarly, the string at height 2 end up at height 4 and vice-versa. Closing this braid, we end up with a 2-component link (just because walking along the string that starts at height 1 we end up where we started without ever walking along the string that goes from height 2 to 4, so these must be disjoint components).

Thus "permutations on n things" is a group (the identity element is the trivial permutation "1 goes to 1, 2 goes to 2, etc.", which we denote by 1 from now on).

Let us make a definition: a permutation of the form $(1\ n\ n-1\ \ldots\ 3\ 2)$ is called an **n-cycle**. We now make the following claim: the braid closure of $(\sigma_1 \ldots \sigma_{n-1})^m$ is a knot iff the associated permutation on heights is an n-cycle.

Proof: we have essentially proved one implication of this assertion in the parenthetical remark at the end of the previous paragraph: if the permutation is not an n-cycle then, after closing the braid, we can walk along a string and get back where we started without ever touching some part of the link. Conversely, if the permutation is an n-cycle then, while travelling along the knot, we reach every string in the braid before ending up back where we started, proving that we walked along a single circle, i.e. that the link is a knot. This completes the proof.

So we have transformed our claim about knots and braids into the following equivalent but slightly easier algebraic claim: $(1\ n\ n-1\ \ldots\ 3\ 2)^m$ is an n-cycle iff n and m are coprime. Suppose, without loss of generality that $1 \leq m \leq n$.

First, if $\gcd(m, n) = 1$, then, by applying the division algorithm successively, it is possible to find integers a, b such that $am + bn = 1$ (this is called Bézout's lemma). Therefore

$$(1\ n\ n-1\ \ldots\ 3\ 2)^{am+bn} = (1\ n\ n-1\ \ldots\ 3\ 2),$$

which means that

$$((1\ n\ n-1\ \ldots\ 3\ 2)^{am} = (1\ n\ n-1\ \ldots\ 3\ 2).$$

But then $(1\ n\ n-1\ \ldots\ 3\ 2)^m$ cannot be a product of disjoint cycles $c_1 \ldots c_k$, since a power of such a product is $(c_1 \ldots c_k)^a = c_1^a \ldots c_k^a$ (think about this). Thus $(1\ n\ n-1\ \ldots\ 3\ 2)^m$ is an n-cycle.

Conversely, if $\gcd(m, n) \neq 1$, then we can find an integer a with $1 \leq a < n$ such that $am \equiv 0 \pmod{n}$ (take a to be $n/\gcd(m, n)$ and recall that $mn = \operatorname{lcm}(m, n) \cdot \gcd(m, n)$). Therefore

$$((1\ n\ n-1\ \ldots\ 3\ 2)^m)^a = 1,$$

so $(1\ n\ n-1\ \ldots\ 3\ 2)^m$ has order a, which is less than n. It follows that it is not an n-cycle because n-cycles have order n. □

5. When n and m are coprime, the closure of $(\sigma_1 \ldots \sigma_{n-1})^m$ is the (m, n)-torus knot.

   Proof. Draw the braid on a cylinder, as in figure 9. We can always do this because we can imagine the strings of the braid starting from a circle instead of a straight line; then our braid is just a rotation of all the strings by $2\pi/n$, so they live on a cylinder (without any self-intersections). Closing the cylinder by attaching the ends together to form a torus, we obtain a knot (by the previous exercise) which intersects the meridian of the torus n times, so that it runs n times around the longitude. We can also draw a longitude and count the intersections to find that t it runs m times along the meridian direction. Thus it is the (m, n)-torus knot. □

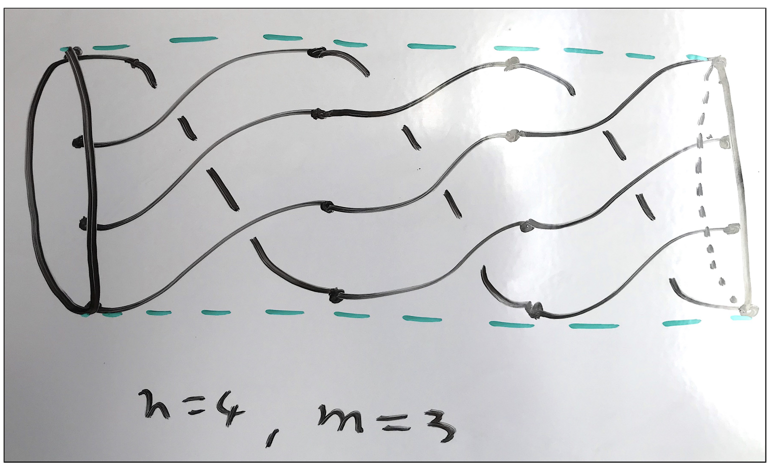


Figure 9: The braid drawn on a cylinder